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Coherent States and Wavelets: A Unified Approach - III

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Abstract

By analogy with the one- and two-dimensional wavelet groups, we introduce the quaternionic affine group, look at some of its properties, its representations on complex and quaternionic Hilbert spaces, the associated wavelet transforms and coherent states.

Contents

Useful facts

We list some useful facts about quaternions and the matrix representation that we shall use.

Let $\mathbb{H}$ denote the field of all quaternions and $\mathbb{H}^*$ the group (under quaternionic multiplication) of all invertible quaternions. A general quaternion can be written as

$$\mathfrak{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, \qquad q_0, q_1, q_2, q_3 \in \mathbb{R},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the three quaternionic imaginary units, satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$. The quaternionic conjugate of $\mathfrak{q}$ is

$$\overline{\mathfrak{q}} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3.$$

We shall use the 2×2 matrix representation of the quaternions, in which

$$\mathbf{i} = \sqrt{-1}\sigma_1, \quad \mathbf{j} = -\sqrt{-1}\sigma_2, \quad \mathbf{k} = \sqrt{-1}\sigma_3,$$

and the σ's are the three Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Useful facts

to which we add

$$\sigma_0 = \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We shall also use the matrix valued vector $\boldsymbol{\sigma} = (\sigma_1, -\sigma_2, \sigma_3)$. Thus, in this representation,

$$\mathfrak{q} = q_0\sigma_0 + i\mathbf{q}\cdot\boldsymbol{\sigma} = \begin{pmatrix} q_0 + iq_3 & -q_2 + iq_1 \\ q_2 + iq_1 & q_0 - iq_3 \end{pmatrix}, \quad \mathbf{q} = (q_1, q_2, q_3).$$

In this representation, the quaternionic conjugate of $\mathfrak{q}$ is given by $\mathfrak{q}^\dagger$.
Introducing two complex variables, which we write as

$$z_1 = q_0 + iq_3, \qquad z_2 = q_2 + iq_1,$$

we may also write

$$\mathfrak{q} = \begin{pmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{pmatrix}. \tag{2.1}$$

Useful facts

From this it is clear that the group $\mathbb{H}^*$ is isomormphic to the *affine* $SU(2)$ *group*, i.e., $\mathbb{R}^{>0} \times SU(2)$, which is the group $SU(2)$ together with all (non-zero) dilations.
As a set $\mathbb{H}^* \simeq \mathbb{R}^{>0} \times S(4)$, where $S(4)$ is the surface of the sphere in $\mathbb{R}^4$, or more simply, $\mathbb{H}^* \simeq \mathbb{R}^4 \backslash \{\mathbf{0}\}$.

Action of $\mathbb{H}^*$ on $\mathbb{H}$

Consider the action of $\mathbb{H}^*$ on $\mathbb{H}$ by right (or left) quaternionic (in our representation matrix) multiplication. It is clear that there are only two orbits under this action, $\{\mathfrak{o}\}$ (the zero quaternion) and $\mathbb{H}^*$. Furthermore, this latter orbit is *open and free*. Let

$$\mathfrak{a} = \begin{pmatrix} w_1 & -\overline{w}_2 \\ w_2 & \overline{w}_1 \end{pmatrix} \in \mathbb{H}^* \quad \text{and} \quad \mathfrak{x} = \begin{pmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{pmatrix} \in \mathbb{H}.$$

Then under left action

$$\mathfrak{x} \longmapsto \mathfrak{x}' = \mathfrak{a}\mathfrak{x} = \begin{pmatrix} w_1 z_1 - \overline{w}_2 z_2 & -\overline{w}_2 \overline{z}_1 - w_1 \overline{z}_2 \\ w_2 z_1 + \overline{w}_1 z_2 & \overline{w}_1 \overline{z}_1 - w_2 \overline{z}_2 \end{pmatrix}. \tag{2.2}$$

We take $w_1 = a_0 + ia_3,\ w_2 = a_2 + ia_1$ and $z_1 = x_0 + ix_3,\ z_2 = x_2 + ix_1$ and consider $\mathfrak{x}$ as the vector

$$\mathbf{x} = \begin{pmatrix} x_0 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} := \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \in \mathbb{R}^4. \tag{2.3}$$

Action of $\mathbb{H}^*$ on $\mathbb{H}$

On this vector, the left action (2.2) is easily seen to lead to the matrix left action

$$\mathbf{x} \longmapsto \mathbf{x}' = A\mathbf{x} = \begin{pmatrix} a_0 & -a_3 & -a_2 & -a_1 \\ a_3 & a_0 & a_1 & -a_2 \\ a_2 & -a_1 & a_0 & a_3 \\ a_1 & a_2 & -a_3 & a_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} A_1 & -A_2^T \\ A_2 & A_1^T \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}, \qquad (2.4)$$

on $\mathbb{R}^4$.

The matrices A_1 and A_2 are rotation-dilation matrices, and may be written in the form

$$A_1 = \lambda_1 \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} = \lambda_1 R(\theta_1), \qquad A_2 = \lambda_2 \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} = \lambda_2 R(\theta_2) \qquad (2.5)$$

where

$$\theta_1 = \tan^{-1}\left(\frac{a_3}{a_0}\right), \ \theta_2 = \tan^{-1}\left(\frac{a_1}{a_2}\right), \ \lambda_1 = \sqrt{a_0^2 + a_3^2}, \ \lambda_2 = \sqrt{a_1^2 + a_2^2} \text{ and } \lambda_1^2 + \lambda_2^2 \neq 0 \qquad (2.6)$$

and $R(\theta)$ is the 2×2 rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \qquad (2.7)$$

Action of $\mathbb{H}^*$ on $\mathbb{H}$

From the above it is clear that when $\mathbb{H}$ is identified with $\mathbb{R}^4$, the action of $\mathbb{H}^*$ on $\mathbb{H}$ is that of two two-dimensional rotation-dilation groups (rotations of the two-dimensional plane together with radial dilations, where at least one of the dilations is non-zero) acting on $\mathbb{R}^4$.

Consequently, we shall consider elements in $\mathbb{H}^*$ interchangeably as 2×2 complex matrices of the type

$$\mathfrak{a} = \begin{pmatrix} w_1 & -\overline{w}_2 \\ w_2 & \overline{w}_1 \end{pmatrix}, \qquad \det[\mathfrak{a}] = |\mathfrak{a}|^2 \neq 0$$

or 4×4 real matrices of the type A in (2.4),

$$A = \begin{pmatrix} \lambda_1 R(\theta_1) & -\lambda_2 R(-\theta_2) \\ \lambda_2 R(\theta_2) & \lambda_1 R(-\theta_1), \end{pmatrix}, \qquad \det[A] = |\mathfrak{a}|^4 = [\lambda_1^2 + \lambda_2^2]^2 \neq 0. \qquad (2.8)$$

Quaternionic affine group

Let us look at the three affine groups, $G^{\mathbb{R}}_{\text{aff}}$, $G^{\mathbb{C}}_{\text{aff}}$ and $G^{\mathbb{H}}_{\text{aff}}$, of the real line, the complex plane and the quaternions, respectively. These groups are defined as the semi-direct products

$$G^{\mathbb{R}}_{\text{aff}} = \mathbb{R} \rtimes \mathbb{R}^*, \qquad G^{\mathbb{C}}_{\text{aff}} = \mathbb{C} \rtimes \mathbb{C}^*, \qquad G^{\mathbb{H}}_{\text{aff}} = \mathbb{H} \rtimes \mathbb{H}^*.$$

Let $\mathbb{K}$ denote any one of the three fields $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and write $G^{\mathbb{K}}_{\text{aff}} = \mathbb{K} \rtimes \mathbb{K}^*$. A generic element in $G^{\mathbb{K}}_{\text{aff}}$ can be written as

$$g = (b, a) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a \in \mathbb{K}^*, \quad b \in \mathbb{K}.$$

Of these, $G^{\mathbb{R}}_{\text{aff}}$ is the *one-dimensional wavelet group* and $G^{\mathbb{C}}_{\text{aff}}$, which is isomorphic to the similitude group of the plane (translations, rotations and dilations of the 2-dimensional plane), is the *two-dimensional wavelet group.*

Quaternionic affine group

By analogy we shall call the quaternionic affine group $G_{\text{aff}}^{\mathbb{H}}$ the *quaternionic wavelet group*, which we now analyse in some detail. In the 2×2 matrix representation of the quaternions introduced earlier, we shall represent an element of $G_{\text{aff}}^{\mathbb{H}}$ as the 3×3 complex matrix

$$g := (\mathfrak{b}, \mathfrak{a}) = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathbf{0}^T & 1 \end{pmatrix}, \quad \mathfrak{a} \in \mathbb{H}^*, \quad \mathfrak{b} \in \mathbb{H}, \quad \mathbf{0}^T = (0,0). \tag{3.1}$$

Alternatively, if A is the 4×4 real matrix corresponding to $\mathfrak{a}$, through (2.2), and $\mathbf{b} \in \mathbb{R}^4$ the vector made out of the components b_0, b_1, b_2, b_3 of $\mathfrak{b}$ (see (2.3)),

$$\mathbf{b} = \begin{pmatrix} b_0 \\ b_3 \\ b_2 \\ b_1 \end{pmatrix},$$

then g may also be written as the 5×5 real matrix,

$$g := (\mathbf{b}, A) = \begin{pmatrix} A & \mathbf{b} \\ \mathbf{0}^T & 1 \end{pmatrix}, \quad \mathbf{0}^T = (0,0,0,0). \tag{3.2}$$

Quaternionic affine group

In this real form $G_{\text{aff}}^{\mathbb{H}}$ may be called the *group of dihedral similitude transformations of* $\mathbb{R}^4$. We shall use both representations of $G_{\text{aff}}^{\mathbb{H}}$ interchangeably.
For each one of these groups $G_{\text{aff}}^{\mathbb{K}} = \mathbb{K} \rtimes \mathbb{K}^*$ there is exactly one non-trivial orbit of $\mathbb{K}^*$ in the dual of $\mathbb{K}$ and this orbit is open and free. Hence on a complex Hilbert space, each one of these groups has exactly one irreducible representtion.
The irreducible representations of $G_{\text{aff}}^{\mathbb{R}}$ and $G_{\text{aff}}^{\mathbb{C}}$ are well known.
We shall compute below the one irreducible representation of $G_{\text{aff}}^{\mathbb{H}}$, both in a complex and in a quaternionic Hilbert space.

Invariant measures of $\mathbb{H}^*$ and $G_{\text{aff}}^{\mathbb{H}}$

The group $\mathbb{H}^*$ is unimodular. The Haar measure is,

$$d\mu_{\mathbb{H}^*} = \frac{d\mathfrak{x}}{|\mathfrak{x}|^4} = \frac{d\mathfrak{x}}{(\det[\mathfrak{x}])^2} = \frac{d\mathbf{x}}{\|\mathbf{x}\|^4}, \quad \text{where} \quad d\mathfrak{x} = d\mathbf{x} = dx_0\, dx_3\, dx_2\, dx_1. \tag{3.3}$$

The group $G_{\text{aff}}^{\mathbb{H}}$ is non-unimodular. The left invariant measure is

$$d\mu_\ell(\mathbf{b}, A) = \frac{d\mathbf{b}}{\|\mathbf{a}\|^4}\, d\mu_{\mathbb{H}^*}(A) = \frac{d\mathbf{b}\; d\mathbf{a}}{\|\mathbf{a}\|^8} := \frac{d\mathbf{b}\; dA}{(\det[A])^2}, \tag{3.4}$$

which we shall also write as

$$d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = \frac{d\mathfrak{b}\; d\mathfrak{a}}{(\det[\mathfrak{a}])^4}. \tag{3.5}$$

Similarly, the right Haar measure is

$$d\mu_r(\mathbf{b}, A) = d\mathbf{b}\; d\mu_{\mathbb{H}^*}(A) = \frac{d\mathbf{b}\; d\mathbf{a}}{\|\mathbf{a}\|^4} := \frac{d\mathbf{b}\; dA}{\det[A]}, \tag{3.6}$$

or alternatively written,

$$d\mu_r(\mathfrak{b}, \mathfrak{a}) = \frac{d\mathfrak{b}\; d\mathfrak{a}}{(\det[\mathfrak{a}])^2}. \tag{3.7}$$

The *modular function* Δ, such that $d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = \Delta(\mathfrak{b}, \mathfrak{a})\; d\mu_r(\mathfrak{b}, \mathfrak{a})$, is

$$\Delta(\mathfrak{b}, \mathfrak{a}) = \frac{1}{(\det[\mathfrak{a}])^2} = \frac{1}{|\mathfrak{a}|^4} = \frac{1}{\|\mathbf{a}\|^4} = \frac{1}{\det[A]} := \Delta(\mathbf{b}, A). \tag{3.8}$$

UIR of $G_{\mathrm{aff}}^{\mathbb{H}}$ in a complex Hilbert space

From the general theory of semi-direct products of the type $\mathbb{R}^n \rtimes H$, where H is a subgroup of $GL(n, \mathbb{R})$, and which has open free orbits in the dual of $\mathbb{R}^n$, we know that $G_{\mathrm{aff}}^{\mathbb{H}}$ has exactly one unitary irreducible representation on a complex Hilbert space and moreover, this representation is quare-integrable. Consider the Hilbert space $\mathfrak{H}_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathbb{R}^4, d\mathbf{x})$ and on it define the representation $G_{\mathrm{aff}}^{\mathbb{H}} \ni (\mathbf{b}, A) \longmapsto U_{\mathbb{C}}(\mathbf{b}, A)$,

$$(U_{\mathbb{C}}(\mathbf{b}, A)f)(\mathbf{x}) = \frac{1}{(\det[A])^{\frac{1}{2}}} f(A^{-1}(\mathbf{x} - \mathbf{b})), \qquad f \in \mathfrak{H}_{\mathbb{C}}. \tag{3.9}$$

This representation is unitary and irreducible.
The *Duflo-Moore operator* C is given in the Fourier domain as the multiplication operator

$$(\widehat{C}\widehat{f})(\mathbf{k}) = \mathcal{C}(\mathbf{k})\widehat{f}(\mathbf{k}), \quad \text{where} \quad \mathcal{C}(\mathbf{k}) = \left[\frac{2\pi}{\|\mathbf{k}\|}\right]^2. \tag{3.10}$$

A vector $f \in \mathfrak{H}_{\mathbb{C}}$ is *admissible* if it is in the domain of C i.e., if its Fourier transform $\widehat{f}$ satisfies

$$(2\pi)^4 \int_{\mathbb{R}^4} \frac{|\widehat{f}(\mathbf{k})|^2}{\|\mathbf{k}\|^4}\, d\mathbf{k} < \infty.$$

UIR of $G_{\text{aff}}^{\mathbb{H}}$ in a complex Hilbert space

Thus, for any two vectors η_1, η_2 in the domain of C and for arbitrary $f_1, f_2 \in \mathfrak{H}_{\mathbb{C}}$, we have the *orthogonality relation*,

$$\int_{G_{\mathbf{aff}}^{\mathbb{H}}} \langle f_1 \mid U_{\mathbb{C}}(\mathbf{b}, A)\eta_1\rangle\langle \eta_2 \mid U_{\mathbb{C}}(\mathbf{b}, A)^* f_2\rangle \; d\mu_\ell(\mathbf{b}, A) = \langle C\eta_2 \mid C\eta_1\rangle\langle f_1 \mid f_2\rangle, \tag{3.11}$$

which is the same as the operator equation

$$\int_{G_{\mathbf{aff}}^{\mathbb{H}}} U_{\mathbb{C}}(\mathbf{b}, A)|\eta_1\rangle\langle\eta_2| U_{\mathbb{C}}(\mathbf{b}, A)^* \; d\mu_\ell(\mathbf{b}, A) = \langle C\eta_2 \mid C\eta_1\rangle I_{\mathfrak{H}_{\mathbb{C}}} \; . \tag{3.12}$$

If $\langle C\eta_2 \mid C\eta_1\rangle \neq 0$, we have the *resolution of the identity*

$$\frac{1}{\langle C\eta_2 \mid C\eta_1\rangle} \int_{G_{\mathbf{aff}}^{\mathbb{H}}} U_{\mathbb{C}}(\mathbf{b}, A)|\eta_1\rangle\langle\eta_2| U_{\mathbb{C}}(\mathbf{b}, A)^* \; d\mu_\ell(\mathbf{b}, A) = I_{\mathfrak{H}_{\mathbb{C}}} \; . \tag{3.13}$$

Given an admissible vector η, such that $\|C\eta\|^2 = 1$, we define the family of *coherent states or wavelets* as

$$\mathfrak{S}_{\mathbb{C}} = \{\eta_{\mathbf{b},A} = U_{\mathbb{C}}(\mathbf{b}, A)\eta \mid (\mathbf{b}, A) \in G_{\mathbf{aff}}^{\mathbb{H}}\}, \tag{3.14}$$

which then satisfies the resolution of the identity,

$$\int_{G_{\text{aff}}^{\mathbb{H}}} |\eta_{\mathbf{b},A}\rangle\langle\eta_{\mathbf{b},A}| \; d\mu_\ell(\mathbf{b}, A) = I_{\mathfrak{H}_{\mathbb{C}}} \; . \tag{3.15}$$

The above representation could also be realized on the Hilbert space $\mathfrak{K}_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathbb{H}, d\mathfrak{x})$ over the quaternions. We simply transcribe Eqs. (3.9) – (3.15) into this framework. Thus, we define the representation $G_{\text{aff}}^{\mathbb{H}} \ni (\mathfrak{b}, \mathfrak{a}) \longmapsto U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})$,

$$(U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})f)(\mathfrak{x}) = \frac{1}{\det[\mathfrak{a}]} f(\mathfrak{a}^{-1}(\mathfrak{x} - \mathfrak{b})), \qquad f \in \mathfrak{K}_{\mathbb{C}}. \tag{3.16}$$

The *Duflo-Moore operator C* is given in the Fourier domain as the multiplication operator

$$(\widehat{C}\widehat{f})(\mathfrak{k}) = \mathcal{C}(\mathfrak{k})\widehat{f}(\mathfrak{k}), \quad \text{where} \quad \mathcal{C}(\mathfrak{k}) = \left[\frac{2\pi}{|\mathfrak{k}|}\right]^2 . \tag{3.17}$$

The admissibility condition is now

$$(2\pi)^4 \int_{\mathbb{R}^4} \frac{|\widehat{f}(\mathfrak{k})|^2}{|\mathfrak{k}|^4} \, d\mathfrak{k} < \infty,$$

UIR of $G_{\mathrm{aff}}^{\mathbb{H}}$ in a complex Hilbert space

and for any two vectors η_1, η_2 in the domain of C and arbitrary $f_1, f_2 \in \mathfrak{K}_{\mathbb{C}}$, the orthogonality relation becomes

$$\int_{G_{\mathrm{aff}}^{\mathbb{H}}} \langle f_1 \mid U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})\eta_1\rangle\langle\eta_2 \mid U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})^* f_2\rangle \, d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = \langle C\eta_2 \mid C\eta_1\rangle\langle f_1 \mid f_2\rangle, \tag{3.18}$$

with its operator version

$$\int_{G_{\mathrm{aff}}^{\mathbb{H}}} U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})|\eta_1\rangle\langle\eta_2| U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})^* \, d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = \langle C\eta_2 \mid C\eta_1\rangle I_{\mathfrak{K}_{\mathbb{C}}} \,. \tag{3.19}$$

Similarly, for $\langle C\eta_2 \mid C\eta_1\rangle \neq 0$,

$$\frac{1}{\langle C\eta_2 \mid C\eta_1\rangle} \int_{G_{\mathrm{aff}}^{\mathbb{H}}} U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})|\eta_1\rangle\langle\eta_2| U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})^* \, d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = I_{\mathfrak{K}_{\mathbb{C}}} \,. \tag{3.20}$$

The family of *coherent states or wavelets* are

$$\mathfrak{S}_{\mathbb{C}} = \{\eta_{\mathfrak{b},\mathfrak{a}} = U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})\eta \mid (\mathfrak{b}, \mathfrak{a}) \in G_{\mathrm{aff}}^{\mathbb{H}}\}, \tag{3.21}$$

with the resolution of the identity,

$$\int_{G_{\mathrm{aff}}^{\mathbb{H}}} |\eta_{\mathfrak{b},\mathfrak{a}}\rangle\langle\eta_{\mathfrak{b},\mathfrak{a}}| \, d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = I_{\mathfrak{K}_{\mathbb{C}}} \,. \tag{3.22}$$

A quaternionic Hilbert space

We now proceed to construct a unitay irreducible representation of the quaternionic affine group $G_{\text{aff}}^{\mathbb{H}}$ on a quaternionic Hilbert space. It will turn out that this representation has an intimate connection with the representation $U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})$ in (3.16) on $\mathfrak{K}_{\mathbb{C}}$.
We consider the Hilbert space $\mathfrak{H}_{\mathbb{H}}$, of quaternionic valued functions over the quaternions. An element $\mathfrak{f} \in \mathfrak{H}_{\mathbb{H}}$ has the form

$$\mathfrak{f}(\mathfrak{x}) = \begin{pmatrix} f_1(\mathfrak{x}) & -\overline{f_2(\mathfrak{x})} \\ f_2(\mathfrak{x}) & \overline{f_1(\mathfrak{x})} \end{pmatrix}, \quad \mathfrak{x} \in \mathbb{H}, \tag{4.1}$$

The norm in given by

$$\|\mathfrak{f}\|^2_{\mathfrak{H}_{\mathbb{H}}} = \int_{\mathbb{H}} \mathfrak{f}(\mathfrak{x})^\dagger \mathfrak{f}(\mathfrak{x})\, d\mathfrak{x} = \int_{\mathbb{H}} |\mathfrak{f}(\mathfrak{x})|^2\, d\mathfrak{x} = \left[\int_{\mathbb{H}} \left(|f_1(\mathfrak{x})|^2 + |f_2(\mathfrak{x})|^2 \right) d\mathfrak{x} \right] \sigma_0, \tag{4.2}$$

the finiteness of which implies that both f_1 and f_2 have to be elements of $\mathfrak{K}_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathbb{H}, d\mathfrak{x})$, so that we may write

$$\|\mathfrak{f}\|^2_{\mathfrak{H}_{\mathbb{H}}} = \left(\|f_1\|^2_{\mathfrak{H}_{\mathbb{C}}} + \|f_2\|^2_{\mathfrak{H}_{\mathbb{C}}} \right) \sigma_0.$$

In view of this, we may also write $\mathfrak{H}_{\mathbb{H}} = L^2_{\mathbb{H}}(\mathbb{H}, d\mathfrak{x})$.

UIR of $G^{\mathbb{H}}_{\text{aff}}$ in a quaternionic Hilbert space

In using the "bra-ket" notation we shall use the notation and convention:

$$(\mathfrak{f} \mid \; = \begin{pmatrix} \langle f_1 | & \langle f_2 | \\ -\langle \overline{f}_2 | & \langle \overline{f}_1 | \end{pmatrix}, \quad \text{and} \quad \mid \mathfrak{f}) = \begin{pmatrix} |f_1\rangle & -|\overline{f}_2\rangle \\ |f_2\rangle & |\overline{f}_1\rangle \end{pmatrix}, \tag{4.3}$$

The scalar product of two vectors $\mathfrak{f}, \mathfrak{f}' \in \mathfrak{H}_{\mathbb{H}}$ is

$$\begin{aligned} (\mathfrak{f} \mid \mathfrak{f}') &= \int_{\mathbb{H}} \mathfrak{f}(\mathfrak{x})^{\dagger} \mathfrak{f}'(\mathfrak{x}) \, d\mathfrak{x} \\ &= \begin{pmatrix} \langle f_1 \mid f_1' \rangle_{\mathfrak{H}_{\mathbb{C}}} + \langle f_2 \mid f_2' \rangle_{\mathfrak{H}_{\mathbb{C}}} & -\langle f_2' \mid \overline{f}_1 \rangle_{\mathfrak{H}_{\mathbb{C}}} + \langle f_1' \mid \overline{f}_2 \rangle_{\mathfrak{H}_{\mathbb{C}}} \\ \langle \overline{f}_2' \mid f_1 \rangle_{\mathfrak{H}_{\mathbb{C}}} - \langle \overline{f}_1' \mid f_2 \rangle_{\mathfrak{H}_{\mathbb{C}}} & \langle f_1' \mid f_1 \rangle_{\mathfrak{H}_{\mathbb{C}}} + \langle f_2' \mid f_2 \rangle_{\mathfrak{H}_{\mathbb{C}}} . \end{pmatrix} \end{aligned} \tag{4.4}$$

Note that

$$(\mathfrak{f} \mid \mathfrak{f}')^{\dagger} = (\mathfrak{f}' \mid \mathfrak{f}).$$

Multiplication by quaternions on $\mathfrak{H}_{\mathbb{H}}$ is defined from the right:

$$(\mathfrak{H}_{\mathbb{H}} \times \mathbb{H}) \ni (\mathfrak{f}, \mathfrak{q}) \longmapsto \mathfrak{f}\mathfrak{q}, \quad \text{such that} \quad (\mathfrak{f}\mathfrak{q})(\mathfrak{x}) = \mathfrak{f}(\mathfrak{x})\mathfrak{q},$$

i.e., we take $\mathfrak{H}_{\mathbb{H}}$ to be a right quaternionic Hilbert space.

A quaternionic Hilbert space

This convention is consistent with the scalar product (4.4) in the sense that

$$(\mathfrak{f} \mid \mathfrak{f}'\mathfrak{q}) = (\mathfrak{f} \mid \mathfrak{f}')\mathfrak{q} \quad \text{and} \quad (\mathfrak{f}\mathfrak{q} \mid \mathfrak{f}') = \mathfrak{q}^{\dagger}(\mathfrak{f} \mid \mathfrak{f}').$$

On the other hand, the action of operators $\mathbf{A}$ on vectors $\mathfrak{f} \in \mathfrak{H}_{\mathbb{H}}$ will be from the left $(\mathbf{A}, \mathfrak{q}) \longmapsto \mathbf{A}\mathfrak{f}$. In particular, an operator A on $\mathfrak{K}_{\mathbb{C}}$ defines an operator $\mathbf{A}$ on $\mathfrak{H}_{\mathbb{H}}$ as,

$$(\mathbf{A}\mathfrak{f})(\mathfrak{x}) = \begin{pmatrix} (Af_1)(\mathfrak{x}) & -\overline{(Af_2)(\mathfrak{x})} \\ (Af_2)(\mathfrak{x}) & \overline{(Af_1)(\mathfrak{x})} \end{pmatrix}.$$

Multiplication of operators by quaternions will also be from the left. Thus, $\mathfrak{q}\mathbf{A}$ acts on the vector $\mathfrak{f}$ in the manner

$$(\mathfrak{q}\mathbf{A}\mathfrak{f})(\mathfrak{x}) = \mathfrak{q}(\mathbf{A}\mathfrak{f})(\mathfrak{x}).$$

We shall also need the "rank-one operator"

$$\begin{aligned} \mid \mathfrak{f})(\mathfrak{f}' \mid &= \begin{pmatrix} |f_1\rangle & -|\overline{f}_2\rangle \\ |f_2\rangle & |\overline{f}_1\rangle \end{pmatrix} \begin{pmatrix} \langle f_1'| & \langle f_2'| \\ -\langle \overline{f}_2'| & \langle \overline{f}_1'| \end{pmatrix} \\ &= \begin{pmatrix} |f_1\rangle\langle f_1'| + |\overline{f}_2\rangle\langle \overline{f}_2'| & |f_1\rangle\langle f_2'| - |\overline{f}_2\rangle\langle \overline{f}_1'| \\ -|\overline{f}_1\rangle\langle \overline{f}_2'| + |f_2\rangle\langle f_1'| & |\overline{f}_1\rangle\langle \overline{f}_1'| + |f_2\rangle\langle f_2'| \end{pmatrix} \qquad (4.5) \end{aligned}$$

A quaternionic Hilbert space

An orthonormal basis in $\mathfrak{H}_{\mathbb{H}}$ can be built using an orthonormal basis in $\mathfrak{K}_{\mathbb{C}}$. Indeed, let $\{\phi_n\}_{n=0}^{\infty}$ be an orthonormal basis of $\mathfrak{K}_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathbb{H}, d\mathfrak{x})$. Define the vectors

$$| \mathbf{\Phi}_n) = \frac{1}{\sqrt{2}} \begin{pmatrix} |\phi_n\rangle & |\phi_n\rangle \\ -|\overline{\phi}_n\rangle & |\overline{\phi}_n\rangle \end{pmatrix}, \quad n = 0, 1, 2, \dots, \tag{4.6}$$

in $\mathfrak{H}_{\mathbb{H}}$. It is easy to check that these vectors are orthonormal in $\mathfrak{H}_{\mathbb{H}}$. The fact that they form a basis follows from the fact that the vectors $\{\phi_n\}_{n=0}^{\infty}$ are a basis of $L^2_{\mathbb{C}}(\mathbb{H}, d\mathfrak{x})$. Indeed, writing

$$| \mathfrak{f}) = \begin{pmatrix} |f_1\rangle & -|\overline{f}_2\rangle \\ |f_2\rangle & |\overline{f}_1\rangle \end{pmatrix} \in L^2_{\mathbb{H}}(\mathbb{H}, d\mathfrak{x}),$$

we easily verify that

$$| \mathfrak{f}) = \sum_{n=0}^{\infty} | \mathbf{\Phi}_n)\mathfrak{q}_n,$$

where

$$\mathfrak{q}_n = (\mathbf{\Phi}_n \mid \mathfrak{f}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \langle \phi_n \mid f_1\rangle_{\mathfrak{H}_{\mathbb{C}}} - \langle \overline{\phi}_n \mid f_2\rangle_{\mathfrak{H}_{\mathbb{C}}} & -\langle f_2 \mid \overline{\phi}_n\rangle_{\mathfrak{H}_{\mathbb{C}}} - \langle f_1 \mid \phi_n\rangle_{\mathfrak{H}_{\mathbb{C}}} \\ \langle \overline{f}_2 \mid \phi_n\rangle_{\mathfrak{H}_{\mathbb{C}}} + \langle \overline{f}_1 \mid \overline{\phi}_n\rangle_{\mathfrak{H}_{\mathbb{C}}} & \langle f_1 \mid \phi_n\rangle_{\mathfrak{H}_{\mathbb{C}}} - \langle f_2 \mid \overline{\phi}_n\rangle_{\mathfrak{H}_{\mathbb{C}}} \end{pmatrix}.$$

Representation of $G_{\text{aff}}^{\mathbb{H}}$ on $\mathfrak{H}_{\mathbb{H}}$

A representation of $G_{\text{aff}}^{\mathbb{H}}$ on $\mathfrak{H}_{\mathbb{H}}$ can be obtained by simply transcribing (3.16) into the present context. We define the operators $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$ on $\mathfrak{H}_{\mathbb{H}}$:

$$(\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f})(\mathfrak{x}) = \frac{1}{\det[\mathfrak{a}]}\mathfrak{f}(\mathfrak{a}^{-1}(\mathfrak{x} - \mathfrak{b})), \qquad \mathfrak{f} \in \mathfrak{H}_{\mathbb{H}}, \tag{4.7}$$

which by (3.16) and (4.3) can also be written as

$$\mid \mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f}\boldsymbol{)} = \begin{pmatrix} |U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})f_1\rangle & -|\overline{U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})f_2}\rangle \\ U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})|f_2\rangle & |\overline{U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})f_1}\rangle \end{pmatrix}. \tag{4.8}$$

The unitarity of this representation is easy to verify. Indeed,

$$\|\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f}\|^2 == \int_{\mathbb{H}} \left(|(U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})f_1)(\mathfrak{x})|^2 + |(U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})f_2)(\mathfrak{x})|^2\right) \, d\mathfrak{x}\; \sigma_0,$$

which, by the unitarity of the representation $U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})$ on $\mathfrak{K}_{\mathbb{C}}$ gives

$$\|\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f}|^2_{\mathfrak{H}_{\mathbb{H}}} = \left(\|f_1\|^2_{\mathfrak{K}_{\mathbb{C}}} + \|f_2\|^2_{\mathfrak{K}_{\mathbb{C}}}\right)\sigma_0 = \|\mathfrak{f}\|^2_{\mathfrak{H}_{\mathbb{H}}}.$$

Similarly, the irreducibility of $U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})$ on $\mathfrak{K}_{\mathbb{C}}$ leads to the irreducibility of $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$.

Square-integrability of $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$

Using the Duflo-Moore operator C in (3.17) for the representation $U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})$ (see (3.16), we define the Duflo-Moore operator $\mathbf{C}$ for the representation $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$:

$$(\mathbf{C}\mathfrak{f})(\mathfrak{x}) = \begin{pmatrix} (Cf_1)(\mathfrak{x}) & -\overline{(Cf_2)(\mathfrak{x})} \\ (Cf_2)(\mathfrak{x}) & \overline{(Cf_1)(\mathfrak{x})} \end{pmatrix}.$$

We say that the vector $\mathfrak{f}$ is *admissible for the representation* $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$ if it is in the domain of $\mathbf{C}$, i.e., if both f_1 and f_2 are admissible for the representation $U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})$. It is then easy to see that the set of admissible vectors is dense in $\mathfrak{H}_{\mathbb{H}}$.

Let $\mathfrak{f}$ and $\mathfrak{f}'$ be two admissible vectors. Then from (4.8), 4.5) and (3.19) we get

$$\int_{G^{\mathbb{H}}_{\mathbf{aff}}} \mid \mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f})(\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f}' \mid d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = \mathfrak{q}\ I_{\mathfrak{H}_{\mathbb{H}}}, \tag{4.9}$$

where $\mathfrak{q}$ denotes the operator of multiplication from the left, on the Hilbert space $\mathfrak{H}_{\mathbb{H}}$, by the quaternion

$$\mathfrak{q} = \begin{pmatrix} \langle Cf_1' \mid Cf_1\rangle_{\mathfrak{H}_{\mathbb{C}}} + \langle\overline{Cf'}_2 \mid \overline{Cf}_2\rangle_{\mathfrak{H}_{\mathbb{C}}} & \langle Cf_2' \mid Cf_1\rangle_{\mathfrak{H}_{\mathbb{C}}} - \langle\overline{Cf'}_1 \mid \overline{Cf}_2\rangle_{\mathfrak{H}_{\mathbb{C}}} \\ \langle Cf_1' \mid Cf_2\rangle_{\mathfrak{H}_{\mathbb{C}}} - \langle\overline{Cf'}_2 \mid \overline{Cf}_1\rangle_{\mathfrak{H}_{\mathbb{C}}} & \langle\overline{Cf'}_1 \mid \overline{Cf}_1\rangle_{\mathfrak{H}_{\mathbb{C}}} + \langle Cf_2' \mid Cf_2\rangle_{\mathfrak{H}_{\mathbb{C}}} \end{pmatrix}. \tag{4.10}$$

Square-integrability of $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$

Equation (4.9) expresses the square-integrability condition for the representation $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$.

In particular, with $\mathfrak{f} = \mathfrak{f}'$, we get the resolution of the identity,

$$\left[\|\mathbf{C}\mathfrak{f}\|^2_{\mathfrak{H}_{\mathbb{H}}}\right]^{-1} \int_{G^{\mathbb{H}}_{\mathbf{aff}}} \mid \mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f})(\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f} \mid d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = I_{\mathfrak{H}_{\mathbb{H}}}. \tag{4.11}$$

Wavelets and reproducing kernels

Let $\boldsymbol{\eta} \in \mathfrak{H}_{\mathbb{H}}$ be an addmissible vector for the representation $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$, normalized so that

$$\|\mathbf{C}\boldsymbol{\eta}\|^2 = 1.$$

We define the *quaternionic wavelets or coherent states* to be the vectors

$$\mathfrak{S}_{\mathbb{H}} = \{\boldsymbol{\eta}_{\mathfrak{b},\mathfrak{a}} = \mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\boldsymbol{\eta} \mid (\mathfrak{b}, \mathfrak{a}) \in G^{\mathbb{H}}_{\mathbf{aff}}\}, \tag{5.1}$$

By virtue of (4.9) they satisfy the resolution of the identity

$$\int_{G^{\mathbb{H}}_{\mathbf{aff}}} |\boldsymbol{\eta}_{\mathfrak{b},\mathfrak{a}})(\boldsymbol{\eta}_{\mathfrak{b},\mathfrak{a}}| \, d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = I_{\mathfrak{H}_{\mathbb{H}}} \, . \tag{5.2}$$

There is the associated reproducing kernel $\mathbf{K} : G^{\mathbb{H}}_{\mathbf{aff}} \times G^{\mathbb{H}}_{\mathbf{aff}} \longrightarrow \mathbb{H}$,

$$\mathbf{K}(\overline{\mathfrak{b}}, \overline{\mathfrak{a}};\ \mathfrak{b}', \mathfrak{a}') = (\boldsymbol{\eta}_{\mathfrak{b},\mathfrak{a}} \mid \boldsymbol{\eta}_{\mathfrak{b}',\mathfrak{a}'})_{\mathfrak{H}_{\mathbb{H}}} \, , \tag{5.3}$$

with the usual properties,

$$\mathbf{K}(\overline{\mathfrak{b}}, \overline{\mathfrak{a}};\ \mathfrak{b}', \mathfrak{a}') = \overline{\mathbf{K}(\overline{\mathfrak{b}'}, \overline{\mathfrak{a}'};\ \mathfrak{b}, \mathfrak{a})}, \qquad \mathbf{K}(\overline{\mathfrak{b}}, \overline{\mathfrak{a}};\ \mathfrak{b}, \mathfrak{a}) > 0,$$

$$\int_{G^{\mathbb{H}}_{\mathbf{aff}}} \mathbf{K}(\overline{\mathfrak{b}}, \overline{\mathfrak{a}};\ \mathfrak{b}'', \mathfrak{a}'')\, \mathbf{K}(\overline{\mathfrak{b}''}, \overline{\mathfrak{a}''};\ \mathfrak{b}', \mathfrak{a}')\, d\mu_\ell(\mathfrak{b}'', \mathfrak{a}'') = \mathbf{K}(\overline{\mathfrak{b}}, \overline{\mathfrak{a}};\ \mathfrak{b}', \mathfrak{a}'). \tag{5.4}$$

Other questions to consider:

1. Unitary embedding of $\mathfrak{H}_{\mathbb{H}}$ into $L^2_{\mathbb{H}}(G^{\mathbb{H}}_{\mathsf{aff}}, d\mu_\ell)$.

2. Extensions of the representation $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$, e.g. by multiplying from the right by the $SU(2)$ part of $\mathfrak{a}$.

3. Discretization