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Coherent states, POVM, quantization and measurement contd.

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Coherent states, POVM, quantization and measurement

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Contents

[1] H Bergeron and J.-P G. Integral quantizations with two basic examples, *Annals of Physics (NY)*, **344** 43-68 (2014) arXiv:1308.2348 [quant-ph, math-ph]

[2] S.T. Ali, J.-P Antoine, and J.P. G. *Coherent States, Wavelets and their Generalizations* 2d edition, Theoretical and Mathematical Physics, Springer, New York (2013), specially Chapter 11.

[3] H. Bergeron, E. M. F. Curado, J.P. G. and Ligia M. C. S. Rodrigues, *Quantizations from (P)OVM's*, Proceedings of the 8th Symposium on Quantum Theory and Symmetries, El Colegio Nacional, Mexico City, 5-9 August, 2013, Ed. K.B. Wolf, J. Phys.: Conf. Ser. (2014); arXiv: 1310.3304 [quant-ph, math-ph]

[4] H. Bergeron, A. Dapor, J.P. G. and P. Małkiewicz, Smooth big bounce from affine quantization, *Phys. Rev. D* **89**, 083522 (2014) (2014); arXiv:1305.0653 [gr-qc]

[5] M. Baldiotti, R. Fresneda, J.P. G. *Three examples of covariant integral quantization* Proceedings of Science (2014)

1. Weyl-Heisenberg integral quantizations of functions and distributions

Acceptable probes ρ

- How to characterize acceptable operator-valued functions $\mathsf{M}(x)$ and functions f quantizable with respect to the latter?
- For the Weyl-Heisenberg integral quantization, let us restrict to positive unit trace operators (or "probe") $\mathsf{M} = \rho$ and, in particular, the examples $\rho = \rho_s$ in the range $\infty < s \leqslant -1$ including the most manageable CS case.
- We recall that the mean value or lower symbol of A_f is defined by

$$\check{f}(z) = \int_{\mathbb{C}} \mathrm{tr}\big(\rho(z)\rho(z')\big)\, f(z')\, \frac{\mathrm{d}^2 z'}{\pi}\,. \tag{1}$$

- In particular, the resolution of the identity proves that:

$$\int_{\mathbb{C}} \mathrm{tr}\big(\rho(z)\rho(z')\big)\frac{\mathrm{d}^2 z'}{\pi} = 1\,,$$

  i.e. for each z, $\mathrm{tr}\big(\rho(z)\rho(z')\big) = \mathrm{tr}\big(\rho\rho(z-z')\big)$ is a probability distribution on the phase space, and so $\check{f}$ is issued from the corresponding kernel averaging of the original f.

CS case: Gaussian convolution

- In the CS case $\rho(z) = |z\rangle\langle z|$, (1) is the Gaussian convolution (Berezin or heat kernel transform) of the function $f(z)$:

$$\check{f}(z) = \langle z|A_f|z\rangle = \int_{\mathbb{C}} e^{-|z-z'|^2} f(z')\,\frac{\mathrm{d}^2 z'}{\pi}\,. \tag{2}$$

- Does lower symbol $\check{f}$ approximates f better and better at the classical limit? For that we must give the complex plane a physical phase-space content after introducing physical units through

$$z \stackrel{\text{def}}{=} \frac{q}{\ell\sqrt{2}} + i\frac{p\ell}{\hbar\sqrt{2}}\,, \tag{3}$$

where ℓ is an arbitrary length scale, and then take $\hbar \to 0\,,\ \ell \to 0\,,\ \hbar/\ell \to 0$. Then, in the Gaussian case, $\check{f} \to f$ uniformly for regular functions through saddle point approximation. For singular functions, the semi-classical limit is less obvious and has to be verified case by case.

Acceptable probes (continued)

- Motivated by the Gaussian case, we adopt the following classicality requirement on the choice of density operators ρ:

- *A density operator ρ is acceptable from the classical point of view if*

  (i) it obeys the limit condition

$$\mathrm{tr}\big(\rho(z)\rho(z')\big) \to \delta(z - z') \quad \textit{as} \quad \hbar \to 0\,,\ \ell \to 0\,,\ \hbar/\ell \to 0\,,$$

  This implies a suitable $\hbar$ and ℓ dependence on all other parameters involved in the expression of ρ,

  (ii) the matrix elements $\langle e_n|\rho(z)|e_{n'}\rangle$ (w.r.t. some orthonormal basis $\{e_n\}$) are C^∞ functions in z with rapid decrease.

- Condition (ii) will appear natural for the quantization of distributions.

Quantizable functions

- Inspired by the CS case in which with mild constraints on f its transform $\check{f}$ inherits infinite differentiability from the Gaussian, let us adopt the second acceptance criterium, which concerns the function f to be quantized.

- *Given an acceptable density operator* ρ*, a function* $\mathbb{C} \ni z \mapsto f(z) \in \mathbb{C}$ *is* ρ-quantizable *along the map* $f \mapsto A_f$ *with* $\mathsf{M} = \rho$*, if the map* $\mathbb{C} \ni z = \frac{1}{\sqrt{2}}(q + ip) \sim (q,p) \mapsto \check{f}(z)$ *is a* C^∞ *function with respect to the* (q,p) *coordinates of the complex plane.*

- This definition is reasonable insofar as differentiability properties of $\check{f}(z)$ are those of the displacement operator $D(z)$. We will extend this definition to distributions $T \in \mathcal{D}'(\mathbb{R}^2)$ in the next subsection.

- In the CS case, the fact that the Berezin transform $f \mapsto \check{f}$ is a Gaussian convolution is of great importance. It explains the robustness of CS quantization, since it is well defined for a very large class of non smooth functions and even, as is shown below, for a class of distributions including the tempered ones.

Illustration: quantum angle or phase

- With $z = \sqrt{J}\, e^{i\gamma}$ in action-angle (J,γ) notations for the harmonic oscillator, quantization of $f(J,\gamma)$, 2π-periodic in γ, yields formally

$$A_f = \int_0^{+\infty} \mathrm{d}J \int_0^{2\pi} \frac{\mathrm{d}\gamma}{2\pi} f(J,\gamma)\rho\left(\sqrt{J}e^{i\gamma}\right) \,. \tag{4}$$

- With the unitary representation $\theta \mapsto U_{\mathbb{T}}(\theta)$ of the unit circle $\mathbb{S}^1$ on the Hilbert space $\mathcal{H}$ which was defined by $U_{\mathbb{T}}(\theta)|e_n\rangle = e^{i(n+\nu)\theta}|e_n\rangle$, ν real, one verifies, in the case of diagonal ρ (i.e. ϖ isotropic), the angular covariance property:

$$U_{\mathbb{T}}(\theta) A_f U_{\mathbb{T}}(-\theta) = A_{T(\theta)f}\,, \quad T(\theta)f(J,\gamma) = f(J,\gamma-\theta)\,. \tag{5}$$

Quantum angle[a] or phase (continued)

- As an example, let us quantize with coherent states, $\rho(z) = |z\rangle\langle z|$, the discontinuous 2π-periodic angle function $\daleth(\gamma) = \gamma$ for $\gamma \in [0, 2\pi)$.

- In terms of the action-angle variables standard CS read as

$$|z\rangle \equiv |J, \gamma\rangle = \sum_n \sqrt{p_n(J)} e^{in\gamma} |e_n\rangle \,, \tag{6}$$

  where $n \mapsto p_n(J) = e^{-J} J^n / n!$ is the Poisson distribution.

- The action variable is precisely the Poisson average of the discrete variable n, $\langle n \rangle_{\text{poisson}} = J$. Note that in electromagnetism, the variables J and γ represent the field intensity and the phase, respectively.

- Since the angle function is real and bounded, its quantum counterpart $A_{\daleth}$ is a bounded self-adjoint operator, and it is covariant in the above sense.

[a] P. Carruthers and M. M. Nieto, Phase and angle variables in quantum mechanics, *Rev. Mod. Phys.* **40** (1968) 411–440; A. Royer, Phase states and phase operators for the quantum harmonic oscillator, *Phys. Rev. A* **53** (1996) 70–108; JPG, F. Szafraniec, Quantum angle operator, in progress

Quantum phase and its classical portrait

- In the basis $|e_n\rangle$, quantum phase or angle operator $A_{\daleth}$ is given by the infinite matrix:

$$A_{\daleth} = \pi\, 1_{\mathcal{H}} + i \sum_{n\neq n'} \frac{\Gamma\left(\frac{n+n'}{2}+1\right)}{\sqrt{n!n'!}}\, \frac{1}{n'-n}\, |e_n\rangle\langle e_{n'}|\, . \tag{7}$$

This operator has spectral measure with support $[0, 2\pi]$.

- The corresponding "lower symbol" reads as the Fourier sine series:

$$\langle J,\gamma|A_{\daleth}|J,\gamma\rangle = \pi - 2\sum_{q=1}^{\infty} d_q(\sqrt{J})\, \frac{\sin q\gamma}{q}\, ,$$

with $d_q(r) = e^{-r^2} r^q \frac{\Gamma(\frac{q}{2}+1)}{\Gamma(q+1)}\, {}_1F_1(\frac{q}{2}+1; q+1; r^2)$ balances the trigonometric Fourier coefficient $2/q$ of the angle function $\daleth$. It can be shown [a] that this positive function is bounded by 1.

[a] JPG and M. del Olmo, *q-coherent states quantization of the harmonic oscillator,* Annals of Physics (NY) **330** 220-245 (2013); arXiv:1207.1200 [quant-ph]

Semi-classical behavior

- At small J, the lower symbol oscillates around its average value π with amplitude equal to $\sqrt{\pi J}$:

$$\langle J,\gamma|A_{\daleth}|J,\gamma\rangle \approx \pi - \sqrt{\pi J}\,\sin\gamma\,.$$

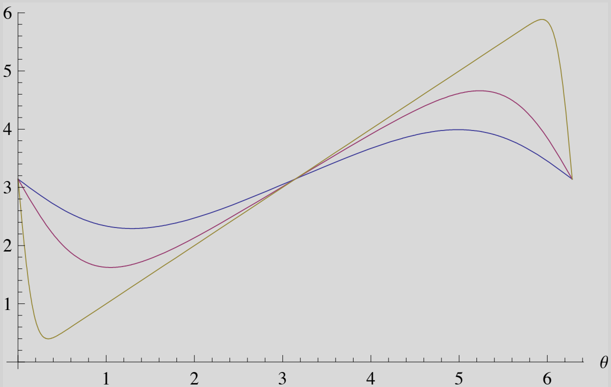
- At large J, we recover the Fourier series of the 2π-periodic angle function:

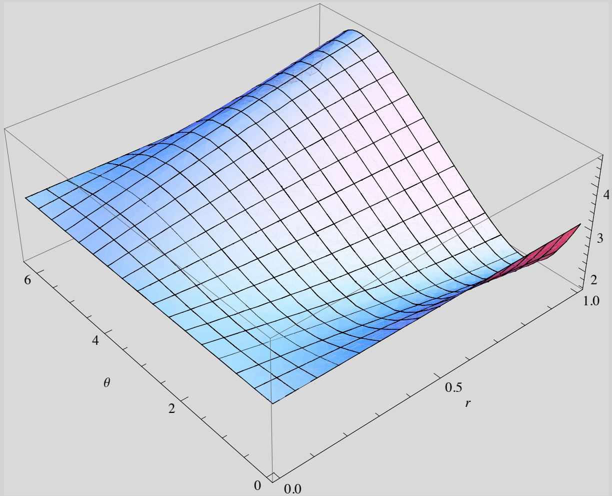
$$\langle J,\gamma|A_{\daleth}|J,\gamma\rangle \approx \pi - 2\sum_{q=1}^{\infty}\frac{1}{q}\sin q\gamma = \daleth(\gamma) \quad \text{for} \quad \gamma\in[0,2\pi)\,.$$

- By re-injecting physical dimensions, $|z|^2 = J$ is an action and should appear in the formulas as divided by $\hbar$: the limit $J\to\infty$ is the classical limit $\hbar\to 0$.

Lower symbol of the phase operator

Behavior of $\langle J, \gamma | A_{\daleth} | J, \gamma \rangle$ as a function of $\theta \equiv \gamma$ for different values of J. Observe how much it becomes close to the classical one at the largest value of J.





Lower symbol of the angle operator for $\sqrt{J} = \{0.5, 1, 5\}$ and $\gamma \equiv \theta \in [0, 2\pi)$ (left) and for $\left(\sqrt{J}, \gamma\right) \in [0, 1] \times [0, 2\pi)$ (right).

Semi-classical behavior continued

- The number operator $\hat{N} = a^\dagger a$ is, up to a constant shift, the quantization of the classical action, $A_J = \hat{N} + 1$: $A_J = \sum_n (n+1)|e_n\rangle\langle e_n|$.
- Are the commutator (if properly defined as an operator) of action and angle operators and its lower symbol close to the canonical value iI?

$$[A_\beth, A_J] = i \sum_{n \neq n'} \frac{\Gamma\left(\frac{n+n'}{2}+1\right)}{\sqrt{n!n'!}} |n\rangle\langle n'| \,,$$

$$\langle J,\gamma|[A_\beth, A_J]|J,\gamma\rangle = 2i \sum_{q=1}^{\infty} d_q(\sqrt{J})\, \cos q\gamma \equiv i\, \mathcal{C}(J,\gamma)\,.$$

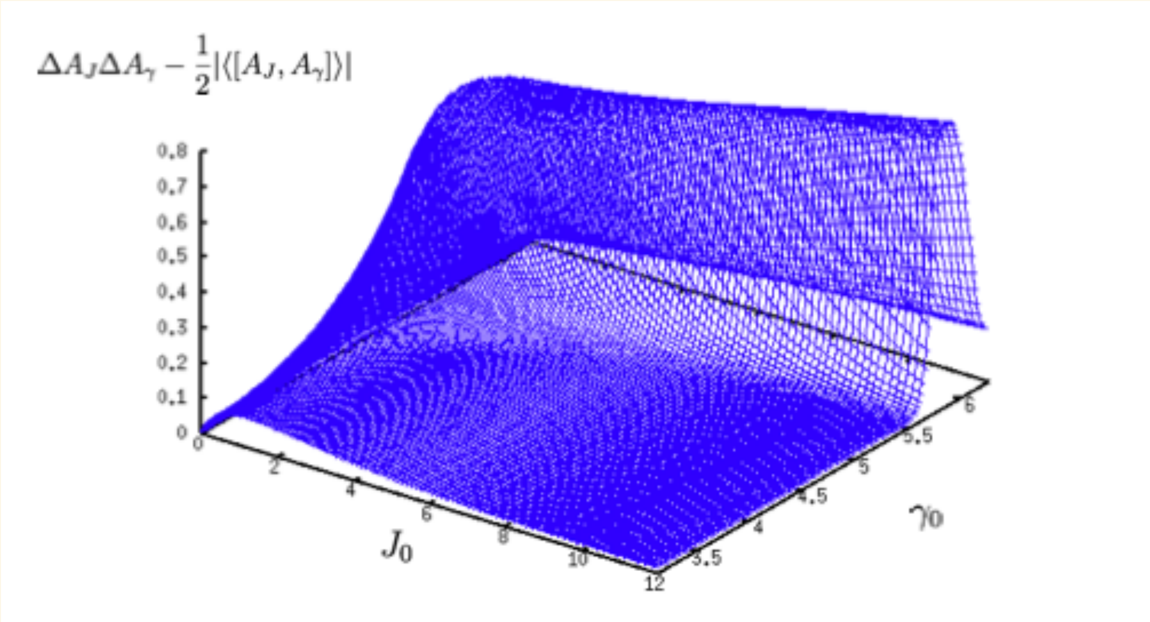
- At small J, the function $\mathcal{C}(J,\gamma)$ oscillates around 0 with amplitude equal to $\sqrt{\pi}\sqrt{J}$: $\mathcal{C}(J,\gamma) \approx \sqrt{\pi}\sqrt{J}\ \cos\gamma$. Applying the Poisson summation formula, we get at $J \to \infty$ (or $\hbar \to 0$) the expected "canonical" behavior for $\gamma \in [0, 2\pi)$:

$$\langle J,\gamma|[A_\beth, A_J]|J,\gamma\rangle \approx -i + 2\pi i \sum_{n\in\mathbb{Z}} \delta(\gamma - 2\pi n)\,.$$

Accept non-canonical commutation rules!

- The fact that the action-angle commutator is not canonical was expected (e.g. see Dirac)
- More precisely, Pauli theorem[a] and its correct forms prevent the corresponding quantum commutator from being exactly canonical.
- At $J \to \infty$ the commutator symbol becomes canonical for $\gamma \neq 2\pi n$, $n \in \mathbb{Z}$. Dirac singularities are located at the discontinuity points of the 2π periodic function $\beth(\gamma)$.
- An interesting exploration concerns the action-angle Heisenberg inequalities $\Delta A_J \Delta A_\gamma \geqslant \frac{1}{2}|\langle J_0, \gamma_0|[A_J, A_\gamma]|J_0, \gamma_0\rangle|$ with dispersions $\Delta(\cdot)$ calculated in CS $|J_0, \gamma_0\rangle$.

[a] E. Galapon, Pauli's theorem and quantum canonical pairs: The consistency of a bounded, self-adjoint time operator canonically conjugate to a Hamiltonian with non-empty point spectrum, Proc. R. Soc. Lond. A, **458** (2002) 451–472



action-angle Heisenberg inequalities

Quantizable distributions

- In most of quantization schemes, e.g. canonical quantization, original function $f(z)$ is forced to be infinitely differentiable functions on $\mathbb{R}^2$, essentially because of the prerequisite Lagrangian and Hamiltonian structures.

- Then how to get through quantization operators like $\Pi_{n,n'} \stackrel{\text{def}}{=} |e_n\rangle\langle e_{n'}|$?

- For that, extend the class of quantizable objects to distributions on $\mathbb{R}^2$ (for canonical coordinates (q,p)) or possibly on $\mathbb{R}^+ \times [0, 2\pi)$ (for polar coordinates (r,θ)).

- Examining matrix elements of A_f:

$$
\begin{aligned}
f \mapsto A_f &= \int_{\mathbb{C}} f(z)\, |z\rangle\langle z| \, \frac{\mathrm{d}^2 z}{\pi} \\
&= \sum_{n,n'=0}^{\infty} |e_n\rangle\langle e_{n'}| \, \frac{1}{\sqrt{n!n'!}} \int_{\mathbb{C}} f(z)\, e^{-|z|^2} z^n \bar{z}^{n'} \, \frac{\mathrm{d}^2 z}{\pi} \stackrel{\text{def}}{=} \sum_{n,n'=0}^{\infty} \left(A_f\right)_{nn'} |e_n\rangle\langle e_{n'}| \,,
\end{aligned} \tag{8}
$$

  lets us think of tempered distributions on the plane as acceptable objects.

- Indeed functions like

$$
\phi_{n,n'}(z) := \langle e_n|z\rangle\langle z|e_{n'}\rangle = \rangle e^{-|z|^2} \, z^n \, \bar{z}^{n'} / \sqrt{n!n'!} \tag{9}
$$

  are rapidly decreasing C^∞ functions on the plane with respect to (q,p), or equivalently with respect to $(z,\bar{z})$: they belong to the Schwartz space $\mathcal{S}(\mathbb{R}^2)$.

Tempered distributions on $\mathbb{C}$

- Any function $f(z)$ which is slowly increasing and locally integrable with respect to the Lebesgue measure d^2z defines a regular tempered distribution T_f, i.e., a continuous linear form on the vector space $\mathcal{S}(\mathbb{R}^2)$ equipped with the usual topology of uniform convergence at each order of partial derivatives multiplied by polynomial of arbitrary degree.

- This definition rests on the map,

$$\mathcal{S}(\mathbb{R}^2) \ni \psi \mapsto \langle T_f, \psi \rangle \stackrel{\text{def}}{=} \int_{\mathbb{C}} f(z)\, \psi(z)\, \mathrm{d}^2 z\,. \tag{10}$$

- For any tempered distribution $T \in \mathcal{S}'(\mathbb{R}^2)$ we define the quantization map $T \mapsto A_T$ as

$$T \mapsto A_T \stackrel{\text{def}}{=} \frac{1}{\pi} \sum_{n,n'=0}^{\infty} \langle T, \phi_{n,n'} \rangle |e_n\rangle\langle e_{n'}|\,, \tag{11}$$

  where the convergence is assumed to hold in a weak sense.

- In the sequel the integral notation will be kept, in the usual abusive manner, for all (tempered or not) distributions T:

$$\int_{\mathbb{C}} T(z)\, |z\rangle\langle z|\, \frac{\mathrm{d}^2 z}{\pi} \stackrel{\text{def}}{=} \frac{1}{\pi} \sum_{n,n'=0}^{\infty} \langle T, \phi_{n,n'} \rangle |e_n\rangle\langle e_{n'}|\,. \tag{12}$$

Quantization of (tempered) distributions

- With $\mathsf{M} \equiv \rho$ a positive unit trace operator, formally define the quantization of a distribution T as

$$T \mapsto A_T = \int_{\mathbb{C}} T(z)\,\rho(z)\,\frac{\mathrm{d}^2 z}{\pi} \stackrel{\text{def}}{=} \frac{1}{\pi}\sum_{n,n'=0}^{\infty} \langle T, \psi_{n,n'}\rangle |e_n\rangle\langle e_{n'}|\,, \tag{13}$$

  where the

$$\psi_{n,n'}(z) := \langle e_n|\rho(z)|e_{n'}\rangle \tag{14}$$

  are assumed to belong to $\mathcal{S}(\mathbb{R}^2)$.

- The resultant lower symbol is

$$\check{T}(z) = \int_{\mathbb{C}} T(z')\,\mathrm{tr}(\rho(z)\rho(z'))\,\frac{\mathrm{d}^2 z'}{\pi} = \frac{1}{\pi}\langle\, T, \mathrm{tr}(\rho(z)\rho(\cdot))\,\rangle\,. \tag{15}$$

- Extended definition of quantizable objects: *Given an acceptable density operator ρ, a distribution $T \in \mathcal{D}'(\mathbb{R}^2)$ is ρ-*quantizable *along the map $T \mapsto A_T$ defined by (13) if the map $\mathbb{C} \ni z = \frac{1}{\sqrt{2}}(q+ip) \sim (q,p) \mapsto \check{T}(z)$ is a smooth (C^∞) function with respect to the (q,p) coordinates of the complex plane.*

Quantization of distributions (continued, 1)

- In the case of CS quantization, above definitions are mathematically justified for all tempered distributions. The following result allows one to extend the set of such acceptable observables.

- *A distribution $T \in \mathcal{D}'(\mathbb{R}^2)$ is* CS quantizable *if there exists $\eta < 1$ such that the product $e^{-\eta |z|^2} T \in \mathcal{S}'(\mathbb{R}^2)$, i.e. is a tempered distribution.*

- In the general ρ case, one expects to have a similar result with suitably chosen weight functions $\varpi(z)$. In the sequel, we suppose that such a choice has been made.

- Therefore, WH integral quantization is extended to locally integrable functions $f(z)$ increasing like $e^{\eta |z|^2} p(z)$ for some $\eta < 1$ and some polynomial p, and, in this way, to distributions.

- Indeed, the latter are characterized as derivatives (in the distributional sense) of such functions. Here we recall here that partial derivatives of distributions are given by

$$\left\langle \frac{\partial^r}{\partial z^r} \frac{\partial^s}{\partial \bar{z}^s} T, \psi \right\rangle = (-1)^{r+s} \left\langle T, \frac{\partial^r}{\partial z^r} \frac{\partial^s}{\partial \bar{z}^s} \psi \right\rangle . \tag{16}$$

  We also recall that the multiplication of distributions T by smooth functions $\alpha(z) \in C^\infty(\mathbb{R}^2)$ is understood through:

$$C^\infty(\mathbb{R}^2) \ni \psi \mapsto \langle \alpha T, \psi \rangle := \langle T, \alpha\, \psi \rangle . \tag{17}$$

Quantization of distributions (continued, 2)

- All compactly supported distributions like Dirac's and its derivatives, are tempered and so are expected to be ρ-quantizable. The Dirac distribution supported by the origin of the complex plane is denoted as usual by δ (and in the present context by $\delta(z)$) :

$$C^\infty(\mathbb{R}^2) \ni \psi \mapsto \langle \delta, \psi \rangle = \int_{\mathbb{C}} \delta(z)\, \psi(z)\, \mathrm{d}^2 z \overset{\text{def}}{=} \psi(0)\,. \tag{18}$$

- As a first example, let us ρ-quantize the Dirac distribution.

$$\int_{\mathbb{C}} \rho(z)\, \pi\delta(z)\, \frac{\mathrm{d}^2 z}{\pi} = \rho(0) \equiv \rho\,. \tag{19}$$

- In particular, in the CS case, we find that the ground state projector is the quantized version of the Dirac distribution supported at the origin of the phase space.

$$A_{\pi\delta} = |e_0\rangle\langle e_0|\,. \tag{20}$$

Quantization of distributions (continued, 3)

- Similarly, the quantization of the Dirac distribution $\delta_{z_0} \equiv \delta(z - z_0)$ at the point z_0 yields the displaced density matrix:

$$A_{\pi\delta_{z_0}} = D(z_0)\rho D^{\dagger}(z_0) = \rho(z_0)\,. \tag{21}$$

- In the CS case, we find the CS projector with parameter z_0:

$$A_{\pi\delta_{z_0}} = D(z_0)|e_0\rangle\langle e_0|D^{\dagger}(z_0) = |z_0\rangle\langle z_0|\,. \tag{22}$$

- Thus, the density matrix ρ, which is, besides the measure ν, the main ingredient of our quantization procedure is precisely the quantized version of the Dirac distribution supported at the origin of the phase space. We have here the key for understanding the deep meaning of this type of quantization: replacing the classical states δ_{z_0}, i.e. the highly abstract points of the phase space, physically unattainable, by a more realistic object, $\rho(z_0)$, a kind of "inverted glasses" chosen by us, whose the probabilistic content takes into account the measurement limitations of any localization apparatus. The operator ρ can be viewed as a probe whose the displaced versions give a quantum portrait of the euclidean plane.

- The obtention of all possible projections $\Pi_{nn} = |e_n\rangle\langle e_n|$ or even all possible simple operators $\Pi_{nn'} = |e_n\rangle\langle e_{n'}|$ is based on the quantization of partial derivatives of the δ distribution.