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Coherent states, POVM, quantization and measurement contd.

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Coherent states, POVM, quantization and measurement

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Contents

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1. Other example of integral quantization: with Pöschl-Teller coherent states

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Play it again with infinite wells?

- The amount of recent works on quantum dots and quantum wells in nanophysics [1] strongly motivates construction of quantum states for infinite wells with localization properties comparable to those of Schrödinger states.
- Infinite wells are often modeled by Pöschl-Teller (also known as trigonometric Rosen-Morse) confining potentials [2, 3] used e.g. in quantum optics [4, 5].
- The infinite square well is a limit case of this family referred to in what follows as $\mathcal{T}$-potentials.
- The question is to find a family of normalized states:
  - (a) phase-space labelled,
  - (b) yielding a resolution of the identity, <u>and</u> the latter holding with respect to the usual uniform measure,
  - (c) allowing a reasonable classical-quantum correspondence (CS quantization)
  - (d) and exhibiting semi-classical phase space properties with respect to $\mathcal{T}$-Hamiltonian time evolution.

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Pöschl-Teller ($\mathcal{T}$) potentials

- $\mathcal{T}$-potentials belong to the class of *shape invariant*[a] potentials intensively studied within the framework of supersymmetric quantum mechanics (SUSYQM)[1, 2, 3, 4, 5, 6, 7, 8].
- Various semi-classical states adapted to $\mathcal{T}$-potentials have been proposed in previous works [9, 10, 12, 13]. However, they do not verify simultaneously (a), (b), (c), and (d).
- Moreover, correspondence between classical and quantum momenta requires a thorough analysis since there exists well-known ambiguity in the definition of a quantum momentum operator [14, 10]. This is due to the confinement of the system in an interval, unlike the harmonic oscillator case.
- The construction of coherent states for $\mathcal{T}$-potentials presented here is based on a general approach given by Bergeron and Valance [15].
- Classical-quantum correspondence based on these states ("CS quantization") show satisfying comparison with the Schrödinger CS in terms of semi-classical time behavior.

[a] for which it is possible to construct a super-family whose members have the same functional form

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Stationary Schrödinger equation with $\mathcal{T}$-potential

- Let us consider the motion of a particle confined in the interval $[0, L]$ and submitted to the repulsive symmetric $\mathcal{T}$-potential

$$V_\nu(x) = \mathcal{E}_0 \frac{\nu(\nu+1)}{\sin^2 \frac{\pi}{L}x}\,,$$

- $\nu \geq 0$: dimensionless parameter. Limit $\nu \to 0$ corresponds to the infinite square well. Factor $\mathcal{E}_0 = \hbar^2\pi^2(2mL^2)^{-1} \geqslant 0$ is chosen as the ground state energy of the infinite square well.

- Quantum Hamiltonian acts in the Hilbert space $\mathcal{H} = L^2([0, L], dx)$ as:

$$\mathbf{H}_\nu = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_\nu(x)\,.$$

Solutions to the eigenenergy problem

- The eigenvalues $E_{n,\nu}$ and corresponding eigenstates $| \phi_{n,\nu} >$ of $\mathbf{H}_\nu$ read as

$$E_n = \mathcal{E}_0 (n+\nu+1)^2, \quad n = 0, 1, 2...,$$

$$\phi_n(x) = Z_n \sin^{\nu+1}\left(\frac{\pi}{L}x\right) \mathrm{C}_n^{\nu+1}\left(\cos\frac{\pi}{L}x\right)$$

- $C_n^{\nu+1}$ is a Gegenbauer polynomial and the normalization constant reads as:

$$Z_n = \Gamma(\nu+1)\frac{2^{\nu+1/2}}{\sqrt{L}}\sqrt{\frac{n!(n+\nu+1)}{\Gamma(n+2\nu+2)}}$$

- Eigenfunctions ϕ_n obey the Dirichlet boundary conditions $\phi_n(0) = \phi_n(L) = 0$. A detailed mathematical discussion on the boundary conditions and self-adjoint extensions for the $\mathcal{T}$-Hamiltonian can be found in [F. Gesztesy and W. Kirsch, Journal für die reine und angewandte Mathematik $1985(1985)28$] and [10].

- In particular, the ground state eigenfunction ϕ_0 is $Z_0 \sin^{\nu+1}\frac{\pi}{L}x$ and the eigenfunctions for the infinite square well ($\nu = 0$) reduce to $\sqrt{\frac{2}{L}}\sin\frac{(n+1)\pi}{L}x$.

Supersymmetric quantum mechanics content

- Superpotential $W_\nu(x)$:

$$W_\nu(x) \stackrel{\text{def}}{=} -\hbar\frac{\phi_0'(x)}{\phi_0(x)} = -\frac{\hbar\pi}{L}(\nu+1)\cot\frac{\pi}{L}x$$

- Lowering and raising operators:

$$\mathbf{A}_\nu \stackrel{\text{def}}{=} W_\nu(x) + \hbar\frac{d}{dx} \text{ and } \mathbf{A}_\nu^\dagger \stackrel{\text{def}}{=} W_\nu(x) - \hbar\frac{d}{dx}$$

  Darboux factorisation of $\mathcal{T}$-Hamiltonian $\mathbf{H}_\nu$ can be rewritten in terms of these operators:

$$\mathbf{H}_\nu = \frac{1}{2m}\mathbf{A}_\nu^\dagger\mathbf{A}_\nu + E_0.$$

- Supersymmetric partner $\mathbf{H}_\nu^{(S)}$:

$$\mathbf{H}_\nu^{(S)} = \frac{1}{2m}\mathbf{A}_\nu\mathbf{A}_\nu^\dagger + E_0.$$

- It coincides with the original Hamiltonian with increased ν: $\mathbf{H}_\nu^{(S)} = \mathbf{H}_{\nu+1}$.

$\mathcal{T}$ coherent states

- Classical phase space for the motion in a $\mathcal{T}$-potential is defined as the infinite band in the plane: $\mathcal{K} = \{(q,p)\,|\,q \in [0,L] \text{ and } p \in \mathbb{R}\}$.

- Introduce operators $\mathbf{Q} : \psi(x) \mapsto x\psi(x)$ and $\mathbf{P} : \psi \mapsto -i\hbar \frac{d}{dx}\psi(x)$.

- $\mathcal{T}$ coherent states $|\eta_{q,p}\rangle$ are defined as normalized eigenvectors of $\mathbf{A}_\nu = W_\nu(\mathbf{Q}) + i\mathbf{P}$ with eigenvalue $W_\nu(q) + ip$:

$$| \, \eta_{q,p} >= N_\nu(q) \left| \xi^{[\nu]}_{W(q)+ip} \right\rangle \,, \ (q,p) \in \mathcal{K} \,,$$

  where $\xi_z(x) = e^{zx/\hbar} \sin^{\nu+1}\left(\frac{\pi}{L}x\right)$ for $x \in [0,L]$.

- Normalization coefficient $N_\nu(q)$:

$$N_\nu(q) = \frac{2^{\nu+1}|\Gamma(\nu+2-i(\nu+1)\cot\frac{\pi}{L}q)|}{\sqrt{L}\sqrt{\Gamma(2\nu+3)}} \times \exp\left[\frac{\pi}{2}(\nu+1)\cot\frac{\pi}{L}q\right].$$

- Function $x \mapsto |\eta_{q,p}(x)|$ reaches its maximal value for $x = q$ and $\langle \mathbf{P} \rangle_{p,q} = p$.

- Uncertainty relation $\Delta W_\nu(\mathbf{Q})\Delta\mathbf{P} \geqslant \frac{\hbar}{2}\langle W'_\nu(\mathbf{Q})\rangle$ is minimized by these CS.

$\mathcal{T}$ CS Quantization and expected values

- $\mathcal{T}$ CS's resolve of identity with respect to the uniform measure on the phase space $\mathcal{K}$:

$$\int_{\mathcal{K}} \mid \eta_{q,p} >< \eta_{q,p} \mid \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi\hbar} = \mathbb{I}\,.$$

- This renders possible CS quantization of "classical observables" $f(q,p)$ through the correspondence [Klauder,Berezin]

$$f(q,p) \to \mathbf{F} = \int_{\mathcal{K}} f(q,p) \mid \eta_{q,p} >< \eta_{q,p} \mid \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi\hbar}\,.$$

- Remind that this operator-valued integral is understood as the sesquilinear form,

$$B_f(\psi_1,\psi_2) = \int_{\mathcal{K}} \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi\hbar} f(q,p) \langle\,\psi_1\,|\,\eta_{q,p}\,\rangle\langle\,\eta_{q,p}\,|\,\psi_2\,\rangle.$$

- *The form B_f is assumed to be defined on a dense subspace of the Hilbert space. If f is real and at least semi-bounded, the Friedrich's extension of B_f univocally defines a self-adjoint operator. However, if f is not semi-bounded, there is no natural choice of a self-adjoint operator associated with B_f. In this case, we can consider directly the symmetric operator* $\mathbf{F}$ *enabling us to obtain a self-adjoint extension (unique for particular operators)*

Some $\mathcal{T}$ CS quantized classical observables

Name	f	$\mathbf{A}_f$	Operator action	Properties
Position	q	$F(\mathbf{Q})$ (*)	multiplication	bounded self-adjoint
Superpotential	$W_\nu(q)$	$W_\nu(\mathbf{Q})$	multiplication	unbounded self-adjoint
Potential	$\dfrac{1}{\sin^2 \pi q/L}$	$\dfrac{(2\nu+3)(2\nu+2)^{-1}}{\sin^2 \pi\mathbf{Q}/L}$	multiplication	unbounded self-adjoint
"Momentum"	p	$\mathbf{P}$	$\mathbf{P}\phi_n = -i\hbar\phi'_n$	unbounded symmetric
Hamiltonian	$\dfrac{p^2}{2m} + \dfrac{2\nu-1}{2\nu+3}\dfrac{\mathcal{E}_0(\nu+1)^2}{\sin^2 \pi q/L}$	$\mathbf{H}_\nu$	Schrödinger operator	semi-bounded self-adjoint $(\nu \geqslant 1/2)$

$$(*) F(x) = \sin^{2\nu+2}(\pi x/L) \int_0^L \mathrm{d}q\, q N_\nu^2(q) \exp(2W_\nu(q)x/L).$$

“Lower” (or “covariant”) symbols of operators

They are the expectation values of operators in the $\mathcal{T}$ CS.

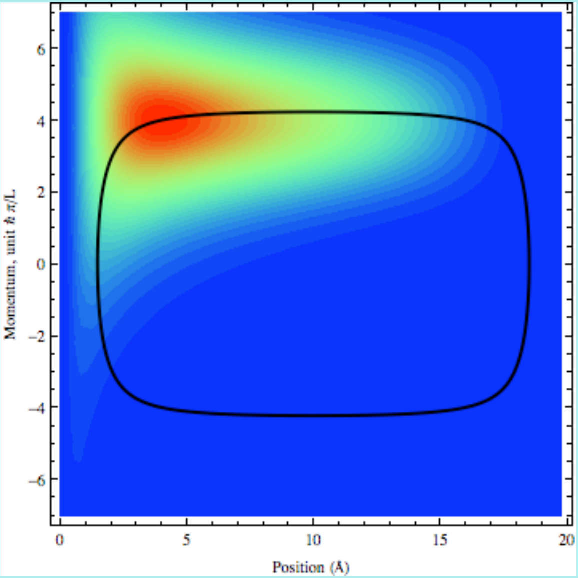
Name	$\mathbf{A}$	f
Position	$\mathbf{Q}$	$N_\nu^2(q) \times \int_0^L \mathrm{d}x\, x \sin^{2\nu+2} \frac{\pi x}{L} e^{\frac{2W_\nu(q)x}{L}}$
“Momentum” (*)	$\mathbf{P}$	p
Superpotential	$W_\nu(\mathbf{Q})$	$W_\nu(q)$
Potential	$\frac{1}{\sin^2 \pi\mathbf{Q}/L}$	$\frac{2\nu+2}{2\nu+1}\frac{1}{\sin^2 \pi q/L}$
Kinetic energy	$\frac{\mathbf{P}^2}{2m}$	$\frac{p^2}{2m} + \frac{1}{2\nu+1}\frac{\mathcal{E}_0(\nu+1)^2}{\sin^2 \pi q/L}$

(*) The operator $\mathbf{P}$ is the one given in table 1.

Semi-classical behavior

For any normalized state $\phi \in \mathcal{H} = L^2([0, L], \mathrm{d}x)$, the resolution of identity allow us to build a probability distribution on the phase space $\mathcal{K}$:

$$\mathcal{K} \ni (q, p) \mapsto \frac{1}{2\pi\hbar} |\langle \eta_{q,p} | \phi \rangle|^2 = \rho_\phi(q, p)$$



Phase space localization distribution for $\nu = 0$ of the state η_{q_0,p_0} with $q_0 = L/5$, $p_0 = 4\pi\hbar/L$ and $L = 20\text{Å}$. The thick curve is the expected trajectory in the infinite square well, deduced from the semi-classical hamiltonian . The particle is an electron, its mean energy is $E = 1.6\,\mathrm{eV}$. Increasing values of the function are encoded by the colors from blue to red (this should be compared with the Gaussian

Time behavior

- Time behavior $t \mapsto \rho_{\phi(t)}(q,p)$ for a state $\phi(t)$ evolving under the action of the infinite square well Hamiltonian $\mathbf{H}_0$:

$$\mid \phi(t) >= e^{-i\mathbf{H_0}t/\hbar} \mid \phi >= \sum_{n=0}^{\infty} e^{-i\mathcal{E}_0(n+1)^2 t/\hbar} \langle\, \phi_{n,0} \,|\, \phi \,\rangle \mid \phi_{n,0} >$$

where, e.g., $\phi_{n,0} \equiv \sqrt{\frac{2}{L}} \sin \frac{(n+1)\pi}{L} x$.

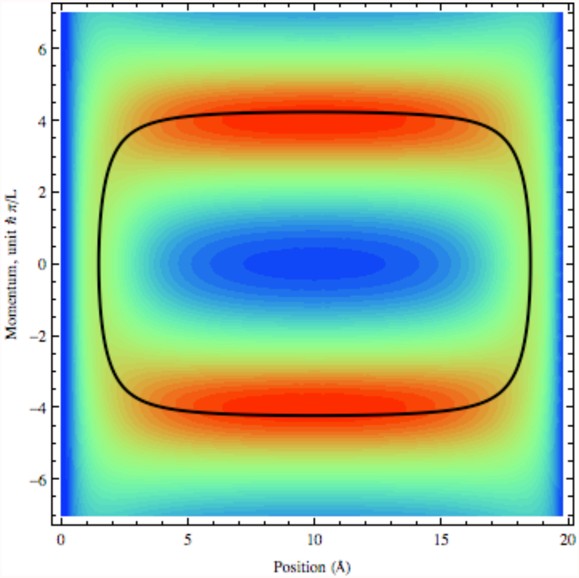
- With $\phi = \eta_{q_0,p_0}$ as an initial state, we have for a given ν

$$\langle\, \phi(t) \,|\, \mathbf{H}_0 | \phi(t) \,\rangle = \frac{p_0^2}{2m} + \frac{1}{2\nu+1} \frac{\mathcal{E}_0(\nu+1)^2}{\sin^2 \frac{\pi}{L} q_0}.$$

- Since the lower symbols of $W_\nu(\mathbf{Q})$ and $\mathbf{P}$ correspond to their classical original functions $W_\nu(q)$ and p, one can expect that the time average of the probability law $\rho_{\eta_{q_0,p_0}(t)}(p,q)$ corresponds to some fuzzy extension in phase space of the classical trajectory corresponding to the time-independent Hamiltonian $\frac{p^2}{2m} + \dfrac{1}{2\nu+1} \frac{\mathcal{E}_0(\nu+1)^2}{\sin^2 \pi q/L}$ which is the lower symbol of the kinetic energy.

Time average distribution

The time average distribution $\bar{\rho}$ is defined as $\bar{\rho}(q,p) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \rho_{\eta_{q_0,p_0}(t)}(q,p)\,\mathrm{d}t$,



Time average of the phase space representation of $\eta_{q_0,p_0}(t)$ evolving under the Hamiltonian of the infinite square well. The values of parameters and the thick curve are those of the first figure. Increasing values of the function are encoded by the colors from blue to red.

The time average distribution $\bar{\rho}$ allows us to compare the quantum behavior with the classical trajectory, but its expression hides the complex details of the wave-packet dynamics CSPT time evolution. The latter exhibits a splitting of the initial wave-packet into secondary ones during the sharp reflection phase, each of them following the classical trajectory, before they amalgamate to reconstitute a unique packet (revival time). This important point makes the difference with the time behavior of the Schrödinger states for the harmonic oscillator.

Conclusion(s)

- The presented family of CS's for the Pöschl-Teller potentials sets a natural bridge between the phase space and its quantum counterpart.
- These CS's share with the Shrödinger ones some of their most striking properties, e.g. resolution of identity with uniform measure, saturation of uncertainty inequalities.
- They also possess remarkable evolution stability features (not to be confused with CS temporal stability in the sense of Klauder corresponding to the time parametric evolution): their time evolution generated by $\mathbf{H}_\nu$ is localized on the classical phase space trajectory.
- The approach developed in this paper can be easily extended to higher dimensional bounded domains, provided that the latter be symmetric enough (e.g. square, equilateral triangle, etc) to allow shape invariance integrability.

2. Affine quantization

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[1] H Bergeron, A Dapor, J-P G and P Małkiewicz, *Wavelet Quantum Cosmology* (2013); arXiv:1305.0653 [gr-qc]

[2] H Bergeron, A Dapor, J-P G and P Małkiewicz, *Towards singularity-free cosmology: coherent state quantization* (2013)

Affine or Wavelet Quantization

- Set X is the upper half-plane $\Pi_+ := \{(q,p)\,|\,p \in \mathbb{R}\,,\ q > 0\}$ equipped with measure $\mathrm{d}q\,\mathrm{d}p$. It is the phase space for the motion on the half-line.
- Equipped with the multiplication $(q,p)(q_0,p_0) = (qq_0, p_0/q + p)$, $q \in \mathbb{R}_+^*$, $p \in \mathbb{R}$, X is viewed as the affine group $\mathrm{Aff}_+(\mathbb{R})$ of the real line.
- $\mathrm{Aff}_+(\mathbb{R})$ has two non-equivalent UIR, $U_\pm$. Both are square integrable $\Rightarrow$ *continuous wavelet analysis*.
- $U_+ \equiv U$ carried on by Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+^*, dx)$:

$$U(q,p)\psi(x) = (e^{ipx}/\sqrt{q})\psi(x/q)\,.$$

- unit-norm state $\psi \in L^2(\mathbb{R}_+^\dagger,\ \mathrm{d}x) \cap L^2(\mathbb{R}_+^\dagger,\ \mathrm{d}x/x)$ ("fiducial vector") produces all wavelet $\Leftrightarrow$ CS defined as $|q,p\rangle = U(q,p)|\psi\rangle$ and yielding the crucial

$$\int_{\Pi_+} |q,p\rangle\langle q,p|\,\frac{\mathrm{d}q\,\mathrm{d}p}{2\pi c_{-1}} = I\,,\ c_\gamma := \int_0^\infty \mathrm{d}x|\psi(x)|^2/x^{2+\gamma}\,.$$

Wavelet Quantization continued

- Covariant quantization from resolution of the identity[a]

$$f \;\mapsto\; A_f = \int_{\Pi_+} f(q,p)|q,p\rangle\langle q,p|\, \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi c_{-1}}$$

- Quantization is canonical (up to a multiplicative constant) for q and p:

$$A_p = P = -i\partial/\partial x\,,\; A_{q^\beta} = (c_{\beta-1}/c_{-1})\, Q^\beta\,,\; Qf(x) = xf(x)\,,$$

- Quantization of kinetic energy:

$$A_{p^2} = P^2 + KQ^{-2}\,, \quad K = K(\psi) = \int_0^\infty (\psi'(u))^2\, \frac{u\,\mathrm{d}u}{c_{-1}}$$

  Thus wavelet quantization forbids a quantum free particle moving on the positive line to reach the origin.

- Operator $P^2 = -d^2/dx^2$ alone in $L^2(\mathbb{R}_+^*, \mathrm{d}x)$ is not essentially self-adjoint whereas the above regularized operator, defined on the domain of smooth compactly supported functions, is for $K \geqslant 3/4$[b]. Then quantum dynamics of the free motion is possible.

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[a] Proceeding in quantum theory with an "affine" quantization instead of the Weyl-Heisenberg quantization was already present in Klauder's work devoted the question of dealing with singularities in quantum gravity (see An Affinity for Affine Quantum Gravity, *Proc. Steklov Inst. of Math.* **272**, 169-176 (2011); gr-qc/1003.261 for recent references). The procedure rests on the representation of the affine Lie algebra. In this sense, it remains closer to the canonical one and it is not of the integral type.

[b] Reed M. and Simon B., Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness Volume 2 Academic Press, New York, 1975

Semi-classical aspects in phase space

- Quantum states and their dynamics have phase space representation through wavelet symbols. For state $|\phi\rangle$:

$$\Phi(q,p) = \langle q,p|\phi\rangle/\sqrt{2\pi}$$

- Associated probability distribution on phase space:

$$\rho_\phi(q,p) = \frac{1}{2\pi c_{-1}}|\langle q,p|\phi\rangle|^2$$

- With (energy) eigenstates of some quantum Hamiltonian H at our disposal, we can compute the time evolution

$$\rho_\phi(q,p,t) := \frac{1}{2\pi c_{-1}}|\langle q,p|e^{-iH}|\phi\rangle|^2$$

for any state ϕ.

Wavelet Quantization for FLRW Quantum Cosmology

- FLRW models filled with barotropic fluid with equation of state $\mathrm{p} = w\rho$ and resolving Hamiltonian constraint leads to a model of singular universe $\sim$ particle moving on the half-line $(0, \infty)$ with Hamiltonian.

$$\{q, p\} = 1, \;\; h(q,p) = \alpha(w)p^2 + 6\tilde{k}q^{\beta(w)}, \;\; q > 0\,.$$

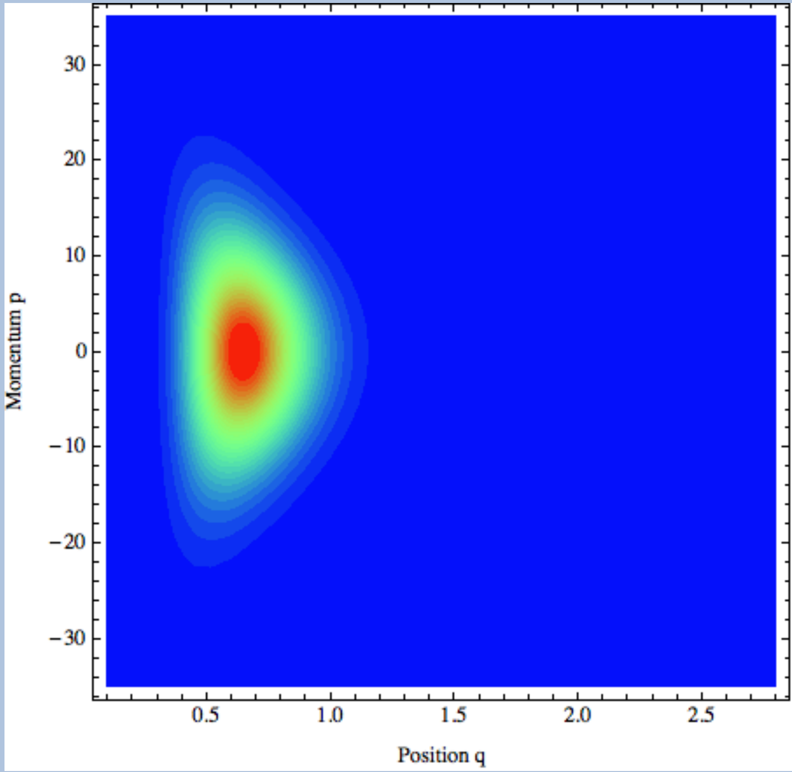
with $\tilde{k} = (\int \mathrm{d}\omega)^{2/3}k$, $\alpha(w) = 3(1-w)^2/32$ and $\beta(w) = 2(3w+1)/(3(1-w))$. $k = 0, -1$ or 1 (in suitable unit of inverse area) depending on whether the universe is flat, open or closed.

- Assume a closed universe with radiation content : $w = 1/3$ and $k = +1$. Affine quantization with a fiducial vector like $\psi(x) \propto \exp(-(\alpha(\nu)x + \beta(\nu)/x)$, whith parameter $\nu > 0$, on $\mathbb{R}_+^*$ yields the quantum Hamiltonian

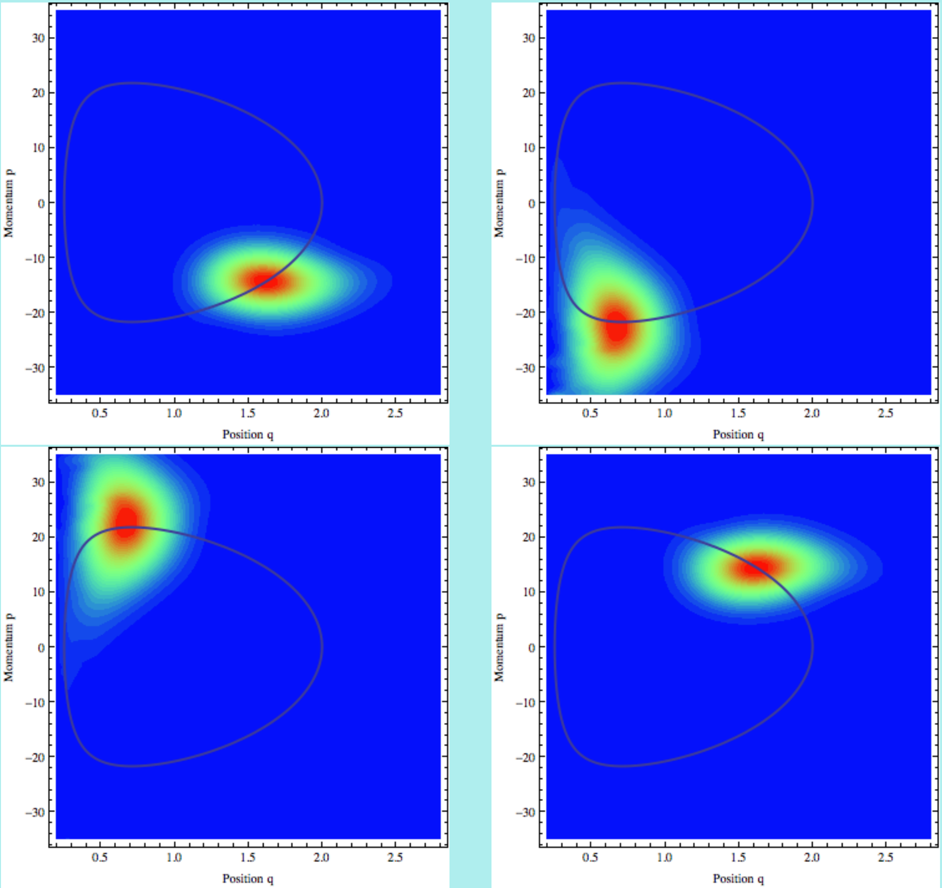
$$A_h = H = \frac{1}{24}P^2 + \frac{a_P^2 K(\nu)}{24}\frac{1}{Q^2} + 6\frac{a_P^2}{\sigma^2}\frac{c_1}{c_{-1}}Q^2\,,$$

a_P is a Planck area.

- For $K(\nu) \geqslant 3/4$ wavelet quantization removes quantum singularity and well-defined quantum evolution exists, at the difference with canonical quantization



Phase space distribution of the ground state with a certain choice of ν. $a_P = 1$. This stationary quantum state of the universe is distributed around the equilibrium point q_e (minimum of the potential curve involved in the Hamiltonian). The existence of the semi-classical equilibrium point $q_e \neq 0$ is a consequence of the repulsive part of the potential.



Phase space distribution $\rho_{q_0,p_0,t}(q,p)$ for some selected values of time t. (Fluid configuration variable is chosen as a clock of universe). The thick curve is the phase trajectory obtained from the effective dynamics.

A "semiclassical" Friedmann equation

- In general lower symbol $\check{f}(q,p)$ differs from its classical counterpart $f(q,p)$: it is a quantum-corrected effective observable.
- Thus, computing lower symbol of Hamiltonian leads to the semiclassical Friedmann equation for scale factor $a(t)$:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} + c^2 a_P^2 (1-w)^2 \frac{\nu}{128}\frac{1}{V^2} = \frac{8\pi G}{3c^2}\rho$$

- Note that the repulsive potential depends explicitly on volume. This excludes non-compact universes from quantum modeling.
- Singularity resolution is confirmed: as the singular geometry is approached ($a \to 0$), the repulsive potential grows faster ($\sim a^{-6}$) than the density of fluid ($\sim a^{-3(1+w)}$) and therefore at some point the two terms become equal and the contraction is brought to a halt.
- **The form of the repulsive potential does not depend on the state of fluid filling the universe: the origin of singularity avoidance is quantum geometrical.**

3. Covariant integral quantizations

Covariant integral quantization with UIR of a group

- Let G be a Lie group with left Haar measure $\mathrm{d}\mu(g)$, and let $g \mapsto U(g)$ be a unitary irreducible representation (UIR) of G in a Hilbert space $\mathfrak{H}$.

- Let M be a bounded operator on $\mathfrak{H}$. Suppose that the operator

$$R := \int_G \mathsf{M}(g)\, \mathrm{d}\mu(g)\,, \quad \mathsf{M}(g) := U(g)\, \mathsf{M}\, U^\dagger(g)\,, \tag{1}$$

  is defined in a weak sense. From the left invariance of $\mathrm{d}\mu(g)$ we have $U(g_0)\, R\, U^\dagger(g_0) = \int_G \mathrm{d}\mu(g)\, \mathsf{M}(g_0 g) = R$ and so R commutes with all operators $U(g)$, $g \in G$. Thus, from Schur's Lemma, $R = c_{\mathsf{M}} I$ with

$$c_{\mathsf{M}} = \int_G \mathrm{tr}\left(\rho_0\, \mathsf{M}(g)\right)\, \mathrm{d}\mu(g)\,, \tag{2}$$

  where the unit trace positive operator ρ_0 is chosen in order to make the integral convergent.

- Resolution of the identity follows:

$$\int_G \mathsf{M}(g)\, \mathrm{d}\nu(g) = I\,, \quad \mathrm{d}\nu(g) := \mathrm{d}\mu(g)/c_{\mathsf{M}}\,. \tag{3}$$

Covariant integral quantization: with square integrable UIR (e.g. affine group)

- For square-integrable UIR U for which $|\eta\rangle$ is an admissible unit vector, i.e. $c(\eta) := \int_G \mathrm{d}\mu(g)\, |\langle\eta|U(g)\eta\rangle|^2 < \infty$.
- Resolution of the identity is obeyed by *coherent states* for G:

$$|\eta_g\rangle = U(g)|\eta\rangle \quad \text{or by} \quad |\eta_g\rangle\langle\eta_g| = \rho(g)\,, \quad \rho := |\eta\rangle\langle\eta|$$

- This allows *covariant* integral quantization of complex-valued functions on the group $f \mapsto A_f = \int_G \rho(g)\, f(g)\, \mathrm{d}\nu(g)$:

$$U(g)A_f U^\dagger(g) = A_{U_r(g)f}\,, \tag{4}$$

  With $f \in L^2(G, \mathrm{d}\mu(g))$, $(U_r(g)f)(g') := f(g^{-1}g')$ is the regular representation.

- Generalization of the Berezin or heat kernel transform on G: $\check{f}(g) := \int_G \mathrm{tr}(\rho(g)\,\rho(g'))\, f(g')\, \mathrm{d}\nu(g')$.

Covariant quantization with UIR square integrable w.r.t. a subgroup (e.g. Weyl Heisenberg group)

- In the absence of square-integrability over G, there exists a definition of square-integrable covariant coherent states with respect to a left coset manifold $X = G/H$, with H a closed subgroup of G, equipped with a quasi-invariant measure ν.[a]

[a] S. T. Ali, J.-P. Antoine, and J.-P. G., *Coherent States, Wavelets and their Generalizations* (Graduate Texts in Mathematics, Springer, New York, 2000). New edition in 2014

4. Conclusion

Beyond the freedom (think to analogy with Signal Analysis where different techniques are complementary) allowed by integral quantization, the advantages of the method with regard to other quantization procedures in use are of four types.

(i) The minimal amount of constraints imposed to the classical objects to be quantized.

(ii) Once a choice of (positive) operator-valued measure has been made, which must be consistent with experiment, there is no ambiguity in the issue, contrarily to other method(s) in use (think in particular to the ordering problem). To one classical object corresponds one and only one quantum object. Of course different choices are requested to be physically equivalent

(iii) The method produces in essence a regularizing effect, at the exception of certain choices, like the Weyl-Wigner integral quantization.

(iv) The method, through POVM choices, offers the possibility to keep a full probabilistic content. As a matter of fact, the Weyl-Wigner integral quantization does not rest on a POVM.

- But what is the real meaning of that freedom granted to us in the choice of POVM or others?
- Such a freedom is governed by our degree of confidence in localizing a pure classical state (q, p) in phase space. The latter is usually viewed as an ideal continuous manifold where all points are physically accessible. As everybody knows, such a view is physically untenable ...
- However, and this is the paradoxical paradigm of contemporary physics, one needs such a leibnizian mathematical ideality (*natura non saltum facit*) to build a more realistic, though more highly mathematical, representation of the physical world.

5. *In complement, as a working example: coherent states for motion on the circle*

5.1. Action & Angle in Classical Mechanics

Action-angle variables[a]

- Consider a conservative one-degree of freedom confined mechanical system with phase space conjugate variables (q, p). For a given motion its Hamiltonian function is fixed to a certain value E of the energy:

$$H(q,p) = E \Rightarrow p = p(q,E)$$

- Action variable

$$J = \oint p(q,E)\, dq = J(E)\,,$$

  where the loop integral is understood as performed over a complete period of libration or rotation.

- Conjugate angle variable from $W = W(J,q) = \int p\, dq$ (*Hamilton characteristic function*) which generates contact transformation $(q,p) \mapsto (J,\gamma)$ at constant Hamiltonian, where

$$\gamma = \frac{\partial W}{\partial J}$$

  with time evolution

$$\gamma = \frac{t}{\tau(E)} + \gamma_0,\,, \quad \tau = \frac{\partial J}{\partial E} = \tau(E)\,.$$

[a] L. Landau et E. Lifchitz, Mechanics, Pergamon, Chapter 7; H. Goldstein, Classical Mechanics, Addison Wesley, Chapter 10

Two extreme cases: free rotator and harmonic oscillator

- **Free rotator**: particle, mass m, freely moving on circle, radius l. Canonical coordinates $(\theta = 2\pi\gamma, p_\theta = ml^2\dot{\theta} = J/2\pi)$, energy $E = \frac{p_\theta^2}{2ml^2}$
- **Harmonic Oscillator**: from $E = \frac{p^2}{2m} + \frac{1}{2}kq^2$,

$$J = \frac{2\pi}{\omega} E\,, \quad \gamma = \pm\frac{\phi}{2\pi} + \gamma_0\,, \quad q = q_{\max} \sin\phi\,.$$

- Prototype of all intermediate sytems: *simple pendulum*

5.2. One typical intermediate case: simple pendulum

Example 3: simple pendulum

- We now consider the "simple pendulum", i.e. a point particle of mass m moving on a vertical circle of radius l and submitted to the potential $V(\theta) = -mg\cos\theta$ where θ is the position angle measured from the lowest position. This problem is pedagogically interesting since it represents a link between the two extreme situations exposed above, pure rotor (at $g = 0$ and harmonic oscillations for small θ).

- The natural canonical coordinates are (p_θ, θ) where $p_\theta = ml^2\dot{\theta}$ is the angular momentum with the dimension of an action. But (p_θ, θ) should not be confounded with the action-angle variables.

- The Hamiltonian reads

$$H = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta\,,$$

  and is conserved: $H = E$

- So the variable p_θ is given in terms of E by $p_\theta = \pm\sqrt{2ml^2}\sqrt{E + mgl\cos\theta}$, the sign ambiguity $\pm$ resulting from the possibility to have clockwise or anticlockwise angular velocity $\dot{\theta}$ on the circle.

Example 3: simple pendulum (continued)

- For the simple pendulum we have to distinguish between 3 regimes:
  - (i) rotation at large enough energy $E > mgl$,
  - (ii) bifurcation separatrix at $E = mgl$,
  - (iii) libration at small enough energy $E < mgl$.
- In view of shortening notations three characteristic frequencies (two of them are energy dependent), an energy-dependent characteristic action, and an energy-dependent characteristic ratio, are introduced:
  - (i) $\omega_r = \omega_r(E) \stackrel{\text{def}}{=} \sqrt{\dfrac{2E}{ml^2}}$: modulus of angular velocity for $E > 0$ at $\theta = \pi/2$,
  - (ii) $\omega_l \stackrel{\text{def}}{=} \sqrt{\dfrac{g}{l}}$, frequency of harmonic small oscillations,
  - (iii) $\omega_0 = \omega_0(E) \stackrel{\text{def}}{=} \sqrt{\omega_r^2 + 2\omega_l^2}$, a kind of quadratic average of the two previous ones,
  - (iv) $\vartheta_0 = \vartheta_0(E) \stackrel{\text{def}}{=} ml^2\omega_0$, which provides a characteristic (energy-dependent) scale for action quantities involved in the model,
  - (v) $k = k(E) \stackrel{\text{def}}{=} 2\dfrac{\omega_l}{\omega_0}$ (also denoted by m by certain authors[a]), the *modulus*[b] of the involved elliptic integrals (see next slide)

[a] Abramowitz, M. and Stegun, I. A. (Eds.) Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, pp. 721-746, 1972.

[b] Magnus, W., Oberhettinger, F., and Soni, R. P., *Formulas and Theorems for the Special Functions of Mathematical Physics*. Springer-Verlag, Berlin, Heidelberg and New York, 1966.

Example 3: simple pendulum (continued)

- The Hamilton characteristic function is given by

$$W = \int_0^\theta p_\theta \, d\theta' + W_0 = \pm\vartheta_0 \int_0^\theta \sqrt{1 - k^2 \sin^2 \frac{\theta'}{2}} \, d\theta' = \pm\vartheta_0 \mathrm{E}_{e\ell\ell}\left(k, \frac{\theta}{2}\right) + W_0 \,,$$

where $\mathrm{E}_{e\ell\ell}(k, \varphi)$ is the *elliptic normal integral of the second kind*[a] (normal Legendre form).

- For the action variable $J = \oint p_\theta \, d\theta$ we have to distinguish between two cases.

(i) Rotation:

$$\begin{aligned} J &= \pm\vartheta_0 \int_0^{2\pi} \sqrt{1 - k^2 \sin^2 \frac{\theta}{2}} \, d\theta = \pm 2\vartheta_0 \int_0^{\pi} \sqrt{1 - k^2 \sin^2 \frac{\theta}{2}} \, d\theta \\ &= \pm 4\vartheta_0 \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 t} \, dt = \pm 4\vartheta_0 \, \mathrm{E}_{e\ell\ell}(k) \,, \end{aligned}$$

where $\mathrm{E}_{e\ell\ell}(k)$ is the *complete elliptic normal integral of the second kind*.

(ii) Libration:

$$J = 4\vartheta_0 \int_0^{2 \arcsin \frac{1}{k}} \sqrt{1 - k^2 \sin^2 \frac{\theta}{2}} \, d\theta = 8\vartheta_0 \, \mathrm{E}_{e\ell\ell}(k, \arcsin \frac{1}{k}) \,.$$

[a] Magnus, W., Oberhettinger, F., and Soni, R. P., *Formulas and Theorems for the Special Functions of Mathematical Physics*. Springer-Verlag, Berlin, Heidelberg and New York, 1966.

Example 3: simple pendulum (continued)

- The angle variable, conjugate to the action variable J, can be directly calculated from $p_\theta = ml^2\dot\theta = \pm\vartheta_0\sqrt{1-k^2\sin^2\frac{\theta}{2}}$:

$$\omega_0\,(t-t_0) = \int_0^\theta \frac{d\theta'}{\sqrt{1-k^2\sin^2\frac{\theta'}{2}}} = 2\mathrm{F}_{\ell\ell}\left(k,\frac{\theta}{2}\right) \equiv 2u\left(k,\frac{\theta}{2}\right)\,,$$

where $\mathrm{F}_{\ell\ell}(k,\varphi)$ is the *elliptic normal integral of the first kind*, and θ is given in terms of u through the elliptic function $\mathrm{sn}(u,k) = \sin\left(\frac{\theta}{2}\right) = \sin\mathrm{am}(u,k)$.

- For the period $\tau = \tau(E)$ and frequency $\nu = 1/\tau = \nu(E)$ of the motion we have to distinguish between two cases.

(i) Rotation ($E > mgl \Leftrightarrow |J| > J_{\mathrm{crit}} \stackrel{\mathrm{def}}{=} 8ml^{3/2}\sqrt{g}$):

$$\tau = \frac{1}{\omega_0}\int_0^{2\pi}\frac{d\theta}{\sqrt{1-k^2\sin^2\frac{\theta}{2}}} = \frac{2}{\omega_0}\int_0^{\pi}\frac{d\theta}{\sqrt{1-k^2\sin^2\frac{\theta}{2}}} = \frac{4}{\omega_0}\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{1-k^2\sin^2 t}} = \frac{4}{\omega_0}\mathrm{K}_{\ell\ell}(k)\,,$$

where $\mathrm{K}_{\ell\ell}(k) \equiv \mathrm{F}_{\ell\ell}\left(k,\frac{\pi}{2}\right)$ is the *complete elliptic normal integral of the first kind*.

(ii) Libration ($0 \leqslant E < mgl \Leftrightarrow 0 \leqslant J < J_{\mathrm{crit}}$):

$$\tau = \frac{4}{\omega_0}\int_0^{2\arcsin\frac{1}{k}}\frac{d\theta}{\sqrt{1-k^2\sin^2\frac{\theta}{2}}}\,d\theta = \frac{8}{\omega_0}\mathrm{F}_{\ell\ell}(k,\arcsin\frac{1}{k}) = \frac{4}{\omega_l}\,\mathrm{K}_{\ell\ell}(k^{-1})\,.$$

- The angle variable is then $w - w_0 = \nu(E)\,(t-t_0)$.

At the origin of quantum mechanics: Bohr-Sommerfeld quantization

- Are permitted only confined motions (in 1d) which obey:

$$J = \oint p(E)\, dq = n\, h\,, \quad n \in \mathbb{N} \quad \text{Bohr-Sommerfeld quantization rule,}$$

  where h is the Planck constant or “quantum of action”.

- This “old” condition is correct for quantization of the angular momentum for the free rotor, $p_\theta = J/(2\pi) = n\hbar$ and so the quantization of the free rotor quantum energy $E = n^2\hbar^2/(2ml^2)$, whereas it gives $E = \hbar\omega n$ for the harmonic oscillator energy and so does not provide the observed ground state one-half quantum.

- The motivation for the Bohr-Sommerfeld quantum condition was the *correspondence principle*: the behavior of a quantum system reproduces classical physics in the limit of large quantum numbers. This principle has to be complemented by the physical observation that the quantity to be quantized must be adiabatic invariant (i.e. stays constant when changes occur slowly).

The adiabatic rationale (...)

- The motivation for the Bohr-Sommerfeld quantum condition was the *correspondence principle*: the behavior of a quantum system reproduces classical physics in the limit of large quantum numbers. This principle has to be complemented by the physical observation that the quantity to be quantized must be adiabatic invariant (i.e. stays constant when changes occur slowly).

- In classical mechanics, an adiabatic change is a slow deformation of the Hamiltonian, where the fractional rate of change of the energy is much slower than the orbital frequency. The area enclosed by the different motions in phase space are the adiabatic invariants, which is precisely the case for the classical action. In quantum mechanics, an adiabatic change is one that occurs at a rate much slower than the difference in frequency between energy eigenstates. In this case, the energy states of the system do not make transitions, so that the quantum number is an adiabatic invariant. **The Bohr-Sommerfeld rule consists in equating the quantum number of a system with its classical adiabatic invariant. Hence, the quantum number is the area in phase space of the classical orbit.**

5.3. Action-angle coherent states and related quantizations for the motion on the circle (“quantum free rotator”)

Action angle coherent states on the cylinder with Gaussian distributions

- Set $X = \{(J,\gamma)\,,\ J \in \mathbb{R}\,,\ \gamma \in [0,2\pi)\}$: cylindric phase space for the motion on the circle.
- Measure $\mathrm{d}\mu(x) = \frac{\mathrm{d}J\ \mathrm{d}\gamma}{2\pi}$
- Adopt the construction of a family of coherent states described in Lesson 1
- Choice of orthonormal set in $L^2(X,\ \mathrm{d}J\ \mathrm{d}\gamma/2\pi)$:

$$\left\{\phi_n(x) = \left(\frac{\epsilon}{\pi}\right)^{1/4} e^{-\frac{\epsilon}{2}(\tilde{J}-n)^2}\, e^{i\alpha_n\gamma}\,,\ n \in \mathbb{Z}\right\},$$

  where $\alpha_n = n$ or $\alpha_n = n^2$ and ϵ controls the width of the Gaussian.

- Let $\mathcal{H}$ be a separable (complex) Hilbert space with orthonormal basis $\{e_n\,,\ n \in \mathbb{Z}\}$,

Action angle coherent states on the cylinder with Gaussian distributions (continued)

- The coherent states read, with $\tilde{J} = J/h$, h is Planck constant,

$$|x\rangle \equiv |J,\gamma\rangle = \frac{1}{\sqrt{\mathcal{N}(J)}} \left(\frac{\epsilon}{\pi}\right)^{1/4} \sum_{n\in\mathbb{Z}} e^{-\frac{\epsilon}{2}(\tilde{J}-n)^2}\, e^{-i\alpha_n\gamma}|e_n\rangle\,,$$

- By construction they solve the identity in $\mathcal{H}$:

$$\int_{-\infty}^{\infty} \mathrm{d}J\,\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\gamma\,\mathcal{N}(J)\,|J,\gamma\rangle\langle J,\gamma| = I\,.$$

- The normalization function $\mathcal{N}(J)$ is given in two forms:

$$\mathcal{N}(J) = \sqrt{\frac{\epsilon}{\pi}}\sum_{n\in\mathbb{Z}} e^{-\epsilon(\tilde{J}-n)^2} \underset{\text{Poisson}}{=} \sum_{n\in\mathbb{Z}} e^{2\pi i n\tilde{J}}\, e^{-\frac{\pi^2}{\epsilon}n^2}\,,$$

and satisfies $\lim_{\epsilon\to 0}\mathcal{N}(J) = 1$.

Quantization with action-angle coherent states on the cylinder

- The corresponding integral quantization reads:

$$f(J,\gamma) \mapsto A_f = \int_{-\infty}^{\infty} \mathrm{d}J\, \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\gamma\, f(J,\gamma)\, \mathcal{N}(J)\, |J,\gamma\rangle\langle J,\gamma|$$
$$= \sum_{n,n'} (A_f)_{nn'}\, |e_n\rangle\langle e_{n'}|\,,$$

$$(A_f)_{nn'} = \left(\frac{\epsilon}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}J\, e^{-\frac{\epsilon}{2}(\tilde{J}-n)^2+\frac{\epsilon}{2}(\tilde{J}-n')^2)} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\gamma\, e^{i(\alpha_n-\alpha_{n'})\gamma}\, f(J,\gamma)\,.$$

- This CS quantization with the simplest choice $\alpha_n = n$ of the action J gives the usual quantum angular momentum

$$J \mapsto A_J \quad \text{with} \quad (A_J)_{nn'} = \delta_{nn'} hn \equiv J_n$$

- and it gives for the energy of the free particle on the circle

$$E \propto J^2 \mapsto A_E \quad \text{with} \quad (A_E)_{nn'} \propto \delta_{nn'}(n^2 + const.)$$

as expected ...

Overlap

- The overlap between two of them is given by:

$$\langle J',\gamma'|J,\gamma\rangle = \frac{e^{-\frac{\epsilon(\tilde{J}-\tilde{J}')^2}{4}}}{\sqrt{\mathcal{N}(J)\mathcal{N}(J')}} \left(\frac{\epsilon}{\pi}\right)^{1/2} \sum_{n\in\mathbb{Z}} e^{-\epsilon\left(\frac{\tilde{J}+\tilde{J}'}{2}-n\right)^2} e^{i\alpha_n(\gamma'-\gamma)}\,,$$

- From the Poisson summation formula,

$$\sum_{n\in\mathbb{Z}} \Phi(n) = \sqrt{2\pi} \sum_{k\in\mathbb{Z}} \widehat{\Phi}(2\pi k)\,, \quad \widehat{\Phi}(k) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}t\, \Phi(t)\, e^{-ikt}\,,$$

  one infers the alternative formula,

$$\langle J',\gamma'|J,\gamma\rangle = \frac{e^{-\frac{\epsilon(\tilde{J}-\tilde{J}')^2}{4}}}{\sqrt{\mathcal{N}(J)\mathcal{N}(J')}} (2\epsilon)^{1/2} \sum_{n\in\mathbb{Z}} \widehat{\Phi}(2\pi n)\,, \quad \Phi(t) = e^{-\epsilon\left(t-\frac{\tilde{J}+\tilde{J}'}{2}\right)^2} e^{i(\gamma'-\gamma)\,\alpha(t)}\,,$$

  with $\alpha(n) \equiv \alpha_n$.

- One can also use the comparison series-integral $\sum_{n\in\mathbb{Z}} |\Phi(n)| \approx \int_{-\infty}^{+\infty} |\Phi(t)|\, dt$ which gives, for ϵ not too large:

$$|\langle J',\gamma'|J,\gamma\rangle| \leqslant \frac{e^{-\frac{\epsilon(\tilde{J}-\tilde{J}')^2}{4}}}{\sqrt{\mathcal{N}(J)\mathcal{N}(J')}} \left(\frac{\epsilon}{\pi}\right)^{1/2} \sum_{n\in\mathbb{Z}} e^{-\epsilon\left(\frac{\tilde{J}+\tilde{J}'}{2}-n\right)^2} \approx \frac{e^{-\frac{\epsilon(\tilde{J}-\tilde{J}')^2}{4}}}{\sqrt{\mathcal{N}(J)\mathcal{N}(J')}}\,.$$

The alternative for the free rotator

- The choice $\alpha_n \propto n$ is the usual one and yields the exact correspondence between the classical Poisson bracket $\{J, e^{i\phi}\} = ie^{i\phi}$ and the commutator $[A_J, A_{e^{i\phi}}] = A_{e^{i\phi}}$
- The choice $\alpha_n \propto n^2$ is appropriate for the quantization of the energy, since, with it, it yields temporal evolution stability

Overlap with the two simple choices: $\alpha_n = 2\pi n/\tau$, $\alpha_n = 2\pi n^2/\tau$

- Two useful Fourier transforms:

$$f_1(t) = e^{-\nu(t-\mu)^2}\, e^{i\lambda t} \overset{\text{Fourier}}{\longrightarrow} \widehat{f}_1(k) = \frac{1}{\sqrt{2\nu}} e^{-\frac{(k-\|ambda)^2}{4\nu}} \, e^{-i\mu(k-\lambda)},$$

$$f_2(t) = e^{-\nu(t-\mu)^2}\, e^{i\lambda t^2} \overset{\text{Fourier}}{\longrightarrow} \widehat{f}_2(k) = \frac{1}{\sqrt{2(\nu - i\lambda)}} e^{-\frac{\nu}{4(\nu^2+\lambda^2)}(k-2\lambda\mu)^2} \, e^{i\lambda\frac{(k+2\mu\nu^2/\lambda)^2}{4(\nu^2+\lambda^2)}} \, e^{-i\mu^2\nu^2/\lambda}.$$

- With $\alpha_n = 2\pi n/\tau$ the overlap is given by:

$$\langle J', \gamma' | J, \gamma\rangle = \frac{e^{-\frac{\epsilon(\tilde{J}-\tilde{J}')^2}{4}}}{\sqrt{\mathcal{N}(J)\mathcal{N}(J')}} \left(\frac{\epsilon}{\pi}\right)^{1/2} \sum_{n\in\mathbb{Z}} e^{-\epsilon\left(\frac{\tilde{J}+\tilde{J}'}{2}-n\right)^2} \, e^{i\alpha_n(\gamma'-\gamma)},$$

Quantum angle, with $\alpha_n = n$

- The integral quantization of the discontinuous 2π-periodic angle function $\daleth(\gamma) = \gamma$ for $\gamma \in [0, 2\pi)$

$$A_{\daleth} = \pi I + \sum_{n \neq n'} i \frac{e^{-\frac{\epsilon}{4}(n-n')^2}}{n - n'} |e_n\rangle\langle e_{n'}| \,,$$

- Corresponding lower symbols at the limit $\epsilon \to 0$

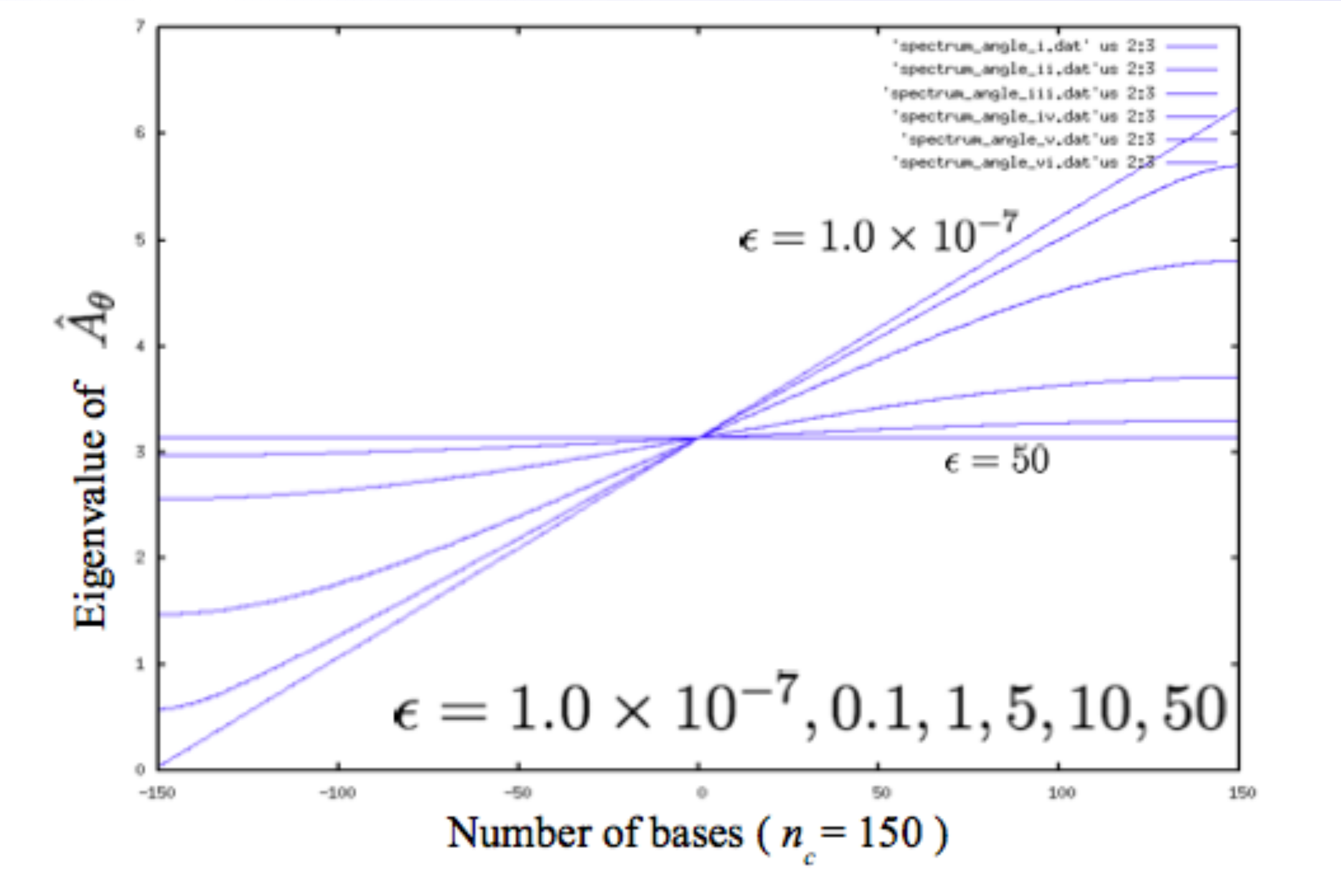
$$\langle J_0, \gamma_0 | A_{\daleth} | J_0, \gamma_0 \rangle = \pi + \frac{1}{2}\left(1 + \frac{\mathcal{N}(J_0 - \frac{1}{2})}{\mathcal{N}(J_0)}\right) \sum_{n \neq 0} i \, \frac{e^{-\frac{\epsilon}{2}n^2 + in\gamma_0}}{n}$$
$$\underset{\epsilon \to 0}{\sim} \pi + \sum_{n \neq 0} i \, \frac{e^{in\gamma_0}}{n} \,,$$

where we recognize at the limit the Fourier series of $\daleth(\gamma_0)$.

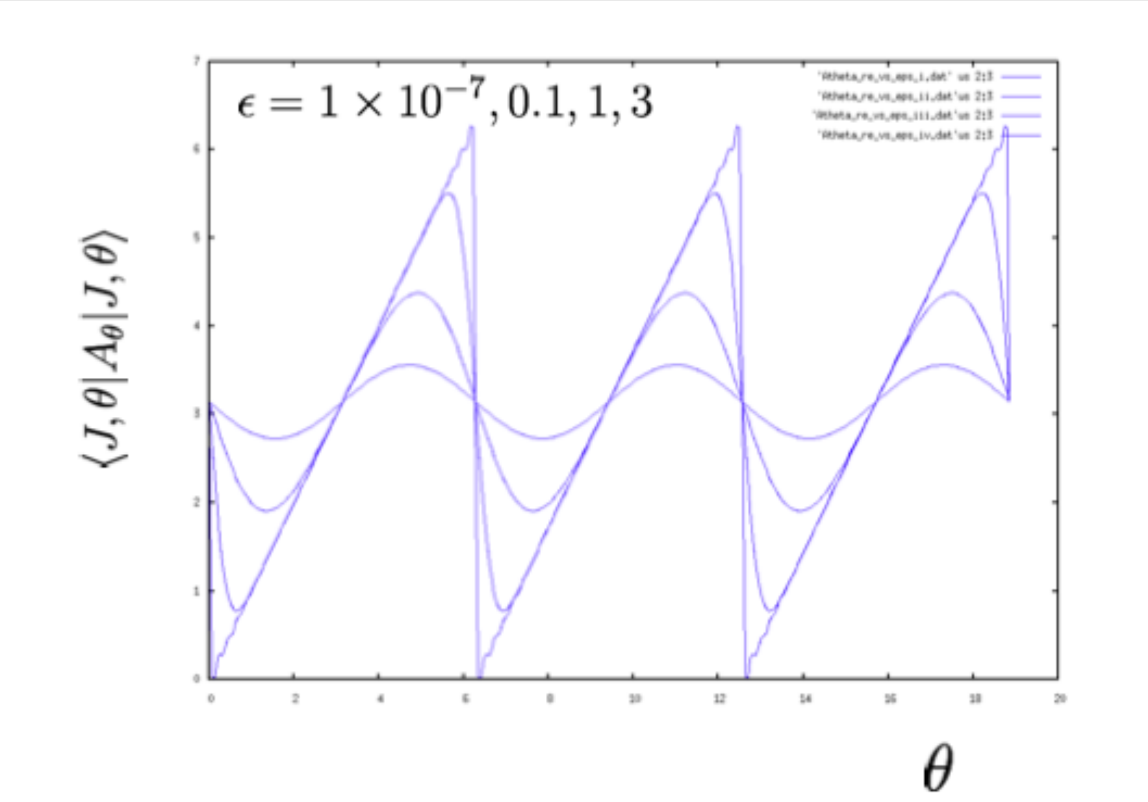
- For the commutator with the action,

$$\langle J_0, \beta_0 | [A_J, A_{\daleth}] | J_0, \gamma_0 \rangle = \frac{1}{2}\left(1 + \frac{\mathcal{N}(J_0 - \frac{1}{2})}{\mathcal{N}(J_0)}\right)\left(-i + \sum_{n \in \mathbb{Z}} i e^{-\frac{\epsilon}{2}n^2 + in\gamma_0}\right)$$
$$\underset{\epsilon \to 0}{\sim} -i + i \sum_n \delta(\gamma_0 - 2\pi n) \,.$$

- So we (almost) recover the canonical commutation rule except for the singularity at the origin $\bmod 2\pi$.



Spectrum of the angle operator $A_\daleth$, here denoted A_θ obtained by CS quantization of the angle of rotation on the circle



Lower symbol of the angle operator obtained by CS quantization of the angle of rotation on the circle

Action angle coherent states on the cylinder with general probability distributions[a]

- For the probability distribution we can actually choose a non-negative, even, well localized and normalized integrable function

$$\mathbb{R} \ni J \mapsto p^{\sigma}(J)\,, \quad p^{\sigma}(J) = p^{\sigma}(-J)\,, \quad \int_{-\infty}^{+\infty} dJ\, p^{\sigma}(J) = 1\,,$$

  where $\sigma > 0$ is some "width" parameter, and obeying $0 < \mathcal{N}^{\sigma}(J) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} p_n^{\sigma}(J) < \infty$ for all $J \in \mathbb{R}$, where $p_n^{\sigma}(J) \stackrel{\text{def}}{=} p_0^{\sigma}(J-n)$. Ex.: $p^{\sigma}(J) = \frac{1}{2\sigma}\chi_{[-\sigma,\sigma]}(J)$.

- The functions $\phi_n(x)$, for $n \in \mathbb{Z}$, are now given by:

$$\phi_n(x) = \sqrt{p_n^{\sigma}(J)}\, e^{in\varphi}\,, \quad n \in \mathbb{Z}\,.$$

- The correspondent family of coherent states on the circle reads as:

$$|J,\varphi\rangle = \frac{1}{\sqrt{\mathcal{N}^{\sigma}(J)}} \sum_{n\in\mathbb{Z}} \sqrt{p_n^{\sigma}(J)}\, e^{-in\varphi} |e_n\rangle\,.$$

- They are normalized, resolve the unity and give the correct quantization for action J and energy J^2.

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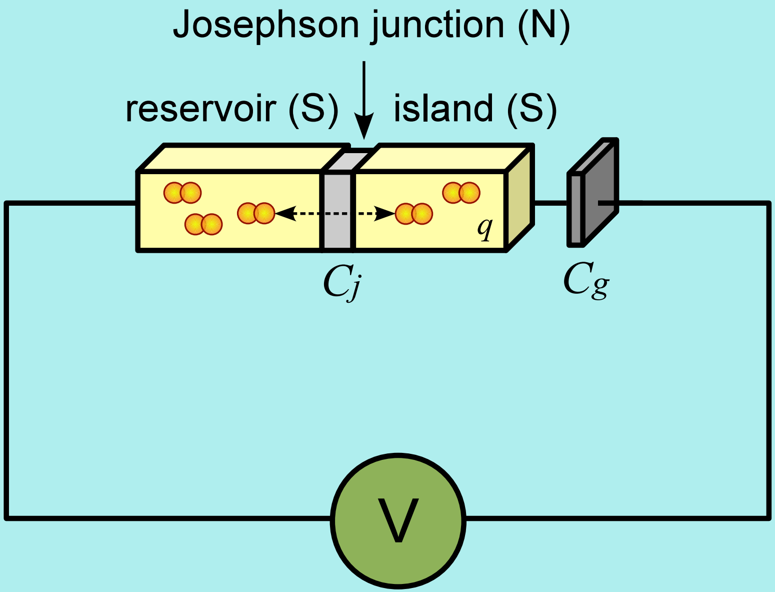
[a] *Action-angle coherent states for quantum systems with cylindric phase space*, I. Aremua, J.P.G, and M. N. Hounkonnou, J. Phys. A: Math. Theor. **45** 335302-1-16 (2012) arXiv:1111.4908v1 [quant-ph]

Experimental evidence of action-angle: superconducting box[a]

- Single superconducting island connected to a superconducting electron reservoir by a tunnel junction with capacitance C_j. Electrons can be transferred from the reservoir to the island by voltage source V connected between the reservoir and the island via a gate capacitance C_g. Both the island and the reservoir are taken to be good Bardeen-Cooper-Schrieffer (BCS) superconductors in conditions such that all electrons in the island are paired.



[a] e.g. Quantum Coherence with a single Cooper Pair, V. Bouchiat et al, *Phys. Scr.* **T76**, 165-170 (1998)

Experimental evidence of action-angle: superconducting box (continued)

- Total number of excess Cooper pairs $n \in \mathbb{Z}$, with total charge $q = -2en$ of the island, is element of the spectrum of a quantum observable $\hat{n}$ analogue to A_J.
- In eigenbasis $|n\rangle$ of $\hat{n}$, the quantum Hamiltonian reads as a quantum pendulum one in rotation (not in libration):

$$\hat{H} = \hat{H}_{\text{el}} + \hat{H}_{\text{Jos}} = E_C \sum_n (n - n_g)^2 |n\rangle\langle n| - \frac{E_J}{2} \sum_n (|n\rangle\langle n+1| + |n+1\rangle\langle n|)$$

(E_C: Coulomb energy, E_J: Josephson energy)

- $\hat{H}_{\text{Jos}}$ is $\propto$ to $A_{\cos\varphi}$.
- Experimental access to $[A_J, A_\phi]$?

5.4. A Bayesian probabilistic construction of action-angle coherent states and related quantizations

Conditional posterior probability distribution

- Suppose that measurement of a confined one-dimensional system yields the sequence of values for the energy observable (up to a constant shift):

$$E_0 < E_1 < \cdots < E_n < \cdots \, .$$

- Let h be a constant characteristic action of the considered system (e.g. the Planck constant). We define a corresponding sequence of probability distributions $J \mapsto p_n(J)$, i.e. $\int_{\mathbb{R} \text{ or } \mathbb{R}^+} d\tilde{J}\, p_n(J) = 1$, with $\tilde{J} \stackrel{\text{def}}{=} J/h$, supposing a (prior) *uniform* distribution on the range of the action variable J, obeying the two conditions:

$$0 < \mathcal{N}(J) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z} \text{ or } \mathbb{N}} p_n(J) < \infty \, , \quad E_n + \text{cst} = \int_{\mathbb{R} \text{ or } \mathbb{R}^+} d\tilde{J}\, E(J)\, p_n(J)$$

  where $\mathbb{R}$ and $\mathbb{Z}$ (resp. $\mathbb{R}^+$ and $\mathbb{N}$) stand for the rotation (resp. libration) type of motion.

- The finiteness condition allows to consider the map $n \mapsto p_n(J)/\mathcal{N}(J)$ as a probabilistic model referring to the discrete data, which might viewed in the present context as a *prior distribution*.

Action-angle coherent states

- Let $\mathcal{H}$ be a complex separable Hilbert space with orthonormal basis $\{|e_n\rangle \; n \in \mathbb{Z} \, \text{or} \, \mathbb{N}\}$
- Let $\tau > 0$ be a rescaled period of the angle variable and $X = \{(J, \gamma)\,,\; J \in \mathbb{R} \text{ or } \mathbb{R}^+,\; 0 \leqslant \gamma < \tau\}$[a] be the action-angle phase space for a rotation (resp. libration) motion with measured energies the discrete sequence $E_0 < E_1 < \cdots < E_n < \cdots$.
- Let $(p_n(J))_{n \in \mathbb{Z} \, \text{or} \, \mathbb{N}}$ be the sequence of probability distributions associated with these energies. We suppose $p_{-n}(J) = p_n(-J)$ in the rotation case.
- One then constructs the family of states in $\mathcal{H}$ for the considered motion as the following continuous map from X into $\mathcal{H}$:

$$X \ni (J, \gamma) \mapsto |J, \gamma\rangle = \frac{1}{\sqrt{\mathcal{N}(J)}} \sum_n \sqrt{p_n(J)}\, e^{-i\alpha_n \gamma}\, |e_n\rangle \in \mathcal{H}\,,$$

where the choice of the real sequence $n \mapsto \alpha_n$ is left to us in order to comply with some if not all criteria previously listed.

[a] Actually we keep the freedom of making γ vary from $-\infty$ to $+\infty$ as we do for any angle variable.

Fundamental properties of action-angle coherent states

In both cases the coherent states $|J,\gamma\rangle$

(i) are unit vector : $\langle J,\gamma|J,\gamma\rangle = 1$

(ii) resolve the unity operator in $\mathcal{H}$ with respect a measure "in the Bohr sense" $\mu_B(dJ\, d\gamma)$ on the phase space X :

$$\int_X \mu_B(dJ\, d\gamma)\,\mathcal{N}(J)\,|J,\gamma\rangle\langle J,\gamma| \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} d\tilde{J}\,\mathcal{N}(J)\,\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} d\gamma|J,\gamma\rangle\langle J,\gamma| = 1_{\mathcal{H}}\,.$$

(iii) allow a "coherent state quantization" of classical observables $f(J,\gamma)$ which is "energy" compatible with our construction of the posterior distribution $J\mapsto p_n(J)$:

$$f(J,\gamma)\mapsto \int_X \mu_B(dJ\, d\gamma)\,\mathcal{N}(J)\,f(J,\gamma)\,|J,\gamma\rangle\langle J,\gamma| \stackrel{\text{def}}{=} A_f\,.$$

(iv) **since it is trivially verified that in both cases the quantum Hamiltonian is exactly what we expect:**

$$A_{E(J)} = \sum_n (E_n + \mathrm{cst})|e_n\rangle\langle e_n|\,.$$

Quantization of action and angle coordinates

- The quantization of any function $h(J)$ of the action variable only yields the diagonal operator:

$$h(J) \mapsto A_h = \sum_n \langle h \rangle_n |e_n\rangle\langle e_n| \,,$$

where

$$\langle h \rangle_n = \int_{-\infty}^{+\infty} d\tilde{J}\, h(J)\, p_n(J) \quad \text{for} \quad X_r \,,$$
$$\langle h \rangle_n = \int_{0}^{+\infty} d\tilde{J}\, h(J)\, p_n(J) \quad \text{for} \quad X_l \,.$$

- Remind that we have already defined $J_n = \langle J \rangle_n$ (for X_r) and $J_{n+1} = \langle J \rangle_n$ (for X_l).

Quantization of action and angle coordinates (continued I)

- The quantization of any τ periodic function $g(\gamma)$ of the angle variable only yields the operator:

$$g(\gamma) \mapsto A_g = \sum_{n,n'} [A_g]_{nn'} |e_n\rangle\langle e_{n'}| \,,$$

where the matrix elements$[A_g]_{nn'}$ are formally given by:

$$[A_g]_{nn'} = \int_{-\infty}^{+\infty} d\tilde{J}\,\sqrt{p_n(J)\,p_{n'}(J)}\,\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} d\gamma\, e^{-i(\alpha_n-\alpha_{n'})\gamma}\, g(\gamma)$$

$$\text{for} \quad X_r \,,$$

$$[A_g]_{nn'} = \int_{0}^{+\infty} d\tilde{J}\,\sqrt{p_n(J)\,p_{n'}(J)}\,\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} d\gamma\, e^{-i(\alpha_n-\alpha_{n'})\gamma}\, g(\gamma)$$

$$\text{for} \quad X_l \,.$$

- In particular the CS quantization procedure provides, for a given choice of the sequence (α_n) a self-adjoint angle operator corresponding to the angle function $\mathcal{A}(\gamma)$ defined on the real line as the τ-periodic extension of $\mathcal{A}(\gamma) = \gamma$ on the interval $[0, \tau)$.

Quantization of functions of angle (continued II)

- The explicit form of the matrix elements is given by specifying the parameter T as $T = 2M\tau$ with $M \in \mathbb{N}$ and letting $M \to \infty$. From the general formula for any integrable τ-periodic function integrable on a period interval,

$$\frac{1}{2M\tau}\int_{-M\tau}^{M\tau} d\gamma\, g(\gamma)\, e^{-i\lambda\gamma} = \left[\frac{1}{\tau}\int_0^{\tau} d\gamma\, g(\gamma)\, e^{-i\lambda\gamma}\right] \times \left(\frac{1}{2M}\sum_{m=-M}^{M-1} e^{-im\lambda\tau}\right)$$
$$= \left[\frac{1}{\tau}\int_0^{\tau} d\gamma\, g(\gamma)\, e^{-i\lambda\gamma}\right] \times \frac{1}{M}\left(\cos((M-1)\lambda\tau/2)\,\frac{\sin((M\lambda\tau/2)}{\sin(\lambda\tau/2)} + e^{iM\lambda\tau}\right),$$

we have at the limit:

$$\lim_{M\to\infty}\frac{1}{2M\tau}\int_{-M\tau}^{M\tau} d\gamma\, g(\gamma)\, e^{-i\lambda\gamma} = \begin{cases} 0 & \text{if } \lambda \notin (2\pi/\tau)\mathbb{Z}\,, \\ c_k(g;\tau) & \text{if } \lambda = 2\pi k/\tau\,, \quad k \in \mathbb{Z}\,, \end{cases}$$

where $c_k(g;\tau) = \frac{1}{\tau}\int_0^{\tau} d\gamma\, g(\gamma)\, e^{-i2\pi k\gamma/\tau}$ is the kth Fourier coefficient of $g(\gamma)$.

- The matrix elements$[A_g]_{nn'}$ are then given by:

$$[A_g]_{nn'} = 0 \text{ if } \alpha_n - \alpha_{n'} \notin (2\pi/\tau)\mathbb{Z} \quad \text{or} \quad = \varpi_{nn'}\, c_k(g;\tau) \text{ if } \alpha_n - \alpha_{n'} = 2\pi k/\tau\,,\ k \in \mathbb{Z}\,,$$

where $\varpi_{nn'} = \int_{-\infty}^{+\infty} d\tilde{J}\,\sqrt{p_n(J)\,p_{n'}(J)}$ (resp. $\int_0^{+\infty} d\tilde{J}\,\sqrt{p_n(J)\,p_{n'}(J)}$) measures the correlation between the two distributions $J \mapsto p_n(J)$, $J \mapsto p_{n'}(J)$.

- Note the diagonal values are all equal to the average of $g(\gamma)$ over one period. Also the infinite matrix can be sparse, even just diagonal, depending on the choice of the α_n's. In the latter case, the quantization transforms classical observables into a commutative algebra of operators.

Localization probability distributions

- The action-angle phase space representation of a particular coherent state $|J_{\rm i}, \gamma_{\rm i}\rangle$, as a function of (J, γ), is precisely given by the "normalized" overlap

$$\Psi_{|J_{\rm i},\gamma_{\rm i}\rangle}(J,\gamma) \stackrel{\rm def}{=} \sqrt{\mathcal{N}(J)}\,\langle J,\gamma|J_{\rm i},\gamma_{\rm i}\rangle = \frac{1}{\sqrt{\mathcal{N}(J_i)}}\sum_n \sqrt{p_n(J)\,p_n(J_i)}\,e^{i\alpha_n(\gamma-\gamma_i)}\,,$$

- Hence, the map X_r (resp. X_l) $\ni (J,\gamma) \mapsto \rho^{\rm phase}_{|J_{\rm i},\gamma_{\rm i}\rangle}(J,\gamma) \equiv |\Psi_{|J_{\rm i},\gamma_{\rm i}\rangle}(J,\gamma)|^2 = \mathcal{N}(J)\,|\langle J,\gamma|J_{\rm i},\gamma_{\rm i}\rangle|^2$ represents a localization probability distribution, namely a generalized version of the Husimi distribution, on the phase space provided with the pseudo-measure μ_B. Indeed, the resolution of the identity gives immediately

$$\int_{X_r\ (\text{resp. } X_l)} \mu_B(dJ\,d\gamma)\,\rho^{\rm phase}_{|J_{\rm i},\gamma_{\rm i}\rangle}(J,\gamma) = 1\,.$$

- If we choose instead a specific realization of the Hilbert space $\mathcal{H}$, like that one generated by eigenfunctions of the quantum Hamiltonian A_H in "q" or "configuration" representation, $|e_n\rangle \mapsto \psi_n(q)$, the corresponding representation of the state $|J_{\rm i},\gamma_{\rm i}\rangle$ reads as

$$\psi_{|J_{\rm i},\gamma_{\rm i}\rangle}(q) = \frac{1}{\sqrt{\mathcal{N}(J_{\rm i})}}\sum_n \sqrt{p_n(J_{\rm i})}\,e^{-i\alpha_n\,\gamma_{\rm i}}\,\psi_n(q)\,,$$

  with corresponding probability density of localization on the range of the q-variable given by $\rho^{\rm circ}_{|J_{\rm i},\gamma_{\rm i}\rangle}(q) \equiv |\psi_{|J_{\rm i},\gamma_{\rm i}\rangle}(q)|^2$

Time evolution

- Let us now examine the time evolution of the coherent states $|J,\gamma\rangle$. Since the CS quantized version A_H of the classical Hamiltonian H is diagonal in the basis $\{|e_n\rangle\,,\, n\in\mathbb{Z}\ (\text{resp. } \mathbb{N})\,\}$, the time evolution of the CS in both representations is given respectively by

$$
\begin{aligned}
e^{-i\tilde{A}_H t}\Psi_{|J_\mathrm{i},\gamma_\mathrm{i}\rangle}(J,\gamma) &= \sqrt{\mathcal{N}(J)}\,\langle J,\gamma|e^{-i\tilde{A}_H t}|J_\mathrm{i},\gamma_\mathrm{i}\rangle \\
&= \frac{1}{\sqrt{\mathcal{N}(J_\mathrm{i})}}\sum_n \sqrt{p_n(J_\mathrm{i})p_n(J)}\, e^{i(\alpha_n(\gamma-\gamma_\mathrm{i})-\tilde{E}_n\, t)}\,, \\
e^{-i\tilde{A}_H t}\psi_{|J_\mathrm{i},\gamma_\mathrm{i}\rangle}(q) &= \frac{1}{\sqrt{\mathcal{N}(J_\mathrm{i})}}\sum_n \sqrt{p_n(J_\mathrm{i})}\, e^{-i(\alpha_n\gamma_\mathrm{i}+\tilde{E}_n\, t)}\,\psi_n(q)\,.
\end{aligned}
$$

  Here we have put $\tilde{A}_H = A_H/\kappa$ and $\tilde{E}_n = E_n/\kappa$ for dimensional purposes.

- From these expressions stems the need to show various snapshots of the time evolution of the corresponding probability densities $|\sqrt{\mathcal{N}(J)}\,\langle J,\gamma|e^{-i\tilde{A}_H t}|J_\mathrm{i},\gamma_\mathrm{i}\rangle|^2$ and $|e^{-i\tilde{A}_H t}\psi_{|J_\mathrm{i},\gamma_\mathrm{i}\rangle}(q)|^2$.

- Note that time evolution stability is granted with the choice $\alpha_n = \alpha_{-n} = E_n$.

The quest for explicit probabilities: the two limit cases

- The central question raised by our construction is the determination of the discretely indexed probability distribution $J \mapsto p_n(J)$. We have two limit situations, the free rotor and the harmonic oscillator, for which the energies are respectively $E_n \propto n^2$+const. and $E_n \propto n$+const.

- In the first case, and with the notations for the classical free rotor, a familiar solution is the normal law centered at each integer, with dimensionless width parameter σ or ϵ:

$$p_n(J) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} e^{-\frac{1}{2\sigma^2\kappa^2}(J-\kappa n)^2} \equiv \left(\frac{\epsilon}{\pi}\right)^{1/2} e^{-\epsilon\,(\tilde{J}-n)^2}\,, \quad n \in \mathbb{Z}\,.$$

  This gives $J_n = \kappa n$ and $E_n = \dfrac{\kappa^2 n^2}{8\pi^2 m l^2} + \dfrac{\sigma^2\kappa^2}{8\pi^2 m l^2}$, the constant shift being the average value of the classical energy with respect the distribution $p_0(J)$.

- In the second case, and with the notations for the classical oscillator, another familiar solution is the discretely indexed gamma distribution:

$$p_n(J) = e^{-\tilde{J}}\,\frac{\tilde{J}^n}{n!}\,, \quad n \in \mathbb{N}\,.$$

  This gives $J_n = \kappa(n+1)$ and $E_n = \dfrac{\omega\,\kappa}{2\pi}(n+1)$, the constant shift being the average value of the classical energy with respect to the distribution $p_0(J)$.

The quest for explicit probabilities: the general case

- The central question is, given the classical relation $E = E(J)$ between action variable and energy, and given the observational or computed sequence (E_n), to find solution(s) $p_n(J)$ (at least with a satisfying approximation), for each $n \in \mathbb{Z}$ or $\in \mathbb{N}$, to the 2 equations

$$E_n = \int_{\mathbb{R}\,\mathrm{or}\,\mathbb{R}^+} d\tilde{J}\, E(J)\, p_n(J)\,,$$
$$1 = \int_{\mathbb{R}\,\mathrm{or}\,\mathbb{R}^+} d\tilde{J}\, p_n(J)\,,$$

  which moreover fulfill the nonzero and finiteness conditions:

$$0 < \mathcal{N}(J) \stackrel{\text{def}}{=} \sum_n p_n(J) < \infty\,.$$

  Recall that the relation $E = E(J)$ is the inverse function of $J(E) = \oint p(E, q)\, dq$.

- For the rotation case, the departure point could be a normal-like law, possibly modified along a perturbation scheme with expansion parameter the strength of the potential energy U.

- In the libration case, the departure could be a Poisson-like law, possibly modified along a perturbation scheme with expansion parameter the strength of the potential energy U

An example of solution given by K.R. Parthasarathy:

linear convex combinations of uniform distributions

- Let $H(t)$ be continuous, $\geqslant 0$, with Range $H = [0, \infty)$. Let μ be any positive number.
- Consider for $H^{-1}([\alpha, \beta])$ for $\alpha < \beta < \mu$ and $H^{-1}([\gamma, \delta])$ for $\mu < \gamma < \delta$.
- Define
$$p_a(t) = \frac{1_{H^{-1}([\alpha,\beta])}}{\mu_L\left(H^{-1}([\alpha, \beta])\right)} \quad \text{and} \quad p_b(t) = \frac{1_{H^{-1}([\gamma,\delta])}}{\mu_L\left(H^{-1}([\gamma, \delta])\right)}\,.$$
where $\mu_L(X)$ is Lebesgue measure of X and 1_X is the indicator function of X.
- Define $\int_{\mathbb{R}} H(t)\, p_a(t)\, dt = a$, $\int_{\mathbb{R}} H(t)\, p_b(t)\, dt = b$. Then $a < \mu < b$.
- Let $\mu = \lambda a + (1-\lambda)b$ and define $p_\mu(t) = \lambda\, p_a(t) + (1-\lambda)\, p_b(t)$. Then
$$\int_{\mathbb{R}} H(t)\, p_\mu(t)\, dt = \mu \quad \forall \mu \in \text{Range}\, H\,, \mu > 0\,,$$
i.e. $\mu \in (0, \infty)$.
- Consequence: for a sequence $\{\mu_n\} \subset (0, \infty)$, then
$$\int_{\mathbb{R}} H(t)\, p_{\mu_n}(t)\, dt = \mu_n\,.$$

An example of solution given by K.R. Parthasarathy (continued)

- Furthermore, the non-zero and finiteness conditions have to be fulfilled. For that, supposing that $H(t)$ is strictly increasing and unbounded (this is the case for the energy function $E(J)$ for $J > 0$), one can choose intertwining sequences $\{\alpha_n\}$ and $\{\beta_n\}$, such as

$$\mu_{n-1} \leqslant \alpha_{n+1} \leqslant \beta_n\,, \quad \mu_{-1} \stackrel{\text{def}}{=} 0\,,$$

while the sequences $\{\gamma_n\}$ and $\{\delta_n\}$ remain free apart from the constraints

$$\mu_n < \gamma_n < \delta_n < \mu_{n+N_l}\,,$$

for a fixed $N_l \geqslant 1$.

- It is then clear that $\mathcal{N}(J) = \sum_n p_n(J)$
  - $\star$ never vanishes, since $p_{a_n}(t) \stackrel{\text{def}}{=} \dfrac{1_{H^{-1}([\alpha_n,\beta_n])}}{\mu_L\left(H^{-1}([\alpha_n,\beta_n])\right)}$ overlaps on a non-empty interval with $p_{a_{n+1}}$,
  - $\star$ and is finite, since a finite set of p_n's overlap on non-empty intervals.

Approximations for the simple pendulum: the rotation case

- Starting from the computed Mathieu eigenvalues E_n, a very empirical approach consists in starting from the sequence of computed action variables $J_n^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} J(E_n)$ and to impose in the rotation case, the normal law

$$p_n(J) = \left(\frac{1}{2\pi\sigma_n^2\kappa^2}\right)^{1/2} e^{-\frac{1}{2\sigma_n^2\kappa^2}(J-J_n^{\mathrm{cl}})^2}, \quad n \in \mathbb{Z},$$

by "adjusting" σ_n in order to suitably approximate the E_n's with the computed quantities

$$E_n^{\mathrm{app}} \stackrel{\mathrm{def}}{=} \int_{-\infty}^{+\infty} E(J)\, p_n(J)\, d\tilde{J}\,.$$

- Note that the consistency condition on the quantization of the action variable is automatically fulfilled:

$$\int_{-\infty}^{+\infty} J\, p_n(J)\, d\tilde{J} = J_n = J_n^{\mathrm{cl}}\,.$$

Approximations for the simple pendulum: the libration case

- Like in the preceding case, starting from the computed Mathieu eigenvalues E_n, a very empirical approach consists in starting from the sequence of computed action variables $J_n^{\rm cl} \stackrel{\rm def}{=} J(E_n)$ and to impose in the libration case (the most delicate one!), the Poisson-like distribution

$$p_n(J) = \frac{w_n(J)}{\mathcal{E}(J)} \frac{\tilde{J}^n}{\tilde{J}_n^{\rm cl}!}\,, \quad n \in \mathbb{N}\,, \quad \tilde{J}_n^{\rm cl}! \stackrel{\rm def}{=} \tilde{J}_1^{\rm cl} \tilde{J}_2^{\rm cl} \cdots \tilde{J}_n^{\rm cl}\,, \quad \tilde{J}_0^{\rm cl}! \stackrel{\rm def}{=} 1\,, \quad \mathcal{E}(J) \stackrel{\rm def}{=} \sum_{n=0}^{\infty} \frac{\tilde{J}^n}{\tilde{J}_n^{\rm cl}!}\,,$$

by "adjusting" the weights $w_n(J)$ (which should not differ appreciably from 1) in order to, not only suitably approximate the E_n's with the computed quantities

$$E_n^{\rm app} \stackrel{\rm def}{=} \int_0^{+\infty} E(J)\, p_n(J)\, d\tilde{J}\,,$$

but also comply with the probability normalization

$$\int_0^{+\infty} p_n(J)\, d\tilde{J} = 1\,,$$

and, hopefully, the approximate consistency condition

$$J_{n+1} = \int_0^{+\infty} J\, p_n(J)\, d\tilde{J} = \int_0^{+\infty} J\, \frac{w_n(J)}{w_{n+1}(J)}\, p_{n+1}(J)\, dJ \approx J_{n+1}^{\rm cl}\,.$$

Concluding points

- Integrable systems provide a variety of such families of action-angle coherent states
- The question is the "good" choice of probability distributions $n_i \mapsto p_{n_i}(J_i)$
- The question is the physical (in terms of physical measurement) equivalence of such frames from quantization point of view
- Extension to non confined systems and subsequent continuous spectra is possible (see below) [a]

[a] *Coherent states and related quantizations for unbounded motions*, V. G. Bagrov, JPG, D. Gitman, and A. Levine, J. Phys. A: Math. Theor. (2012), arXiv:1201.0955v2 [quant-ph]