The Abdus Salam
International Centre
for Theoretical Physics
50th Anniversary 1964-2014

2585-33

Joint ICTP-TWAS School on Coherent State Transforms, Time-Frequency and Time-Scale Analysis, Applications

2 - 20 June 2014

Coherent state and wavelets on manifolds: Modular wavelets

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Index

CWT on the sphere $\mathbb{S}^2$

- Antoine and Vandergheynst defined satisfactorily a dilation on the sphere $\mathbb{S}^2$. They used a group-theoretical approach based on the Lorentz group $G = SO(3,1)$
- $G = KAN$ with K compact, A Abelian and N nilpotent subgroups. The parameter space X of the CWT is the quotient G/N
- The expression for the dilation, with parameter $a > 0$, of the colatitude angle θ is
$$\theta_a = 2\arctan(a\tan(\theta/2)),$$
- Geometrical interpretation as a dilation around the North Pole of the sphere, lifted from the tangent plane by inverse stereographic projection
- A unitary representation of this dilation is given by

$$[D_a^{\mathbb{S}^2} f](\theta,\varphi) = \lambda(a,\theta)^{1/2} f(\theta_{1/a},\varphi),$$

CWT on the sphere $\mathbb{S}^2$ (II)

- The multiplier (Radon-Nikodym derivative) is:

$$\lambda(a,\theta) = \frac{d\cos\theta_{1/a}}{d\cos\theta} = \frac{4a^2}{((a^2-1)\cos\theta + a^2+1)^2}$$

- Points of X are pairs (β, a) with $\beta \in SO(3)$ (rotations) and $a \in \mathbb{R}^+$ (dilations).
- Given $f \in L^2(\mathbb{S}^2)$, the representation

$$f_{\beta,a}(\theta,\varphi) := [U_\beta^{\mathbb{S}^2} \circ D_a^{\mathbb{S}^2} f](\theta,\varphi)$$

is unitary, where $[U_\beta^{\mathbb{S}^2} f](\theta,\varphi) = f(\beta^{-1}(\theta,\varphi))$ is the quasi-regular representation of $SO(3)$.

Admissibility and Frame conditions

- A non-zero function $f \in L^2(\mathbb{S}^2)$ is called admissible (mod(N,σ)) iff the condition
$$0 < \int_X d\nu(\beta, a) |\langle f_{\beta,a} | \psi \rangle|^2 < \infty$$
is satisfied for any $\psi \in L^2(\mathbb{S}^2)$, where $d\nu(\beta, a) = \frac{da}{a^3} d\mu(\beta)$ is the measure on X and $d\mu(\beta)$ is the Haar measure on $SO(3)$.
- A weaker (necessary but not sufficient) admissibility condition is
$$\int_{\mathbb{S}^2} \frac{f(\theta, \varphi)}{1 + \cos\theta} d\Omega = 0.$$
- Given an admissible $f \in L^2(\mathbb{S}^2)$, the family $\{f_{\beta,a}, \beta \in SO(3), a > 0\}$ is a frame iff there exist $0 < A \leq B$ such that
$$A\|\psi\|^2 \leq \int_X d\nu(\beta, a) |\langle f_{\beta,a} | \psi \rangle|^2 \leq B\|\psi\|^2, \ \forall \psi \in L^2(\mathbb{S}^2).$$

Lifting admissible functions from the Euclidean plane

- Any function $\phi \in L^2(\mathbb{R}^2)$ fulfilling the (weak) zero mean admissibility condition

  $$\int_{\mathbb{R}^2} \phi(r,\varphi) r dr d\varphi = 0$$

  (in polar coordinates), provides a function f on the sphere by inverse stereographic projection

  $$f(\theta,\varphi) = [\Pi_{\mathbb{S}^2}^{-1}\phi](\theta,\varphi) = \frac{2\phi(2\tan(\theta/2),\varphi)}{1+\cos\theta} \qquad (1)$$

  that satisfies the weak admissibility condition for the sphere.
- This result also holds for the (strong) admissibility condition.

Hilbert space of functions on the torus

- Hilbert space $L^2(\mathbb{T}^2, d\omega)$ of square integrable functions on the torus $\mathbb{T}^2$, with measure $d\omega = d\theta_1 d\theta_2$
- θ_1, θ_2 angles parametrizing the "meridional" and "equatorial" circles.
- $d\omega$ invariant under translations $\theta_{1,2} \to \theta_{1,2} + \vartheta_{1,2}$, derived from the Haar measure on $SO(2,2)$.
- Inner product with respect to this measure:

$$\langle f|g\rangle := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \overline{f(\theta_1,\theta_2)}g(\theta_1,\theta_2)d\omega, \qquad \forall f,g \in L^2(\mathbb{T}^2)$$

- Orthonormal basis of $L^2(\mathbb{T}^2)$ in terms of "plane waves"

$$\begin{aligned}\phi_{n_1 n_2}(\theta_1,\theta_2) &= \frac{1}{2\pi} e^{in_1\theta_1} e^{in_2\theta_2}, \quad n_1, n_2 \in \mathbb{Z} \\ \langle \phi_{n_1,n_2} | \phi_{n_1',n_2'}\rangle &= \delta_{n_1,n_1'}\delta_{n_2,n_2'}.\end{aligned}$$

- The coefficients $\widehat{f}^{n_1,n_2} := \langle \phi_{n_1,n_2} | f\rangle$ are the usual Fourier coefficients of $f \in L^2(\mathbb{T}^2)$.

Dilations on the torus

- The group of "conformal" transformations of the torus is:

$$SO(2,2) = (SO(2,1) \times SO(2,1))/\mathbb{Z}_2$$

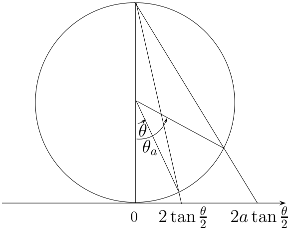
- $SO(2,1) \approx SL(2,\mathbb{R})$, and any 2×2 matrix of determinant one can be decomposed as

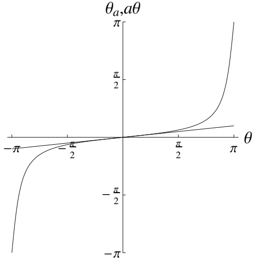
$$\begin{pmatrix} \cos(\vartheta/2) & \sin(\vartheta/2) \\ -\sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} \begin{pmatrix} \sqrt{a} & 0 \\ 0 & 1/\sqrt{a} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

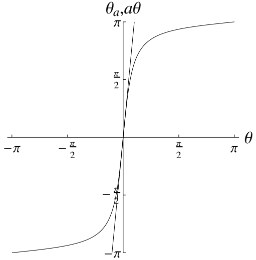
  then the KAN decomposition of $SL(2,\mathbb{R})$ is given by $K_1 = \mathbb{T}^1 = \mathbb{S}^1$, $A_1 = \mathbb{R}^+$ and $N_1 = \mathbb{R}$.
- Since $SO(2,2)$ is locally the direct product of two copies of $SO(2,1)$, the parameter space of the CWT is $X = KAN/N = \mathbb{T}^2 \times (\mathbb{R}^+)^2$ labeled by $(\vartheta_1, \vartheta_2, a_1, a_2)$, with $\vartheta_i \in (-\pi, \pi)$, $a_i \in \mathbb{R}^+$ for $i = 1, 2$.

The action of the dilation group A on the torus is:

$$\theta_a = 2\arctan(a\tan(\theta/2)), \;\; \theta = \theta_k, \; a = a_k, \; k = 1, 2.$$







Dilations on the torus

- The dilation of each angle of the torus is similar to the one for the colatitude angle in the sphere, but in our case $\theta_k \in (-\pi, \pi)$ instead of $(0, \pi)$.
- As for the sphere, this transformation are interpreted as independent dilations around the points $\theta_i = 0$, $i = 1, 2$, lifted from the tangent lines to each circle by inverse stereographic projections.
- For $f \in L^2(\mathbb{T}^2)$, a pure dilation is

$$[D_{a_1,a_2} f](\theta_1, \theta_2) = \lambda(a_1, \theta_1)^{1/2} \lambda(a_2, \theta_2)^{1/2} f((\theta_1)_{1/a_1}, (\theta_2)_{1/a_2}),$$

- The multiplier (Radon-Nikodym derivative) is

$$\lambda(a, \theta) = \frac{d\theta_{1/a}}{d\theta} = \frac{2a}{(a^2 - 1)\cos\theta + a^2 + 1}$$

- Given $f \in L^2(\mathbb{T}^2)$, the action

$$\begin{aligned} f_{a_1,a_2}^{\vartheta_1,\vartheta_2}(\theta_1, \theta_2) &= [U_{\vartheta_1,\vartheta_2} \circ D_{a_1,a_2} f](\theta_1, \theta_2) \\ &= \lambda(a_1, \theta_1 - \vartheta_1)^{\frac{1}{2}} \lambda(a_2, \theta_2 - \vartheta_2)^{\frac{1}{2}} f((\theta_1 - \vartheta_1)_{\frac{1}{a_1}}, (\theta_2 - \vartheta_2)_{\frac{1}{a_2}}) \end{aligned}$$

is unitary, where $U_{\vartheta_1,\vartheta_2}$ is the representation of translations on the torus.

Admissibility condition on the torus

Definition

A non-zero function $\gamma \in L^2(\mathbb{T}^2)$ is admissible (mod(N,σ)) iff the condition

$$0 < \int_X d\nu(\vartheta_1, \vartheta_2, a_1, a_2) |\langle \gamma_{a_1,a_2}^{\vartheta_1,\vartheta_2} | \psi \rangle|^2 < \infty$$

is satisfied for all $0 \neq \psi \in L^2(\mathbb{T}^2)$.

- The measure on X is
$$d\nu(\vartheta_1, \vartheta_2, a_1, a_2) = \frac{da_1}{a_1^2} \frac{da_2}{a_2^2} \frac{d\vartheta_1 d\vartheta_2}{(2\pi)^2}.$$
- The admissibility condition can be restated as follows:

Proposition

A non-zero function $\gamma \in L^2(\mathbb{T}^2)$ is admissible iff there exist $C \in \mathbb{R}$ such that

$$0 < \Lambda_{n_1,n_2} \equiv \int_0^\infty \int_0^\infty \frac{da_1}{a_1^2} \frac{da_2}{a_2^2} |\widehat{\gamma}_{a_1,a_2}^{n_1,n_2}|^2 < C < \infty \quad \forall (n_1, n_2) \in \mathbb{Z}^2$$

where $\widehat{\gamma}_{a_1,a_2}^{n_1,n_2} = \langle \phi_{n_1,n_2} | \gamma_{a_1,a_2} \rangle$ are the Fourier coefficients of $\gamma_{a_1,a_2} = D_{a_1,a_2}\gamma$.

A necessary admissibility condition on the torus

Proposition

A non-zero function $\gamma \in L^2(\mathbb{T}^2)$ *is admissible only if it fulfills the condition*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Gamma(\theta_1,\theta_2)d\theta_1 d\theta_2 = 0\,.$$

where $\Gamma(\theta_1,\theta_2) := \gamma(\theta_1,\theta_2)/\sqrt{(1+\cos\theta_1)(1+\cos\theta_2)}$.

- Admissibility condition does not guarantee a proper reconstruction of a function from its wavelet coefficients, and a frame condition is required.
- However, the admissibility condition is enough (and easier to proof) if γ is localized.
- By "localized" we mean that $\theta_{i,a_i} \approx a_i\theta_i, \forall(\theta_1,\theta_2) \in \mathrm{supp}(\gamma)$ and $a_i \leq 1$ (i.e., a valid approximation in the Euclidean limit).
- For practical purposes, this is not really a restriction since the approximation $\theta_a \approx a\theta$ is quite good for a large range of θ when $a \leq 1$.

Frame condition on the torus

- Let us denote by $Q_q, q = 1, 2, 3, 4$, the four quadrants of the Fourier plane in counterclockwise order.
- Since dilations do not mix quadrants, and translations do not change the support of $\widehat{\gamma}$, it is clear that $\widehat{\gamma}$ must have support on all (four) quadrants in order to be admissible.

Theorem

For any localized admissible function γ, the family $\{\gamma_{a_1,a_2}^{\vartheta_1,\vartheta_2},\ (\vartheta_1,\vartheta_2,a_1,a_2) \in X\}$ is a continuous frame; that is, there exist real constants $0 < c \le C$ such that

$$c||\psi||^2 \le \int_X d\nu(\vartheta_1,\vartheta_2,a_1,a_2)|\langle\gamma_{a_1,a_2}^{\vartheta_1,\vartheta_2}|\psi\rangle|^2 \le C||\psi||^2, \ \ \forall\psi \in L^2(\mathbb{T}^2).$$

- In the proof of this theorem we use the property (valid for localized admissible functions)

$$\widehat{\gamma}_{a_1,a_2}^{n_1,n_2} \approx 2\sqrt{a_1 a_2}\,\widehat{\Gamma}^{a_1 n_1, a_2 n_2}, \quad a_1, a_2 << 1$$

CWT on the torus and reconstruction formula

- The CWT of a function $\psi \in L^2(\mathbb{T}^2)$ reads as:

$$\Psi_{a_1,a_2}^{\vartheta_1,\vartheta_2} = \langle \gamma_{a_1,a_2}^{\vartheta_1,\vartheta_2} | \psi \rangle = \iint_{\mathbb{T}^2} \overline{\gamma_{a_1,a_2}^{\vartheta_1,\vartheta_2}(\theta_1,\theta_2)} \psi(\theta_1,\theta_2) d\omega, \ \psi \in L^2(\mathbb{T}^2).$$

- The original function ψ can be reconstructed (in the weak sense) from its wavelet coefficients $\Psi_{a_1,a_2}^{\vartheta_1,\vartheta_2}$ by means of the reconstruction formula:

$$\psi(\theta_1,\theta_2) = \int_X d\nu(a_1,a_2,\vartheta_1,\vartheta_2) \Psi_{a_1,a_2}^{\vartheta_1,\vartheta_2} \, \widetilde{\gamma}_{a_1,a_2}^{\vartheta_1,\vartheta_2}(\theta_1,\theta_2)$$

where $\{\widetilde{\gamma}_{a_1,a_2}^{\vartheta_1,\vartheta_2}\}$ is the dual frame whose Fourier coefficients are given by

$$\langle \phi_{n_1 n_2} | \widetilde{\gamma}_{a_1,a_2}^{\vartheta_1,\vartheta_2} \rangle = \Lambda_{n_1 n_2}^{-1} \langle \phi_{n_1 n_2} | \gamma_{a_1,a_2}^{\vartheta_1,\vartheta_2} \rangle.$$

- Note that the dual frame is well-defined ($0 \neq \widetilde{\gamma}_{a_1,a_2}^{\vartheta_1,\vartheta_2} \in L^2(\mathbb{T}^2)$) since admissibility condition ensures that $0 < c < \Lambda_{n_1 n_2} < C < \infty, \ \forall (n_1,n_2) \in \mathbb{Z}^2$.

Existence of admissible functions

- Existence of admissibile functions on the torus is guaranteed by "lifting" separable admissibile functions on the euclidean plane:

Proposition

A "tensor-product" admissible function $\psi \in L^2(\mathbb{R}^2)$ provides an admissible function on $L^2(\mathbb{T}^2)$ by inverse stereographic projection

$$[\Pi_{\mathbb{T}^2}^{-1}\psi](\theta_1,\theta_2) = \frac{1}{\sqrt{1+\cos\theta_1}\sqrt{1+\cos\theta_2}}\psi\left(2\tan\frac{\theta_1}{2}, 2\tan\frac{\theta_2}{2}\right)$$

- As an example consider Difference of Gaussians (DoG) in 1D

$$\psi_\alpha(x) = e^{-x^2} - \frac{e^{-x^2/\alpha^2}}{\alpha}$$

- A two-dimensional *separable* DoG function on the torus would be

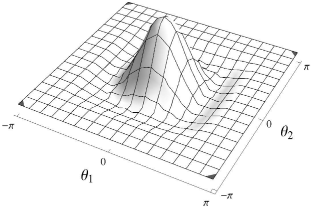
$$[\Pi_{\mathbb{T}^2}^{-1}\psi_{\alpha_1,\alpha_2}](\theta_1,\theta_2) = \frac{1}{\sqrt{1+\cos\theta_1}\sqrt{1+\cos\theta_2}}\psi_{\alpha_1}\left(2\tan\frac{\theta_1}{2}\right)\psi_{\alpha_2}\left(2\tan\frac{\theta_2}{2}\right).$$

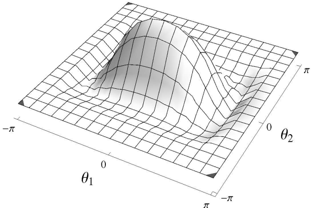
- but also axisymmetric (non-separable) DoG can be constructed

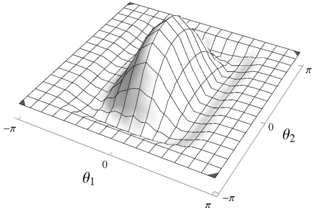
$$[\Pi_{\mathbb{T}^2}^{-1}\psi_{\alpha}](\theta_1,\theta_2) = \frac{1}{\sqrt{1+\cos\theta_1}\sqrt{1+\cos\theta_2}}\psi_{\alpha}\left(2\sqrt{\tan^2\frac{\theta_1}{2}+\tan^2\frac{\theta_2}{2}}\right)$$

Examples of admissible functions on the torus

Axisymmetric DoG on $\mathbb{T}^2$ and its dilation for two cases: $a_1 = 2, a_2 = 1$ and $a_1 = 1, a_2 = 2$:







Can we have wavelets on the torus with a single dilation?

- To prove the frame property it has been essential to have two dilations a_1, a_2.
- We need two dilations to bring any pair (n_1, n_2) to $\mathrm{supp}(\widehat{\Gamma})$ (extended to n_1, n_2 reals), ensuring that $\Lambda_{n_1,n_2} > c$.
- Can we have wavelets on the torus with a single dilation?
- The idea is to restrict to a "single" dilation $(a_1, a_2 = \sigma(a_1))$, with $\sigma' > 0$, usually $\sigma(a) = a$ (also $\sigma(a) = \sqrt{a}$ for shearlets).
- The parameter space X is restricted to $X' = \{(a, b_1, b_2), a > 0, b_{1,2} \in \mathbb{R}\}$. From the measure $d\nu(b_1, b_2, a_1, a_2) = db_1 \, db_2 \frac{da_1^2}{a_1^2} \frac{da_2^2}{a_2^2}$ on X we derive the measure on X'

$$d\nu'(b_1, b_2, a) = \frac{\sigma(a)}{\sigma'(a)} \frac{da}{a^4} db_1 \, db_2.$$

- Is the subset $\{\psi_a^{b_1,b_2} \equiv \psi_{a,\sigma(a)}^{b_1,b_2}\}$ a frame?
- Yes if we impose additional conditions to $\mathrm{supp}(\widehat{\Gamma})$, like extending it to a ring around the origin $(0,0)$, or to introduce extra group parameters like rotations, shears, etc.
- In the discrete case, frames in $\mathbb{R}^n$, with $n \geq 2$, with a single isotropic dilation are constructed from more than one (in fact at least $2^n - 1$) admissible function.

Modular group on the torus

Definition

The modular group on the torus $\mathbb{T}^2$ is the subgroup

$$SL(2,\mathbb{Z}) = \left\{ M = \begin{pmatrix} m & n \\ p & q \end{pmatrix}; m,n,p,q \in \mathbb{Z},\ \det(M) = mq - np = 1 \right\}, \quad (2)$$

of the group $SL(2,\mathbb{R})$ of linear transformations of the plane preserving the area with integer entries.

- The modular group transforms pair of integers (n_1, n_2) into pairs of integers $(n_1', n_2')^t = M(n_1, n_2)^t = (mn_1 + nn_2, pn_1 + qn_2)^t$.
- It preserves the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and its action on functions on the torus is

$$f_M(\theta_1, \theta_2) \equiv f(M^{-1}(\theta_1, \theta_2)^t).$$

- Since M preserves the area, this defines a unitary representation of $SL(2,\mathbb{Z})$ on $L^2(\mathbb{T}^2)$:

$$\begin{aligned} U : \quad & L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2) \\ & f(\theta_1, \theta_2) \mapsto [U(M)f](\theta_1, \theta_2) \equiv f_M(\theta_1, \theta_2)\,. \end{aligned}$$

Modular group on the torus

- This unitary representation is not irreducible, admitting infinite invariant subspaces $\mathcal{V}_g \subset L^2(\mathbb{T}^2), g \in \mathbb{N} \cup \{0\}$.
- To prove this, we first note that the action of the modular group in Fourier space is given by:

$$\widehat{f}_M^{(n_1,n_2)} = \widehat{f}^{(n_1,n_2)M} \quad \forall (n_1, n_2) \in \mathbb{Z}^2 \,, \; M \in SL(2,\mathbb{Z}) \,, \; f \in L^2(\mathbb{T}^2) \,.$$

- The action of a modular transformation M in Fourier space is through its transpose $\vec{n}' = M^t \vec{n}$, which is again a modular transformation.
- Since we shall work mainly in Fourier space, we shall consider the action on row vectors, $(n_1', n_2') = (n_1, n_2)M$.
- The action of the modular group on $\mathbb{Z}^2$ is not transitive, leaving certain subsets invariant. In what follows, g.c.d. stands for greatest common divisor.

Lemma

For each $g \in \mathbb{N} \cup \{0\}$ the subset $\mathcal{G}_g = \{(n_1, n_2) \in \mathbb{Z}^2 \,:\, \mathrm{g.c.d.}(n_1, n_2) = g\}$, with $\mathcal{G}_0 \equiv \{(0,0)\}$, is invariant under the modular group.

Modular group on the torus (II)

- We can think of $\mathbb{Z}^2$ as partitioned into orbits under the action of $SL(2,\mathbb{Z})$.
- Each orbit $\mathcal{G}_g$ is generated by the action of the group on, let us say, the point $(g,g) \in \mathbb{Z}^2$ (or $(g,0)$ or $(0,g)$).
- The action of the modular group in each orbit $\mathcal{G}_g$ is transitive but not free, since the point $(g,g) \neq (0,0)$ has a stabilizer (or isotropy) group:

$$N = \left\{ \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^k, \ k \in \mathbb{Z} \right\} \sim \mathbb{Z}$$

  while the point $(0,0)$, which is an orbit by itself, has as stabilizer the whole group $SL(2,\mathbb{Z})$.

Modular group on the torus (III)

- The stabilizer is the same for all orbits $\mathcal{G}_g\,, g \neq 0$. Also, for $g \neq 0$, if we choose a different point in the orbit (like $(g, 0)$ or $(0, g)$), the stabilizer group is different but isomorphic (in fact conjugate). For example, for $(g, 0)$, the stabilizer is

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^k,\ k \in \mathbb{Z} \right\} \sim \mathbb{Z},$$

  while for $(0, g)$ it is

$$N_2 = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k,\ k \in \mathbb{Z} \right\} \sim \mathbb{Z}.$$

- By the orbit-stabilizer theorem, there is a bijection between each orbit $\mathcal{G}_g\,, g \neq 0$, and the quotient $\mathcal{X} \equiv SL(2, \mathbb{Z})/N$.
- This means that there is also a bijection between any two orbits $\mathcal{G}_g$, $\mathcal{G}_{g'}$ with $g, g' \neq 0$. This bijection can be realized as follows:

Proposition

Given $(n_1, n_2) \in \mathcal{G}_g$, there is only one representative $M^g_{n_1,n_2} \in \mathcal{X}$ (i.e. mod. N) such that $(n_1, n_2)M^g_{n_1,n_2} = (g, g)$.

Modular group on the torus (III)

- $M^g_{n_1,n_2}$ can be written as $M^g_{n_1,n_2} = M^1_{n'_1,n'_2}$, where $n'_1 = n_1/g$, $n'_2 = n_2/g$ are coprime, i.e. g.c.d.$(n'_1, n'_2) = 1$. This allows us to take the representative $M_{n'_1,n'_2} \equiv M^1_{n'_1,n'_2} = M^g_{n_1,n_2}$ for all cases $g \neq 0$, for instance, when writing expressions like $\sum_{M\in\mathcal{X}}$.
- Similar results hold for $(g, 0)$ and $(0, g)$.
- The previous proposition allows us to label pairs $(n_1, n_2) \in \mathbb{Z}^2$ equivalently as $(g, M^{-1}_{\frac{n_1}{g},\frac{n_2}{g}})$, where g.c.d$(n_1, n_2) = g$, for $(n_1, n_2) \neq (0, 0)$.
- For $(n_1, n_2) = (0, 0)$ we can label it as $(g = 0, I_2)$, where I_2 represents the 2×2 identity matrix.
- A similar analysis can be done for functions on the torus, studying the action of the modular group on the Fourier coefficients.
- Denote by $\mathcal{V}_g \subset L^2(\mathbb{T}^2)$ subspace spanned by the states ϕ_{n_1,n_2} with $\gcd(n_1, n_2) = g$, $g = 0, 1, 2, \ldots$. The same considerations as in the case of the subsets $\mathcal{G}_g$ apply here. Thus we have:

Modular group on the torus (IV)

Proposition

The subspaces $\mathcal{V}_g$, $g = 0, 1, 2, \dots$ of $L^2(\mathbb{T}^2)$ given by

$$\mathcal{V}_g = \{\psi \in L^2(\mathbb{T}^2) \;:\; \mathrm{supp}(\widehat{\psi}) \subset \mathcal{G}_g\}$$

are invariant under the action of the modular group $SL(2,\mathbb{Z})$.

- The action of the modular group in each orbit is transitive but not free, the stabilizer group being again N for orbits $\mathcal{V}_g$, $g \neq 0$, and the whole $SL(2,\mathbb{Z})$ for $\mathcal{V}_0$.
- There is a bijection between each orbit $\mathcal{V}_g$, $g \neq 0$ and the quotient $\mathcal{X} \equiv SL(2,\mathbb{Z})/N$, and between any two orbits $\mathcal{V}_g$, $\mathcal{V}_{g'}$ with $g, g' \neq 0$.
- Thus, expressions like $\sum_{n_1,n_2=-\infty}^{\infty} q_{n_1,n_2}$ can be written as $\sum_{g=0}^{\infty} \sum_{M \in \mathcal{X}_g} q_{g,M^{-1}}$, where we mean by $\mathcal{X}_0 = \{I_2\}$ and $\mathcal{X}_g = \mathcal{X}$ for $g \neq 0$.

The previous considerations can be restated as follows:

Proposition

Let $g \in \mathbb{N}$. If $\gamma = \phi_{n_1,n_2}$, with g.c.d$(n_1, n_2) = g$, then $B_{g,\gamma} = \{\gamma_M \,/\, M \in \mathcal{X}\}$ is an orthonormal basis of $\mathcal{V}_g$.

Modular Coherent States

- The question is whether we can extend this "basis" to the whole $L^2(\mathbb{T}^2)$. The answer is given in the following Proposition:

Proposition

Let $\eta \in L^2(\mathbb{T}^1)$ such that $\mathrm{supp}(\widehat{\eta}) = \mathbb{Z}$, and define $\gamma(\theta_1, \theta_2) = \eta(\theta_1 + \theta_2)$. Then the set $F_\gamma = \{\gamma_M^{\vartheta_1,\vartheta_2} \,/\, M \in \mathcal{X}\,,\, \vartheta_1, \vartheta_2 \in \mathbb{T}^2\}$ is a continuous upper semi-frame in the sense that there exist $C > 0$ such that

$$0 < \int \frac{d\vartheta_1 d\vartheta_2}{(2\pi)^2} \sum_{M\in\mathcal{X}} |\langle \gamma_M^{\vartheta_1,\vartheta_2} | \psi \rangle|^2 \leq C||\psi||^2\,,\ \forall \psi \in L^2(\mathbb{T}^2),\ \psi \neq 0\,.$$

- This provides an admissibility condition for modular "coherent states".
- Thanks to modular transformations, now $\widehat{\gamma}$ does not need to have support on the four Fourier quadrants $Q_q, q = 1, 2, 3, 4$, but only on the main diagonal $n_1 = n_2$ (or in $n_1 = 0$ or $n_2 = 0$ lines).
- The set F_γ is not a frame in $L^2(\mathbb{T}^2)$, since $|\widehat{\gamma}^{g,g}| \to 0$ when $g \to \infty$, preventing $|\widehat{\gamma}^{g,g}|$ to be uniformly bounded from below by a positive constant.

Modular Frame

- However if we restrict ourselves to suitable subspaces of $L^2(\mathbb{T}^2)$, like that of band-limited functions

$$W_{L_1,L_2} = \{\psi \in L^2(\mathbb{T}^2) : \widehat{\psi}^{n_1,n_2} = 0, \forall |n_1| > L_1, |n_2| > L_2\} \subset L^2(\mathbb{T}^2)\,,$$

the set F_γ becomes a frame, even for a suitable bandlimited function $\eta \in L^2(\mathbb{T}^1)$. More precisely, we have the following result:

Corollary

Under the conditions of the previous Proposition, the set F_γ is a frame for any subspace W_{L_1,L_2} of band limited functions in $L^2(\mathbb{T}^2)$.

- Note that if γ is chosen such that $\widehat{\eta} = \chi_{[0,g_{\max}]}$, then F_γ is a tight frame, and a Parseval frame if appropriately rescaled.

Modular transformations and dilations

- We shall make use of the modular group to complete the parameter space X' for the case of dependent dilations $a_2 = \sigma(a_1)$ (for simplicity, we shall restrict ourselves to the case $\sigma(a) = a$).
- The action of the modular group on $\mathbb{T}^2$ induces a transformation of functions $f \in L^2(\mathbb{T}^2)$ that completes the previous (dilation and translation) transformations as

$$f_{a,M}^{\vartheta_1,\vartheta_2}(\theta_1,\theta_2) := f_a^{\vartheta_1,\vartheta_2}(M^{-1}(\theta_1,\theta_2)^t) = f_a^{\vartheta_1,\vartheta_2}(q\theta_1 - n\theta_2, -p\theta_1 + m\theta_2), \quad (3)$$

  where we have used the notation $f_a^{\vartheta_1,\vartheta_2} := f_{a,a}^{\vartheta_1,\vartheta_2}$ when restricting to a single dilation.
- Adding the whole modular group $SL(2,\mathbb{Z})$ to the parameter space X' introduces redundancy that is not suitable for admissibility conditions. Therefore, we shall restrict ourselves to the quotient space $\mathcal{X} = SL(2,\mathbb{Z})/N$.
- The choice of N (isotropy subgroup of (g,g)) is in fact connected with the case $\Gamma(\theta_1,\theta_2) = \eta(\theta_1+\theta_2)$, for which the only possible non-zero Fourier coefficients are the diagonal $\widehat{\Gamma}^{l,l}$ (we shall make use of this property when proving the frame condition).

Modular admissibility

The admissibility condition for “modular wavelets” on the torus reads:

Definition

A non-zero function $\gamma \in L^2(\mathbb{T}^2)$ is called “modular-admissible” if there exist $C \in \mathbb{R}$ such that the condition

$$0 < \int_{X'} d\nu'(\vartheta_1, \vartheta_2, a) \sum_{M \in \mathcal{X}} |\langle \gamma_{a,M}^{\vartheta_1,\vartheta_2} | \psi \rangle|^2 < C < \infty$$

is satisfied for every non-zero $\psi \in L^2(\mathbb{T}^2)$.

- This admissibility condition can be equivalently expressed as follows:

Proposition

A non-zero function $\gamma \in L^2(\mathbb{T}^2)$ is “modular-admissible” iff there exist $C \in \mathbb{R}$ such that

$$0 < \widetilde{\Lambda}_{n_1,n_2} \equiv \int_0^\infty \frac{da}{a^3} \sum_{M \in \mathcal{X}} |\widehat{\gamma}_{a,M}^{n_1,n_2}|^2 < C < \infty \,, \forall (n_1, n_2) \in \mathbb{Z}^2$$

where $\widehat{\gamma}_{a,M}^{n_1,n_2} = \langle \phi_{n_1,n_2} | \gamma_{a,M} \rangle$ are the Fourier coefficients of $\gamma_{a,M} \equiv \gamma_{a,M}^{0,0}$.

Modular Frame

Proposition

The necessary (weak) admissibility condition still holds for modular admissible functions.

- We shall restrict ourselves to "diagonal" functions $\Gamma(\theta_1, \theta_2) = \eta(\theta_1 + \theta_2)$, for which $\widehat{\Gamma}^{n_1,n_2} = 0$ if $n_1 \neq n_2$.
- Modular transformations relaxes the requirement that $\widehat{\Gamma}$ must have support on the four quadrants.
- In fact it is just enough that $\widehat{\Gamma}$ has support on the positive main diagonal.

Theorem

For any localized modular-admissible function γ, whose associated function Γ is diagonal, the family

$$\left\{\gamma_{a,M}^{\vartheta_1,\vartheta_2},\ (\vartheta_1, \vartheta_2) \in (-\pi, \pi)^2, a \in \mathbb{R}^+, M \in \mathcal{X}\right\}$$

is a frame, that is, there exist real constants $0 < c \leq C$ such that

$$c||\psi||^2 \leq \sum_{M \in \mathcal{X}} \int_{X'} d\nu'(\vartheta_1, \vartheta_2, a) |\langle \gamma_{a,M}^{\vartheta_1,\vartheta_2} | \psi \rangle|^2 \leq C||\psi||^2, \ \ \forall \psi \in L^2(\mathbb{T}^2).$$

Example of modular wavelets

- Let us provide a particular example of modular admissible function based on DoG functions.
- Consider the diagonal function

$$\Gamma(\theta_1, \theta_2) = \frac{\psi_\alpha\left(2\tan\frac{\theta_1+\theta_2}{2}\right)}{1+\cos(\theta_1+\theta_2)},$$

so that the corresponding admissible function on the torus is the "diagonal DoG"

$$\gamma(\theta_1, \theta_2) = \sqrt{(1+\cos\theta_1)(1+\cos\theta_2)}\,\Gamma(\theta_1, \theta_2).$$

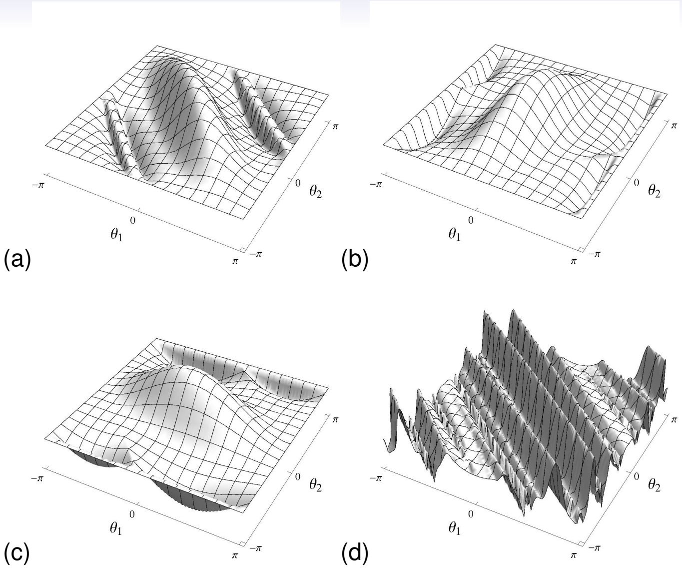


Figure: Modular transformation of the "diagonal DoG" with $\alpha = 10$ for: (a) $M = I_2$, (b) $M_{1,0}$, (c) $M_{0,1}$, (d) $M_{4,5}$.

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