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Coherent States and Wavelets: A Unified Approach – IV

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Abstract

In this lecture we construct coherent states using unitary irreducible representations of locally compact groups on Hilbert spaces.

Contents

The problem

We have seen that the canonical coherent states could be obtained by the action of a unitary irreducible representation of the Weyl-Heisenberg group on a fixed vector in the Hilbert space. The resulting resolution of the identity was a consequence, as we shall now see, a specific property of the representation, its square integrability. An entirely analogous situation held for the affine groups of the real line, the complexes and the quaternions.

It turns out that square integrability is a property shared by all representations, of a locally compact group, which lie in in the discrete series.

We now study square integrable representations in general and construct families of CS or generalized wavelets using these representations.

But first we need to introduce a couple of group theoretical concepts.

Notation

The following notation will be fixed, from now on:

- G : locally compact group.
- $G \ni g \longmapsto U(g)$: unitary irreducible representation of G on a Hilbert space $\mathfrak{H}$.
- $\mu := \mu_\ell$: left invariant Haar measure of G. We shall mostly work with this measure.
- μ_r : right invariant Haar measure of G.
- $G \ni g \longmapsto \boldsymbol{\Delta}(g)$, modular function of G , i.e., $d\mu_\ell = \boldsymbol{\Delta}(g)\, d\mu_r$.

Left and right regular representations

There are two representations of a locally compact group G, both induced representations, which are of great importance in harmonic analysis and in the theory of CS. These are the so-called regular representations of G.

We start with μ, the left Haar measure on G and consider the trivial subgroup $H = \{e\}$, consisting of just the identity element. The representation of G induced by the trivial representation of H is carried by the Hilbert space $L^2(G, d\mu)$. Denoting this representation by U_ℓ, we have for all $f \in L^2(G, d\mu)$,

$$(U_\ell(g)f)(g') = f(g^{-1}g'), \qquad g, g' \in G.$$

This representation is called the left regular representation of G. Similarly, using the right Haar measure μ_r and the Hilbert space $L^2(G, d\mu_r)$, we can construct another unitary representation U_r, the right regular representation:

$$(U_r(g)f)(g') = f(g'g), \qquad g, g' \in G\ , \quad \forall f \in L^2(G, d\mu_r)\ .$$

Left and right regular representations

In general, these representations are reducible. On the other hand, U_ℓ and U_r are unitarily equivalent representations. Indeed, the map

$$V : L^2(G, d\mu) \to L^2(G, d\mu_r), \qquad (Vf)(g) = f(g^{-1}), \qquad g \in G,$$

is easily seen to be unitary, and

$$VU_\ell(g)V^{-1} = U_r(g), \qquad g \in G.$$

The regular representation U_r can also be realized on the Hilbert space $L^2(G, d\mu)$ (rather than on $L^2(G, d\mu_r)$), using the fact that μ and μ_r are related by the modular function $\boldsymbol{\Delta}$. Thus, the map

$$W : L^2(G, d\mu_r) \to L^2(G, d\mu), \qquad (Wf)(g) = \boldsymbol{\Delta}(g)^{-\frac{1}{2}} f(g)$$

is unitary,

Left and right regular representations

and for all $f \in L^2(G, d\mu)$,

$$(\overline{U}_r(g)f)(g') = \mathbf{\Delta}(g)^{\frac{1}{2}} f(g'g), \quad \text{where} \quad \overline{U}_r(g) = WU_r(g)W^{-1}, \quad g \in G.$$

From this we see that the left and right regular representations commute:

$$[U\ell(g), U_r(g)] = 0 \;, \qquad \forall g \in G \;.$$

Clearly, the two representations U_ℓ and $\overline{U}_r$ on $L^2(G, d\mu)$ are also unitarily equivalent. More interesting, however, is the map $J : L^2(G, d\mu) \to L^2(G, d\mu)$,

$$\begin{aligned} (Jf)(g) &= \overline{f(g^{-1})}\mathbf{\Delta}(g)^{-\frac{1}{2}}, \qquad J^2 = I \\ JU_\ell(g)J &= \overline{U}_r(g), \qquad g \in G, \end{aligned}$$

which is an antiunitary isomorphism and leads to a certain modular structure on the corresponding von Neumann algebras.

An extended Schur's lemma

In harmonic analysis, the irreducibility of a unitary group representation is usually determined by an application of Schur's lemma. For our purposes, we need an extended version of this lemma. We state below three lemmata: the classical Schur's lemma, a generalized version of it and an extended Schur's lemma.

Lemma (Classical Schur's lemma)

Let U be a continuous unitary irreducible representation of G on the Hilbert space $\mathfrak{H}$. If $T \in \mathcal{L}(\mathfrak{H})$, and T commutes with $U(g)$, for all $g \in G$, then $T = \lambda I$, for some $\lambda \in \mathbb{C}$,

In order to state the extended lemma we need an couple of additional concepts. Let U_1 and U_2 be two representations of G on the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. A linear map $T : \mathfrak{H}_1 \to \mathfrak{H}_2$ is said to intertwine U_1 and U_2 if

$$TU_1(g) = U_2(g)T, \qquad \forall g \in G.$$

Generalized Schur's lemma

Given two Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, a linear map $T : \mathfrak{H}_1 \rightarrow \mathfrak{H}_2$ is said to be a multiple of an isometry if there exists $\lambda > 0$ such that,

$$\|T\phi\|^2_{\mathfrak{H}_2} = \lambda \|\phi\|^2_{\mathfrak{H}_1}, \qquad \phi \in \mathfrak{H}_1.$$

Lemma (Generalized Schur's lemma)

Let U_1 be a unitary irreducible representation of G on $\mathfrak{H}_1$ and U_2 a unitary, but not necessarily irreducible, representation of G on $\mathfrak{H}_2$. Let $T : \mathfrak{H}_1 \rightarrow \mathfrak{H}_2$ be a bounded linear map which intertwines U_1 and U_2. Then T is either null or a multiple of an isometry.

This is the form in which Schur's lemma is mostly used in the study of infinite dimensional representations in harmonic analysis.

Extended Schur's lemma

The next extended version of Schur's lemma is the one we shall use in our construction of coherent states from group representations.

Lemma (Extended Schur's lemma)

Let U_1 be a unitary irreducible representation of G on $\mathfrak{H}_1$ and U_2 a unitary, but not necessarily irreducible, representation of G on $\mathfrak{H}_2$. Let $T : \mathfrak{H}_1 \to \mathfrak{H}_2$ be a closed linear map, the domain $\mathcal{D}(T)$ of which is dense in $\mathfrak{H}_1$ and stable under U_1 (i.e., $U_1(g)\phi \in \mathcal{D}(T)$, for all $g \in G$ and $\phi \in \mathcal{D}(T)$), and suppose that T intertwines U_1 and U_2. Then T is either null or a multiple of an isometry.

As a corollary, if $\mathfrak{H}_1 = \mathfrak{H}_2$ and $U_1 = U_2$, then as a consequence of the classical Schur's lemma, T is a multiple of the identity.

We proceed now to a systematic analysis of square integrable group representations and coherent states built out of them.

Admissible vectors

Definition (Admissible vector)

A vector $\eta \in \mathfrak{H}$ is said to be admissible if

$$I(\eta) = \int_G |\langle U(g)\eta|\eta\rangle|^2 \, d\mu(g) < \infty.$$

Note that since $d\mu_r(g) = d\mu(g^{-1})$, and since $U(g)$ is unitary,

$$I(\eta) = \int_G |\langle U(g^{-1})\eta|\eta\rangle|^2 \, d\mu_r(g) = \int_G |\langle \eta|U(g)\eta\rangle|^2 \, d\mu_r(g).$$

Hence,

$$I(\eta) = \int_G |\langle U(g)\eta|\eta\rangle|^2 \, d\mu_r(g),$$

so that it is immaterial whether the left or the right invariant Haar measure is used in the definition of admissibility. Note also that if $\eta \neq 0$, then $I(\eta) \neq 0$.

Admissible vectors

Indeed, since $g \mapsto \langle U(g)\eta|\eta\rangle$ is a continuous function, and the measure $d\mu$ is invariant under left translations, $I(\eta) = 0$ implies $\langle U(g)\eta|\eta\rangle = 0$, for *all* $g \in G$. Since $U(g)\eta$, $g \in G$, is a dense set of vectors in $\mathfrak{H}$, this implies that $\eta = 0$.

Lemma

If $\eta \in \mathfrak{H}$ is an admissible vector, then so also is $\eta_g = U(g)\eta$, for all $g \in G$.

Proof. Indeed,

$$\begin{aligned} I(\eta_g) &= \int_G |\langle U(g')\eta_g|\eta_g\rangle|^2 \, d\mu(g') = \int_G |\langle U(g^{-1}g'g)\eta|\eta\rangle|^2 \, d\mu(g') \\ &= \int_G |\langle U(g'g)\eta|\eta\rangle|^2 \, d\mu(g') \quad \text{by the left invariance of } d\mu \\ &= \int_G |\langle U(g')\eta|\eta\rangle|^2 \boldsymbol{\Delta}(g^{-1}) \, d\mu(g') \\ &= \frac{1}{\boldsymbol{\Delta}(g)} \int_G |\langle U(g')\eta|\eta\rangle|^2 \, d\mu(g') \, . \end{aligned}$$

Admissible vectors

Thus,

$$I(\eta_g) = \frac{1}{\mathbf{\Delta}(g)} I(\eta) < \infty.$$

Let $\mathcal{A}$ denote the set of all admissible vectors. Then, as a consequence of this lemma, $\mathcal{A}$ is stable under $U(g)$, $g \in G$. Since U is irreducible, either $\mathcal{A} = \{0\}$, i.e., it consists of the zero vector only, or $\mathcal{A}$ is total in $\mathfrak{H}$. Furthermore, it turns out that

$$\eta \in \mathcal{A} \quad \text{iff} \quad \int_G |\langle U(g)\eta|\phi\rangle|^2 \, d\mu(g) < \infty, \; \forall\, \phi \in \mathfrak{H},$$

and this in turn implies that $\eta_1 + \eta_2$ is admissible if η_1, η_2 are, i.e. $\mathcal{A}$ is a vector subspace of $\mathfrak{H}$. Therefore, either $\mathcal{A} = \{0\}$, or $\mathcal{A}$ is dense in $\mathfrak{H}$. For $\eta \in \mathcal{A}$, $\eta \neq 0$, we shall write

$$c(\eta) = \frac{I(\eta)}{\|\eta\|^2}.$$

Square integrability of a group representation

Definition

The unitary, irreducible representations $G \ni g \longmapsto U(g)$ is said to be square integrable if $\mathcal{A} \neq \{0\}$.

We then have the result

Theorem

Suppose the UIR $g \mapsto U(g)$ of the locally compact group G is square integrable. Then, for any $\eta \in \mathcal{A}$, the mapping

$$W_\eta : \mathfrak{H} \to L^2(G, d\mu), \qquad (W_\eta \phi)(g) = [c(\eta)]^{-\frac{1}{2}} \langle \eta_g | \phi \rangle, \;\; \phi \in \mathfrak{H}, \; g \in G,$$

is a linear isometry onto a (closed) subspace $\mathfrak{H}_\eta$ of $L^2(G, d\mu)$.
On $\mathfrak{H}$ on has the resolution of the identity

$$\frac{1}{c(\eta)} \int_G |\eta_g\rangle\langle\eta_g| \; d\mu(g) = I \; .$$

Square integrability of a group representation

Theorem (contd.)

The subspace $\mathfrak{H}_\eta = W_\eta \mathfrak{H} \subset L^2(G, d\mu)$ *is a reproducing kernel Hilbert space. The corresponding projection operator*

$$\mathbb{P}_\eta = W_\eta W_\eta^*, \qquad \mathbb{P}_\eta L^2(G, d\mu) = \mathfrak{H}_\eta,$$

has the reproducing kernel K_η*:*

$$\begin{aligned} (\mathbb{P}_\eta \widetilde{\Phi})(g) &= \int_G K_\eta(g, g')\widetilde{\Phi}(g')\, d\mu(g'), \qquad \widetilde{\Phi} \in L^2(G, d\mu), \\ K_\eta(g, g') &= \frac{1}{c(\eta)} \langle \eta_g | \eta_{g'} \rangle, \end{aligned}$$

as its integral kernel.

Furthermore, W_η *intertwines* U *and the left regular representation* U_ℓ*,*

$$W_\eta U(g) = U_\ell(g) W_\eta, \quad g \in G.$$

Square integrability of a group representation

Before proving this theorem, we observe that an entirely analogous result holds with the right regular representation U_r. Thus, for each $\eta \in \mathcal{A}$, there exists a linear isometry,

$$W_\eta^r : \mathfrak{H} \to L^2(G, d\mu_r), \qquad (W_\eta^r \phi)(g) = [c(\eta)]^{-\frac{1}{2}} \langle \eta_{g^{-1}} | \phi \rangle, \ \ \phi \in \mathfrak{H}, \ g \in G.$$

The corresponding reproducing kernel is

$$K_\eta^r(g, g') = \frac{1}{c(\eta)} \langle \eta_{g^{-1}} | \eta_{g'^{-1}} \rangle = K_\eta(g^{-1}, g'^{-1}).$$

Proof of the theorem. The domain $\mathcal{D}(W_\eta)$ of W_η is the set of all vectors $\phi \in \mathfrak{H}$ such that

$$\frac{1}{c(\eta)} \int_G |\langle \eta_g | \phi \rangle_{\mathfrak{H}}|^2 \, d\mu(g) < \infty.$$

Square integrability of a group representation

But, for any $\phi \in \mathcal{D}(W_\eta)$ and $g' \in G$, we have

$$\begin{aligned} \frac{1}{c(\eta)} \int_G |\langle \eta_g | U(g')\phi\rangle_{\mathfrak{H}}|^2 \, d\mu(g) &= \frac{1}{c(\eta)} \int_G |\langle \eta_{g'^{-1}g} | \phi\rangle_{\mathfrak{H}}|^2 \, d\mu(g) \\ &= \frac{1}{c(\eta)} \int_G |\langle \eta_g | \phi\rangle_{\mathfrak{H}}|^2 \, d\mu(g), \end{aligned}$$

the last equality following from the invariance of μ. Thus $\mathcal{D}(W_\eta)$ is stable under U, hence dense in $\mathfrak{H}$, since U is irreducible. Moreover, on $\mathcal{D}(W_\eta)$, the operator W_η intertwines $U(g)$ and the left regular representation $U_\ell(g)$, as is easily seen from the definitions.

We prove next that, as a linear map, W_η is closed. Let $\{\phi_n\}_{n=1}^{\infty} \subset \mathcal{D}(W_\eta)$ be a sequence converging to $\phi \in \mathfrak{H}$ and let the corresponding sequence $\{W_\eta \phi_n\}_{n=1}^{\infty} \subset L^2(G, d\mu)$ converge to $\Phi \in L^2(G, d\mu)$. Then, by the continuity of the scalar product in $\mathfrak{H}$,

Square integrability of a group representation

$$\lim_{n\to\infty} W_\eta \phi_n(g) = \lim_{n\to\infty} \langle \eta_g | \phi_n \rangle = \langle \eta_g | \phi \rangle .$$

Thus, since $W_\eta \phi_n \to \Phi$ in $L^2(G, d\mu)$ and $W_\eta \phi_n(g) \to \langle \eta_g | \phi \rangle$ pointwise,

$$\langle \eta_g | \phi \rangle = \Phi(g),$$

almost everywhere (with respect to μ), whence,

$$\int_G |\langle \eta_g | \phi \rangle|^2 \, d\mu(g) < \infty,$$

implying that $\phi \in \mathcal{D}(W_\eta)$ and $W_\eta \phi = \Phi$, i.e., W_η is closed.
Using the extended Schur's lemma, we establish the boundedness of $W_\eta : \mathcal{D}(W_\eta) \to L^2(G, d\mu)$. Hence $\mathcal{D}(W_\eta) = \mathfrak{H}$, and furthermore, W_η is a multiple of the isometry:

$$\| W_\eta \phi \|^2_{L^2(G,d\mu)} = \lambda \| \phi \|^2_{\mathfrak{H}}, \qquad \phi \in \mathfrak{H}\,, \;\; \lambda \in \mathbb{R}^+ \,.$$

To fix λ, take $\phi = \eta$. Then

$$\lambda = \frac{\|W_\eta \eta\|^2_{L^2(G,d\mu)}}{\|\eta\|^2} = \frac{I(\eta)}{c(\eta)\|\eta\|^2} = 1,$$

Thus, W_η is an isometry, i.e. $W_\eta^* W_\eta = I$, which implies that the resolution of the identity holds. Therefore, the range of W_η is a closed subspace of $L^2(G, d\mu)$, and the projection on it is $\mathbb{P}_\eta = W_\eta W_\eta^*$. Then the expression for the reproducing kernel and the intertwining property follow from immediately.

An immediate consequence of this theorem is the following important result.

Corollary

Every square integrable representation of a locally compact group G is unitarily equivalent to a subrepresentation of its left regular representation (and hence also of its right regular representation).

The proof of this corollary consists simply in showing that the projection $\mathbb{P}_\eta$ on the range of W_η commutes with the left regular representation. Indeed:

$$\begin{aligned}\mathbb{P}_\eta U_\ell(g) &= W_\eta W_\eta^* U_\ell(g) = W_\eta (U_\ell(g^{-1}) W_\eta)^* = W_\eta (W_\eta U(g^{-1}))^* \\ &= W_\eta U(g) W_\eta^* = U_\ell(g) W_\eta W_\eta^* = U_\ell(g)\mathbb{P}_\eta .\end{aligned}$$

Since W_η is an isometry, its inverse is equal to its adjoint on its range, i.e. $W_\eta^{-1} = W_\eta^*$ on $\mathfrak{H}_\eta$. Then, applying both sides of the resolution of the identity to an arbitrary vector $\phi \in \mathfrak{H}$, we obtain the reconstruction formula

$$\phi = W_\eta^* \Phi = \frac{1}{[c(\eta)]^{\frac{1}{2}}} \int_G \Phi(g) \eta_g \; d\mu(g).$$

Later we shall obtain a generalized version of this reconstruction formula using two different admissible vectors.

Square integrability of a group representation

A consequence of the above theorem is that we may obtain total set of CS indexed by the points of the group G itself, i.e., if ϕ is an admissible vector then every vector in the set

$$\mathfrak{S}_\phi = \{\phi_g = U(g)\phi \mid g \in G\}$$

is a coherent state or a generalized wavelet and this is a total set in the Hilbert space $\mathfrak{H}$ of the unitary irreducible representation U of the group G.

However, in general, the CS systems of physical interest are supported by a quotient manifold $X = G/H$.

Square integrability of a group representation

We note that a vector $\Phi \in W_\eta \mathfrak{H} = \mathfrak{H}_\eta = \mathbb{P}_\eta L^2(G, d\mu)$, if and only if there exists a vector $\phi \in \mathfrak{H}$ such that $\Phi(g) = [c(\eta)]^{-\frac{1}{2}} \langle \eta_g | \phi \rangle$ for almost all $g \in G$ (with respect to the measure μ).

This also means, in view of the strong continuity of the representation $g \mapsto U(g)$, that $\Phi(g)$ can be identified with the bounded continuous function of G,

$$\begin{aligned} g &\mapsto [c(\eta)]^{-\frac{1}{2}} \langle \eta_g | \phi \rangle = \langle U(g)\eta | \phi \rangle \\ \sup_{g \in G} |\langle \eta_g | \phi \rangle| &\leq \sup_{g \in G} \|U(g)\eta\| \, \|\phi\| = \|\eta\| \, \|\phi\|. \end{aligned}$$

Hence the reproducing kernel subspace $\mathfrak{H}_\eta$ can be identified with a space of bounded, continuous functions on the group G.

In addition, the reproducing kernel $K_\eta(g, g')$ is in the present case a convolution kernel on G : $K_\eta(g, g') = \langle \eta | U(g^{-1} g')\eta \rangle$. This implies that K_η has a regularizing effect.

Square integrability of a group representation

For instance, if G is a Lie group, and η is appropriately chosen, the elements of $\mathfrak{H}_\eta$ can be made to be infinitely differentiable functions, which extend to holomorphic functions on the complexified group G^c.

This gives rise to some of the attractive holomorphic properties of CS, and their geometrical implications. Another consequence of the convolution character of K_η is that the kernel, and hence the elements of $\mathfrak{H}_\eta$, have interpolation properties which prove useful in practical computations.

Finally, it should be emphasized that the reproducing kernel K_η is the main tool for computing the efficiency or resolving power of the transform W_η, in wavelet analysis.

Notice that each admissible vector η determines its own reproducing kernel K_η and reproducing kernel subspace $\mathfrak{H}_\eta$.

In our discussion of square integrable representations so far, the representation U has been assumed to be irreducible. This requirement may be weakened in several ways.

Square integrability of a group representation

A first possibility is to take a direct sum of square integrable representations. In this case one may prove:

Theorem

Let G be a locally compact group, with left Haar measure μ. Let U be a strongly continuous unitary representation of G into a Hilbert space $\mathfrak{H}$, and assume that U is a direct sum of disjoint square integrable representations U_i:

$$U = \bigoplus_i U_i, \quad in \quad \mathfrak{H} = \bigoplus_i \mathfrak{H}_i.$$

Let η be an admissible vector. Then,

$$\int_G |\langle U(g)\eta|\phi\rangle|^2 \, d\mu(g) = \sum_i c_i \|\mathbb{P}_i\phi\|^2, \qquad \phi \in \mathfrak{H},$$

where $\mathbb{P}_i$ is the projection on $\mathfrak{H}_i$ and

Square integrability of a group representation

Theorem (Contd.)

$$c_i = \|\mathbb{P}_i \eta\|^{-2} \int_G |\langle U_i(g)\mathbb{P}_i\eta|\mathbb{P}_i\eta\rangle|^2 \, d\mu(g).$$

If, in addition, all the constants c_i are equal, then the map $W_\eta : \phi \mapsto \langle U(g)\eta|\phi\rangle$ is an isometry (up to a constant) from $\mathfrak{H}$ into $L^2(G, d\mu)$.

Thus, when the conditions of this theorem are satisfied, CS may be built in the usual way. By similar arguments, the same is true if some of the components U_i are mutually unitarily equivalent.

Another generalization is to take for U a cyclic representation, with η a cyclic vector. In this case, assuming the admissibility condition, all the assertions of the above theorem may be recovered.

A more radical approach is to take a direct integral over irreducible representations from the continuous series.

Orthogonality relations

If G is a compact group and U a unitary irreducible representation of G, then according to the Peter-Weyl theorem, the matrix elements $\langle U(g)\psi|\phi\rangle$ of U satisfy certain orthogonality relations, and one may construct an orthonormal basis of $L^2(G, d\mu)$ consisting of such matrix elements.

When G is only locally compact, square integrable representations have the same property. Thus, among all UIR's, the square integrable representations are the direct generalizations of the irreducible representations of compact groups. These orthogonality relations, well-known when G is unimodular, extend to non-unimodular groups as well.

Theorem (Orthogonality relations)

Let G be a locally compact group, U a square integrable representation of G on the Hilbert space $\mathfrak{H}$. Then there exists a unique positive, self-adjoint, invertible operator C in $\mathfrak{H}$, the domain $\mathcal{D}(C)$ of which is dense in $\mathfrak{H}$ and is equal to $\mathcal{A}$, the set of all admissible vectors;

Orthogonality relations

Theorem (Contd.)

if η and η' are any two admissible vectors and ϕ, ϕ' are arbitrary vectors in $\mathfrak{H}$, then

$$\int_G \overline{\langle \eta'_g | \phi' \rangle} \langle \eta_g | \phi \rangle \, d\mu(g) = \langle C\eta | C\eta' \rangle \, \langle \phi' | \phi \rangle .$$

Furthermore $C = \lambda I$, $\lambda > 0$, if and only if G is unimodular.

Proof. Let $\eta, \eta' \in \mathcal{A}$, and consider the corresponding isometries $W_\eta, W_{\eta'}$. With $W_\eta^* : L^2(G, d\mu) \to \mathfrak{H}$ denoting, as before, the adjoint of the linear map $W_\eta : \mathfrak{H} \to L^2(G, d\mu)$, the operator $W_{\eta'}^* W_\eta$ is bounded on $\mathfrak{H}$.
Next, for all $g \in G$,

$$\begin{aligned} W_{\eta'}^* W_\eta U(g) &= W_{\eta'}^* U_\ell(g) W_\eta \\ &= [U_\ell(g^{-1}) W_{\eta'}]^* W_\eta = [W_{\eta'} U(g^{-1})]^* W_\eta \\ &= U(g) W_{\eta'}^* W_\eta . \end{aligned}$$

Orthogonality relations

By Schur's lemma, $W^*_{\eta'} W_\eta$ is therefore a multiple of the identity on $\mathfrak{H}$:

$$W^*_{\eta'} W_\eta = \lambda(\eta, \eta') I, \qquad \lambda(\eta, \eta') \in \mathbb{C}$$

($\lambda(\eta, \eta')$ is antilinear in η and linear in η'). Applying the square-integrability theorem, we find, for $\eta = \eta'$,

$$\lambda(\eta, \eta) = 1, \qquad \eta \in \mathcal{A}.$$

Set

$$q(\eta, \eta') = [c(\eta)c(\eta')]^{\frac{1}{2}} \ \lambda(\eta, \eta'),$$

with $c(\eta)$ as previously defined. Thus,

$$\begin{aligned}
\int_G \overline{\langle \eta'_g | \phi' \rangle} \langle \eta_g | \phi \rangle \, d\mu(g) &= [c(\eta)c(\eta')]^{\frac{1}{2}} \int_G \overline{(W_{\eta'}\phi')(g)} (W_\eta \phi)(g) \, d\mu(g) \\
&= [c(\eta)c(\eta')]^{\frac{1}{2}} \ \langle W_{\eta'}\phi' | W_\eta \phi \rangle_{L^2(G, d\mu)} \\
&= [c(\eta)c(\eta')]^{\frac{1}{2}} \ \langle \phi' | W^*_{\eta'} W_\eta \phi \rangle_{\mathfrak{H}},
\end{aligned}$$

Orthogonality relations

for all $\eta, \eta' \in \mathcal{A}$ and $\phi, \phi' \in \mathfrak{H}$. Hence,

$$\int_G \overline{\langle \eta'_g | \phi' \rangle} \langle \eta_g | \phi \rangle \, d\mu(g) = q(\eta, \eta') \langle \phi' | \phi \rangle_{\mathfrak{H}}.$$

But we also have from the above,

$$W^*_{\eta'} W_\eta = \frac{1}{[c(\eta) c(\eta')]^{\frac{1}{2}}} \int_G |\eta'_g\rangle\langle\eta_g| \, d\mu(g).$$

Also, we see that $q : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ is a positive, symmetric, sesquilinear form on the dense domain $\mathcal{A}$.

Moreover, since q is independent of ϕ, ϕ', taking $\phi = \phi' \neq 0$ we obtain

$$q(\eta, \eta') = \frac{1}{\|\phi\|^2} \int_G \overline{\langle U(g)\eta' | \phi \rangle} \langle U(g)\eta | \phi \rangle \, d\mu(g).$$

Orthogonality relations

We next prove that as a sesquilinear form q is closed on its form domain $\mathcal{A}$. Indeed, on $\mathcal{A}$ consider the scalar product and associated norm:

$$\langle \eta | \eta' \rangle_q = \langle \eta | \eta' \rangle_{\mathfrak{H}} + q(\eta, \eta'), \quad \|\eta\|_q^2 = \|\eta\|_{\mathfrak{H}}^2 + q(\eta, \eta), \qquad \eta, \eta' \in \mathcal{A}.$$

Let $\{\eta_k\}_{k=1}^{\infty} \subset \mathcal{A}$ be a Cauchy sequence in the $\| \ldots \|_q$-norm. Clearly, $\{\eta_k\}_{k=1}^{\infty}$ is also a Cauchy sequence in the norm of $\mathfrak{H}$, implying that there exists a vector $\eta \in \mathfrak{H}$ such that $\lim_{k\to\infty} \|\eta_k - \eta\|_{\mathfrak{H}} = 0$. Also, since the sequence is Cauchy in the $\| \ldots \|_q$-norm, $q(\eta_j - \eta_k,\ \eta_j - \eta_k) \to 0$ for $j, k \to \infty$. From the equation above we infer that the sequence of functions,

$$\{\widetilde{\Phi}_k\}_{k=1}^{\infty} \subset L^2(G, d\mu), \qquad \widetilde{\Phi}_k(g) = \langle U(g)\eta_k | \phi \rangle_{\mathfrak{H}},$$

is a Cauchy sequence in $L^2(G, d\mu)$. Thus there exists a vector $\widetilde{\Phi} \in L^2(G, d\mu)$ satisfying

$$\lim_{k\to\infty} \|\widetilde{\Phi}_k - \widetilde{\Phi}\|_{L^2(G,d\mu)} = 0,$$

and therefore, the sequence $\{\widetilde{\Phi}_k\}_{k=1}^{\infty}$ also converges to $\widetilde{\Phi}$ weakly, with the sequence of norms $\{\|\widetilde{\Phi}_k\|_{L^2(G,d\mu)}\}_{k=1}^{\infty}$ remaining bounded. Moreover, for any $g \in G$,

$$\lim_{k\to\infty} \langle U(g)\eta_k|\phi\rangle_{\mathfrak{H}} = \langle U(g)\eta|\phi\rangle_{\mathfrak{H}} \quad \Rightarrow \quad \lim_{k\to\infty} |\widetilde{\Phi}_k(g) - \widetilde{\Phi}(g)| = 0.$$

Thus, $\widetilde{\Phi}(g) = \langle U(g)\eta|\phi\rangle_{\mathfrak{H}}$, for all $g \in G$ and all $\phi \in \mathfrak{H}$, so that $g \mapsto \langle U(g)\eta|\phi\rangle_{\mathfrak{H}}$ defines a vector in $L^2(G, d\mu)$. Taking $\phi = \eta$, we see that this implies $\eta \in \mathcal{A}$. Next,

$$\begin{aligned} \lim_{k\to\infty} \|\eta_k - \eta\|_q^2 &= \lim_{k\to\infty} \|\eta_k - \eta\|_{\mathfrak{H}}^2 + \lim_{k\to\infty} q(\eta_k - \eta,\ \eta_k - \eta) \\ &= 0 + \lim_{k\to\infty} \frac{1}{\|\phi\|^2}\|\widetilde{\Phi}_k - \widetilde{\Phi}\|_{L^2(G,d\mu)}^2, \quad \text{by } (\mathbf{??}), \\ &= 0. \end{aligned}$$

Consequently, $\mathcal{A}$ is complete in the $\|\ldots\|_q$-norm, so that q is closed.

Orthogonality relations

Since q is a closed, symmetric, positive form, the well known second representation theorem implies that there exists a unique positive self-adjoint operator C, with domain $\mathcal{A}$, such that

$$q(\eta, \eta') = \langle C\eta | C\eta' \rangle_{\mathfrak{H}}.$$

Next, if $\eta \neq 0$, then

$$\|C\eta\|^2 = c(\eta) = \frac{I(\eta)}{\|\eta\|^2} \neq 0$$

So C is injective and consequently it is invertible. Moreover, its inverse C^{-1} is densely defined, as the inverse of an invertible self-adjoint operator (indeed it is easily seen that Ran(C) (the range of C) is dense in $\mathfrak{H}$.

It remains to prove the last statement. Now, for all $g \in G$,

$$\begin{aligned} q(U(g)\eta, U(g)\eta') &= \frac{1}{\|\phi\|^2} \int_G \overline{\langle U(g'g)\eta' | \phi \rangle} \langle U(g'g)\eta | \phi \rangle \, d\mu(g') \\ &= \frac{\boldsymbol{\Delta}(g^{-1})}{\|\phi\|^2} \int_G \overline{\langle U(g')\eta' | \phi \rangle} \langle U(g')\eta | \phi \rangle \, d\mu(g'), \end{aligned}$$

Orthogonality relations

so that finally,

$$q(U(g)\eta, U(g)\eta') = \boldsymbol{\Delta}(g^{-1})\, q(\eta, \eta').$$

Hence, for all $\eta, \eta' \in \mathcal{A}$,

$$\langle CU(g)\eta | CU(g)\eta' \rangle_{\mathfrak{H}} = \frac{1}{\boldsymbol{\Delta}(g)} \langle C\eta | C\eta' \rangle_{\mathfrak{H}}.$$

Now C^2 is positive and densely defined in $\mathfrak{H}$. In addition, its domain is invariant under U. Indeed, let $\eta' \in \mathcal{D}(C^2)$, which implies that $\eta' \in \mathcal{D}(C)$, $C\eta' \in \mathcal{D}(C)$ and $\eta'_g \in \mathcal{D}(C)$. Then the above equation becomes

$$\langle C\eta_g | C\eta'_g \rangle_{\mathfrak{H}} = \frac{1}{\boldsymbol{\Delta}(g)} \langle \eta | C^2\eta' \rangle_{\mathfrak{H}},$$

which shows that $C\eta'_g \in \mathcal{D}(C)$ as well, i.e. $\eta'_g \in \mathcal{D}(C^2)$. Thus, on the dense invariant domain $\mathcal{D}(C^2)$:

$$C^2 U(g) = \frac{1}{\boldsymbol{\Delta}(g)} U(g) C^2.$$

Orthogonality relations

Using the Extended Schur's Lemma, with $U_1 = U_2$, we see that $\mathbf{\Delta}(g) = 1$, for all $g \in G$, that is, G is unimodular if and only if $C = \lambda I$, $\lambda > 0$.
The operator C is known in the mathematical literature as the Duflo-Moore operator, often denoted $C = K^{-1/2}$. Actually, it can be shown that if G is compact, then

$$C = [\dim \mathfrak{H}]^{-\frac{1}{2}} I.$$

(Note that with G compact and U irreducible, $\dim \mathfrak{H}$ is finite.) If G is not compact, but just unimodular, then with $\|\eta\| = 1$,

$$C = [c(\eta)]^{\frac{1}{2}} I,$$

so that the value of $c(\eta)$ does not depend on $\eta \in \mathcal{A}$. In that case, we call $d_U \equiv c(\eta)^{-1}$ the formal dimension of the representation U. In this terminology, when G is a nonunimodular group, the formal dimension of a square integrable representation U is the positive self-adjoint (possibly unbounded) operator C^{-2}.

Orthogonality relations

Finally, we derive a generalized version of the resolution of the identity.

Corollary

Let U be a square integrable representation of the locally compact group G. If η and η' are any two nonzero admissible vectors, then, provided $\langle C\eta | C\eta'\rangle \neq 0$,

$$\frac{1}{\langle C\eta | C\eta'\rangle} \int_G |\eta'_g\rangle\langle\eta_g| \, d\mu(g) = I.$$

Proof. This is mere restatement of the orthogonality relation, since the vectors ϕ and ϕ' are arbitrary.

From here we get the reconstruction formula

$$\phi = \frac{[c(\eta')]^{\frac{1}{2}}}{\langle C\eta | C\eta'\rangle} \int_G \Phi(g)\eta'_g \, d\mu(g), \quad \phi \in \mathfrak{H},$$

provided $\langle C\eta | C\eta'\rangle \neq 0$.

Orthogonality relations

This generalizes our earlier reconstruction formula. Here η is called the analyzing vector and η' the synthesizing vector. An important consequence of the above formula is that there are many kernels associated to a given η, namely all the functions

$$K_{\eta\eta'}(g, g') = \frac{1}{\langle C\eta | C\eta' \rangle} \langle \eta_g | \eta'_{g'} \rangle,$$

each one of which defines the evaluation map on $\mathfrak{H}_\eta \in L^2(G, d\mu)$:

$$\int_G K_{\eta\eta'}(g, g')\Phi(g')\, d\mu(g') = \Phi(g), \qquad \Phi \in \mathfrak{H}_\eta = W_\eta(\mathfrak{H}).$$

It ought to be noted, however, that if $\eta \neq \eta'$, $K_{\eta\eta'}$ is not a positive definite kernel, and hence not a reproducing kernel, although, as an integral operator on $L^2(G, d\mu)$, it is idempotent:

$$\int_G K_{\eta\eta'}(g, g'')K_{\eta\eta'}(g'', g')\, d\mu(g'') = K_{\eta\eta'}(g, g').$$