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Coherent states, (discrete) frames and sampling on manifolds:Applications

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Conformal Wavelets in Complex Minkowski Space

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Realization of $SO(4,2)$ in 3+1D Minkowski spacetime

$$\mathbb{M}^4 = \{x = (x_0 = t, x_1, x_2, x_3) \in \mathbb{R}^4, \; x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2\}$$

- Poincaré transformations $\mathcal{P} = SO(3,1) \circledS \mathbb{R}^4$

$$x'^{\mu} = x^{\mu} + \mathrm{b}^{\mu}, \quad x'^{\mu} = \Lambda^{\mu}_{\nu}(\omega) x^{\nu}$$

Dilations

$$x'^{\mu} = e^{\tau} x^{\mu}, \quad e^{\tau} \in \mathbb{R}_+$$

Special conformal transformations (relativistic uniform accelerations)

$$x'^{\mu} = \frac{x^{\mu} + \mathrm{a}^{\mu} x^2}{1 + 2\mathrm{a}x + \mathrm{a}^2 x^2}, \quad \mathrm{a}^{\mu} \in \mathbb{R}^4$$

$$x^{\mu} \xrightarrow{\mathrm{inv}} \frac{x^{\mu}}{x^2} \xrightarrow{+\mathrm{a}_{\mu}} \frac{x^{\mu} + x^2 \mathrm{a}^{\mu}}{x^2} \xrightarrow{\mathrm{inv}} \frac{(x^{\mu} + x^2 \mathrm{a}^{\mu})/x^2}{(x^{\mu} + x^2 \mathrm{a}^{\mu})^2/x^4} = x'^{\mu}.$$

Isomorphism: $SO(4,2) = SU(2,2)/\mathbb{Z}_4$

$$SU(2,2) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_{4\times 4}(\mathbb{C}) : g^\dagger \Gamma g = \Gamma, \det(g) = 1 \right\}$$

Γ hermitian form with signature $(++--)$. Inverse: $g^{-1} = \Gamma g^\dagger \Gamma$.
Identifying the 3+1D Minkowski space with the space of 2×2 hermitian matrices, $X = x_\mu \sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$, with

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the so called Pauli matrices, then

Local Isomorphism: $SO(4,2) = SU(2,2)/\mathbb{Z}_4$

$$\boxed{X \to X' = (AX+B)(CX+D)^{-1}}, \; g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(2,2)$$

- Lorentz transformations: $B = C = 0$ and $A = D^{-1\dagger} \in SL(2,\mathbb{C})$.

  Dilations: $B = C = 0$ y $A = D^{-1} = e^{\tau/2} I$

  Spacetime translations: $A = D = I$, $C = 0$ y $B = \mathrm{b}_\mu \sigma^\mu$.

  SCT: $A = D = I, B = 0$ y $C = \mathrm{a}_\mu \sigma^\mu$

  Taking into account that: $\det(CX + I) = 1 + 2\mathrm{a}x + \mathrm{a}^2 x^2$

$$X' = X(CX+I)^{-1} \leftrightarrow x'^\mu = \frac{x^\mu + \mathrm{a}^\mu x^2}{1 + 2\mathrm{a}x + \mathrm{a}^2 x^2}$$

Phase space: Future tube domain

$$\mathbb{T}_4 = \mathbb{C}^4_+ \equiv \{W = X + iY \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : \; Y > 0\}$$

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$W = w_\mu \sigma^\mu, \quad w_\mu = x_\mu + i y_\mu \in \mathbb{C}, \; \mu = 0, 1, 2, 3; y^0 > \|\vec{y}\|$$

which can be mapped to the Cartan domain

$$\mathbb{D}_4 = U(2,2)/U(2)^2 = \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : I - ZZ^\dagger > 0\}$$

$$(I - ZZ^\dagger > 0 \Rightarrow \mathrm{tr}(ZZ^\dagger) < 2 \Rightarrow \mathbb{D}_4 \subset \mathbb{B}_8(0, \sqrt{2}))$$

through the Cayley transformation:

$$\begin{array}{lll} W & \to & Z(W) = (I - iW)^{-1}(I + iW), \\ Z & \to & W(Z) = i(I - Z)(I + Z)^{-1}, \end{array}$$

The Shilov boundary (compactified Minkowski space) is:

$$\check{\mathbb{D}}_4 = \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : I - ZZ^\dagger = 0\} = U(2) = (\mathbb{S}^3 \times \mathbb{S}^1)/\mathbb{Z}_2$$

Obtaining the Cartan domain as a homogeneous space of the conformal group. The 0+1 (time-energy) case

Iwasawa decomposition for $g \in SU(1,1) \sim \mathbb{D}_1 \times U(1)$ (loc.)

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \delta & z\delta \\ \bar{z}\delta & \delta \end{pmatrix} \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix},$$

where $z = b/d \in \mathbb{D}_1 \subset \mathbb{C}$ (stereographic projection of the hyperboloid onto the open unit disk), $\delta = (1 - z\bar{z})^{-1/2}$ and $e^{i\beta} = a/|a|$.

This decomposition is adapted to the quotient $\mathbb{D}_1 = SU(1,1)/U(1)$.

Iwasawa decomposition for any $g \in U(2,2) \sim \mathbb{D}_4 \times U(2)^2 (loc.)$

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \Delta_1 & Z\Delta_2 \\ Z^\dagger \Delta_1 & \Delta_2 \end{pmatrix} \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix},$$

with $Z(g) = BD^{-1}, \;\; Z^\dagger(g) = CA^{-1}$ and

$$\begin{aligned} \Delta_1 &= (AA^\dagger)^{1/2} = (I - ZZ^\dagger)^{-1/2}, \; U_1 = \Delta_1^{-1} A, \\ \Delta_2 &= (DD^\dagger)^{1/2} = (I - Z^\dagger Z)^{-1/2}, U_2 = \Delta_2^{-1} D. \end{aligned}$$

This decomposition is adapted to the quotient $\mathbb{D}_4 = U(2,2)/U(2)^2 = \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : I - ZZ^\dagger > 0\}$ of $G = U(2,2)$ by the maximal compact subgroup $H = U(2)^2$.

Hilbert space and invariant measure

Let us consider the space $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ of square-integrable holomorphic functions $\phi(Z)$ on the Cartan domain:

$$\mathbb{D}_4 = \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : I - ZZ^\dagger > 0\},$$

with $SU(2,2)$-invariant integration measure:

$$d\nu_\lambda(Z, Z^\dagger) = W_\lambda(Z, Z^\dagger)|dZ|, \quad \lambda \geq 4,$$

where

$$W_\lambda(Z, Z^\dagger) \equiv \pi^{-4}(\lambda - 1)(\lambda - 2)^2(\lambda - 3)\det(I - ZZ^\dagger)^{\lambda - 4}$$

is a weight function, $|dZ|$ denotes the usual Lebesgue measure on $\mathbb{C}^4$ and $\lambda = 4, 5, 6, \ldots$ is the scale dimension.

The 0+1 (time-energy) case

Hilbert space $L^2_h(\mathbb{D}_1, d\nu_\lambda)$ of square-integrable holomorphic functions $\phi(z)$ on the open unit disk:

$$\mathbb{D}_1 = \{z \in \mathbb{C} : 1 - z\bar{z} > 0\},$$

with $SU(1,1)$-invariant integration measure:

$$d\nu_\lambda(z, \bar{z}) = \frac{2\lambda - 1}{\pi}(1 - z\bar{z})^{2(\lambda-1)}|dz|, \quad \lambda \geq 1,$$

$$\begin{aligned} \langle\phi|\phi'\rangle &= \int_{\mathbb{D}_1} \overline{\phi(z)}\phi'(z)d\nu_\lambda(z, \bar{z}) \\ &= \frac{2\lambda - 1}{\pi} \int_{\alpha=0}^{\alpha=2\pi} \int_{r=0}^{r=1} \overline{\phi(re^{i\alpha})}\phi'(re^{i\alpha})(1 - r^2)^{2(\lambda-1)} r dr\, d\alpha. \end{aligned}$$

Unitary irreducible square integrable representation

Proposition

For any group element $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(2,2)$, *the following (left-)action*

$$\begin{aligned} \phi_g(Z) &\equiv [\mathcal{U}_\lambda(g)\phi](Z) = \det(D^\dagger - B^\dagger Z)^{-\lambda}\phi(Z'), \\ Z' &= g^{-1}Z = (A^\dagger Z - C^\dagger)(D^\dagger - B^\dagger Z)^{-1} \end{aligned}$$

defines a unitary irreducible square integrable representation of $SU(2,2)$ *on* $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ *under the invariant scalar product*

$$\langle\phi|\phi'\rangle = \int_{\mathbb{D}_4} \overline{\phi(Z)}\phi'(Z)d\nu_\lambda(Z, Z^\dagger).$$

Theorem

For any group element $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(2,2)$*, the representation* $\phi_g(Z) \equiv [\mathcal{U}_\lambda(g)\phi](Z) = \det(D^\dagger - B^\dagger Z)^{-\lambda}\phi(Z')$ *with* $Z' = g^{-1}Z = (A^\dagger Z - C^\dagger)(D^\dagger - B^\dagger Z)^{-1}$*, is square integrable, the constant unit function* $\psi(Z) = 1, \forall Z \in \mathbb{D}_4$ *being an admissible vector (fiducial state or mother wavelet), i.e.:*

$$c_\psi = \int_G |\langle \mathcal{U}_\lambda(g)\psi|\psi\rangle|^2 d\mu(g) < \infty \quad (1)$$

and the set of coherent states (or wavelets) $F = \{\psi_g = \mathcal{U}_\lambda(g)\psi, g \in G\}$ *constituting a continuous tight frame in* $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ *satisfying the resolution of the identity:*

$$\mathcal{A} = \int_G |\psi_g\rangle\langle\psi_g| d\mu(g) = c_\psi \mathcal{I}. \quad (2)$$

Wigner's $\mathcal{D}$-matrices

$$\mathcal{D}^j_{q_1,q_2}(Z) = \sqrt{\frac{(j+q_1)!(j-q_1)!}{(j+q_2)!(j-q_2)!}} \sum_{k=\max(0,q_1+q_2)}^{\min(j+q_1,j+q_2)} \binom{j+q_2}{k} \binom{j-q_2}{k-q_1-q_2} \times z_{11}^k z_{12}^{j+q_1-k} z_{21}^{j+q_2-k} z_{22}^{k-q_1-q_2},$$

where $j \in \mathbb{N}/2$ (the spin) runs on all non-negative half-integers, $q_1, q_2 = -j, -j+1, \ldots, j-1, j$.
$\mathcal{D}^j_{q_1,q_2}(Z)$ are homogeneous polynomials of degree $2j$ in z_{ij}.

Iwasawa decomposition for $Z = U \in SU(2) = \mathbb{S}^3 \sim \mathbb{S}^2 \times \mathbb{S}^1$

$$Z = U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \delta & -z\delta \\ \bar{z}\delta & \delta \end{pmatrix} \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix},$$

where $z = -b/d \in \overline{\mathbb{C}} \simeq \mathbb{S}^2$ (the one-point compactification of $\mathbb{C}$ by inverse stereographic projection), $\delta = (1 + z\bar{z})^{-1/2}$ and $e^{i\beta} = a/|a|$. $SU(2)$-invariant measure:

$$ds(U) = (1 + z\bar{z})^{-2} dz d\bar{z} d\beta = \frac{r dr d\alpha d\beta}{(1 + r^2)^2}, \quad z = re^{i\alpha}$$

$$\int_{\mathbb{S}^3} ds(U) \mathcal{D}^j_{q_1,q_2}(\overline{U}) \mathcal{D}^{j'}_{q'_1,q'_2}(U) = \frac{2\pi^2}{2j+1} \delta_{j,j'} \delta_{q_1,q'_1} \delta_{q_2,q'_2}.$$

Wigner's $\mathcal{D}$-matrices: other properties

Multiplication property (representation):

$$\sum_{q'=-j}^{j} \mathcal{D}^j_{qq'}(Z)\mathcal{D}^j_{q'q''}(Z') = \mathcal{D}^j_{qq''}(ZZ').$$

Transpositional symmetry:

$$\mathcal{D}^j_{qq'}(Z) = \mathcal{D}^j_{q'q}(Z^T), \quad (3)$$

MacMahon Master Theorem for Z of size $N = 2$:

$$\sum_{j\in\mathbb{N}/2} \sum_{q,q'=-j}^{j} \mathcal{D}^j_{qq'}(Z)\mathcal{D}^j_{qq'}(Z'^T) = \det(I - ZZ')^{-1}. \quad (4)$$

Orthonormal basis of $L^2_h(\mathbb{D}_4, d\nu_\lambda)$

Proposition

The set of homogeneous polynomials of degree $2j+2m$:

$$\varphi^{j,m}_{q_1,q_2}(Z) \equiv \sqrt{\frac{2j+1}{\lambda-1}\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}}\,\det(Z)^m \mathcal{D}^j_{q_1,q_2}(Z),$$
$$m \in \mathbb{N},\ j \in \mathbb{N}/2,\ q_1, q_2 = -j, -j+1, \ldots, j-1, j,$$

constitutes an orthonormal basis of $L^2_h(\mathbb{D}_4, d\nu_\lambda)$, that is:

$$\langle \varphi^{j,m}_{q_1,q_2} | \varphi^{j',m'}_{q'_1,q'_2} \rangle = \delta_{j,j'}\delta_{m,m'}\delta_{q_1,q'_1}\delta_{q_2,q'_2}.$$

Note that, in general, the number of linearly independent mononomials $\prod_{i,j=1}^{2} z_{ij}^{n_{ij}}$ of degree of homogeneity $n = \sum_{i,j=1}^{2} n_{ij}$ is $(n+1)(n+2)(n+3)/6 = \binom{n+3}{3} = \binom{n+3}{n}$.

$$|{*}{*}|{*}|{*}\,{*}\,{*}|{*}| \qquad \binom{7+3}{3} = CR_4^7 = C_{10}^3$$

This coincides with the number of linearly independent polynomials $\varphi_{q_1,q_2}^{j,m}$ with degree of homogeneity $n = 2m + 2j$. This proves that the set of polynomials $\{\varphi_{q_1,q_2}^{j,m}\}$ is a basis for analytic functions $\phi \in L_h^2(\mathbb{D}_4, d\nu_\lambda)$. Moreover, this basis turns out to be orthonormal.

The set of holomorphic functions on the unit disk $\mathbb{D}_1$,

$$\varphi_n(z) \equiv \binom{2\lambda + n - 1}{n}^{1/2} z^n, \quad n = 0, 1, 2, \ldots, \tag{5}$$

constitutes an orthonormal basis of $L^2_h(\mathbb{D}_1, d\nu_\lambda)$, i.e. $\langle \varphi_m | \varphi_n \rangle = \delta_{m,n}$. These basis functions verify the following closure relation:

$$\sum_{n=0}^{\infty} \overline{\varphi_n(z)} \varphi_n(z') = (1 - \bar{z} z')^{-2\lambda},$$

which is nothing other than the reproducing (Bergman) kernel of $L^2_h(\mathbb{D}_1, d\nu_\lambda)$

Closure relation: "reproducing Bergman kernel"

$$\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j}\overline{\varphi^{j,m}_{q_1,q_2}(Z)}\varphi^{j,m}_{q_1,q_2}(Z') = \frac{1}{\det(I - Z^\dagger Z')^{\lambda}}$$

Can be proved through the λ-Extended MacMahon-Schwinger Master Theorem ($tX = Z^\dagger Z'$)

$$\sum_{j\in\mathbb{N}/2}\frac{2j+1}{\lambda-1}\sum_{n=0}^{\infty}t^{2j+2n}\binom{n+\lambda-2}{\lambda-2}\binom{n+2j+\lambda-1}{\lambda-2}\det(X)^n \times \sum_{q=-j}^{j}\mathcal{D}^j_{qq}(X) = \det(I - tX)^{-\lambda}.$$

$\lambda = 2$: Szegö kernel

(Fundamental result in Combinatorial Analysis)

Theorem

(MacMahon Master Theorem) *Let X be an $N \times N$ matrix of indeterminates x_{ij}, and Y be the diagonal matrix $Y \equiv \mathrm{diag}(y_1, y_2, \ldots, y_N)$. Then the coefficient of $y^\alpha \equiv y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_N^{\alpha_N}$ in the expansion of* $\det(I - XY)^{-1}$ *equals the coefficient of y^α in the product*

$$\prod_{i=1}^{N} (x_{i1} y_1 + x_{i2} y_2 + \cdots + x_{iN} y_N)^{\alpha_i}.$$

Schwinger's Formula ($N = 2$)

MMT turns out to be essentially equivalent to:

Theorem

(Schwinger's Formula). *Let X be any 2×2 matrix X and $Y = tI$, where t is an arbitrary parameter and I stands for the 2×2 identity matrix. Then the following identity holds:*

$$e^{(\partial_u : X : \partial_v)} e^{(u: Y^T : v)} \Big|_{u=v=0} = \sum_{j \in \mathbb{N}/2} t^{2j} \sum_{q=-j}^{j} \mathcal{D}^j_{qq}(X) = \det(I - tX)^{-1}$$

where we denote by

$$(u : X : v) \equiv uXv^T = \sum_{i,j=1}^{N} u_i x_{ij} v_j, \quad \partial_{u_i} \equiv \frac{\partial}{\partial u_i}.$$

Extended MacMahon-Schwinger Master Theorem

Theorem

(λ-Extended SMT) *For every* $\lambda \in \mathbb{N}, \lambda \geq 2$ *and every* 2×2 *matrix* X*, the following identity holds:*

$$\sum_{j \in \mathbb{N}/2} \frac{2j+1}{\lambda - 1} \sum_{n=0}^{\infty} t^{2j+2n} \binom{n+\lambda-2}{\lambda-2} \binom{n+2j+\lambda-1}{\lambda-2} \det(X)^n \times \sum_{q=-j}^{j} \mathcal{D}^j_{qq}(X) = \det(I - tX)^{-\lambda}.$$

Using the λ-Extended MacMahon-Schwinger Master Theorem for $tX = D^{-1}CZ$, we can expand the coherent state $\psi_g = \mathcal{U}_\lambda(g)\psi, g \in G$, as

$$\begin{aligned}\psi_g(Z) &= \det(D^\dagger - B^\dagger Z)^{-\lambda} = \det(D^\dagger)^{-\lambda}\det(I - (BD^{-1})^\dagger Z)^{-\lambda} \\ &= \det(D^\dagger)^{-\lambda}\sum_{j=0}^{\infty}\frac{2j+1}{\lambda-1}\sum_{n=0}^{\infty}\binom{n+\lambda-2}{\lambda-2}\binom{n+2j+\lambda-1}{\lambda-2} \\ &\times \det((BD^{-1})^\dagger Z)^n \sum_{q=-j}^{j}\mathcal{D}^j_{qq}((BD^{-1})^\dagger Z). \end{aligned} \tag{6}$$

Now, taking into account that $\det((BD^{-1})^\dagger Z)^n = \det((BD^{-1})^\dagger)^n \det(Z)^n$ and the Wigner matrix property

$$\mathcal{D}^j_{qq}((BD^{-1})^\dagger Z) = \sum_{q'=-j}^{j}\mathcal{D}^j_{qq'}((BD^{-1})^\dagger)\mathcal{D}^j_{q'q}(Z), \tag{7}$$

Conformal wavelets in terms of basis functions

we recognize the orthonormal basis functions $\varphi^{j,n}_{q'q}(Z)$ in the last expansion, so that we can write the coherent states (wavelets) as an expansion in terms of the basis functions as:

$$\psi_g(Z) = \sum_{j\in\mathbb{N}/2}\sum_{n=0}^{\infty}\sum_{q,q'=-j}^{j}\hat{\psi}^{j,n}_{q',q}(g)\varphi^{j,n}_{q'q}(Z)$$

with “Fourier” coefficients

$$\begin{aligned}\hat{\psi}^{j,n}_{q',q}(g) &= \det(D^\dagger)^{-\lambda}\sqrt{\frac{2j+1}{\lambda-1}\binom{n+\lambda-2}{\lambda-2}\binom{n+2j+\lambda-1}{\lambda-2}} \\ &\times \det((BD^{-1})^\dagger)^n\mathcal{D}^j_{qq'}((BD^{-1})^\dagger) = \overline{\det(D)^{-\lambda}\varphi^{j,n}_{q',q}(BD^{-1})}.\end{aligned}$$

Reconstruction Formula

The reconstruction formula of a given function ϕ adopts the following form:

$$\phi(Z) = \int_G \Phi_\psi(g)\psi_g(Z)d\mu(g),$$

with wavelet coefficients

$$\Phi_\psi(g) = \frac{1}{c_\psi}\langle\psi_g|\phi\rangle = \frac{1}{c_\psi}\int_{\mathbb{D}_4} \det(D - Z^\dagger B)^{-\lambda}\phi(Z)d\nu_\lambda(Z, Z^\dagger).$$

and $d\mu(g)$ the Haar measure on $SU(2,2)$.

Reconstruction Formula

Expanding ϕ in the basis $\{\varphi^{j,n}_{q,q'}(Z)\}$

$$\phi(Z) = \sum_{j\in\mathbb{N}/2}\sum_{n=0}^{\infty}\sum_{q,q'=-j}^{j} \hat{\phi}^{j,n}_{q,q'}\varphi^{j,n}_{q,q'}(Z),$$

and using

$$\psi_g(Z) = \sum_{j\in\mathbb{N}/2}\sum_{n=0}^{\infty}\sum_{q,q'=-j}^{j} \hat{\psi}^{j,n}_{q',q}(g)\varphi^{j,n}_{q'q}(Z)$$

together with the orthogonality properties of $\{\varphi^{j,n}_{q,q'}(Z)\}$, we can write the wavelet coefficients in terms of the Fourier coefficients $\hat{\phi}^{j,n}_{q',q}$ as:

$$\begin{aligned} \Phi_\psi(g) &= \frac{1}{c_\psi}\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q,q'=-j}^{j} \overline{\hat{\psi}^{j,m}_{q,q'}(g)}\hat{\phi}^{j,m}_{q,q'} \\ &= \frac{1}{c_\psi}\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q,q'=-j}^{j} \det(D)^{-\lambda}\varphi^{j,m}_{q,q'}(BD^{-1})\hat{\phi}^{j,m}_{q,q'}. \end{aligned}$$

Connecting the Cartan and Tube domain pictures

Proposition

The correspondence

$$\begin{array}{cccc} \mathcal{S}_\lambda : & L^2_h(\mathbb{D}_4, d\nu_\lambda) & \longrightarrow & L^2_h(\mathbb{C}^4_+, d\tilde{\nu}_\lambda) \\ & \phi & \longmapsto & \mathcal{S}_\lambda \phi \equiv \tilde{\phi}, \end{array}$$

with

$$\tilde{\phi}(W) = 2^{2\lambda} \det(I - iW)^{-\lambda} \phi(Z(W)),$$

$$Z(W) = (I - iW)^{-1}(I + iW), \quad W(Z) = i(I - Z)(I + Z)^{-1},$$

$$d\tilde{\nu}_\lambda(W, W^\dagger) \equiv \frac{c_\lambda}{2^4} \det(\frac{i}{2}(W^\dagger - W))^{\lambda-4} |dW|,$$

is an isometry. Actually

$$\langle \phi | \phi' \rangle_{L^2_h(\mathbb{D}_4, d\nu_\lambda)} = \langle \mathcal{S}_\lambda \phi | \mathcal{S}_\lambda \phi' \rangle_{L^2_h(\mathbb{C}^4_+, d\tilde{\nu}_\lambda)}.$$

As a direct consequence, the set of functions defined by

$$\tilde{\varphi}^{j,m}_{q_1,q_2}(W) \equiv 2^{2\lambda} \det(I - iW)^{-\lambda} \varphi^{j,m}_{q_1,q_2}(Z(W)),$$

constitutes an orthonormal basis of holomorphic rational functions of $L^2_h(\mathbb{C}^4_+, d\tilde{\nu}_\lambda)$ and the closure relation

$$\sum_{j\in\mathbb{N}/2} \sum_{m=0}^{\infty} \sum_{q,q'=-j}^{j} \overline{\tilde{\varphi}^{j,m}_{q',q}(W)} \tilde{\varphi}^{j,m}_{q',q}(W') = \det(\frac{i}{2}(W^\dagger - W'))^{-\lambda},$$

gives the reproducing (Bergman) kernel in $L^2_h(\mathbb{C}^4_+, d\tilde{\nu}_\lambda)$.

Differential Equation

$$W = w_\mu \sigma^\mu, \quad w_\mu = x_\mu + i y_\mu \in \mathbb{C}, \ \mu = 0,1,2,3$$

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{aligned} P_\mu &= \frac{\partial}{\partial w^\mu}, \quad M_{\mu\nu} = w_\mu \frac{\partial}{\partial w^\nu} - w_\nu \frac{\partial}{\partial w^\mu}, \\ D &= w^\mu \frac{\partial}{\partial w^\mu}, \quad K_\mu = -2 w_\mu w^\nu \frac{\partial}{\partial w^\nu} + w^2 \frac{\partial}{\partial w^\mu}, \end{aligned}$$

$$\left(D^2 - \frac{1}{2} M_{\mu\nu} M^{\mu\nu} + \frac{1}{2} (P_\mu K^\mu + K_\mu P^\mu) \right) \tilde{\phi}(w) = m_\lambda^2 \tilde{\phi}(w)$$

$$m_\lambda^2 = \lambda(\lambda + 4)$$

Conclusions

- We construct the Continuous Wavelet Transform (CWT) on the Cartan domain $\mathbb{D}_4 = U(2,2)/U(2)^2$ of the conformal group $SU(2,2)$ in 1+3 dimensions.
  The manifold $\mathbb{D}_4$ can be mapped one-to-one onto the future tube domain $\mathbb{C}^4_+$ of the complex Minkowski space through a Cayley transformation, where other kind of (electromagnetic) wavelets have already been proposed in the literature:
  - G. Kaiser, Space-time-scale Analysis of Electromagnetic Waves, Proceedings of the IEEE-SP International Symposium on Time-Frequency and Time-Scale Analysis, Victoria, (1992) 209-212.
  - G. Kaiser, Wavelet Electrodynamics II: Atomic Composition of Electromagnetic Waves, Appl. Comput. Harmon. Anal. **1** (1994) 246-260.
  - G. Kaiser, Physical wavelets and their sources: real physics in complex spacetime, J. Phys. A: Math. Gen. **36** (2003) R291-R338

Conclusions

- We study the unitary irreducible representations of the conformal group on the Hilbert spaces $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ and $L^2_h(\mathbb{C}^4_+, d\tilde{\nu}_\lambda)$ of square integrable holomorphic functions with scale dimension λ and continuous mass spectrum, prove the isomorphism (equivariance) between both Hilbert spaces, admissibility and tight-frame conditions, provide reconstruction formulas and orthonormal basis of homogeneous polynomials (and discuss symmetry properties and the Euclidean limit of the proposed conformal wavelets).

  For that purpose, we firstly state and prove a λ-extension of Schwinger's Master Theorem (SMT), which turns out to be a useful mathematical tool for us, particularly as a generating function for the unitary-representation functions of the conformal group and for the derivation of the reproducing (Bergman) kernel of $L^2_h(\mathbb{D}_4, d\nu_\lambda)$.

Conclusions

- SMT is related to MacMahon's Master Theorem (MMT) and an extension of both in terms of Louck's $SU(N)$ solid harmonics has also been provided in the first Refeence of the next page. This generalization applies to unirreps of general pseudounitary groups $U(N, N)$.

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