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Motion analysis with the Continuous Wavelet Transform

J-P. Antoine
UCL, Louvain La Neuve
Belgium

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J-P. Antoine

Institut de Recherche en Mathématique et Physique
Université catholique de Louvain
B-1348 Louvain-la-Neuve, Belgique

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A REMINDER ABOUT WAVELET ANALYSIS OF 2-D IMAGES

Wavelet analysis of 2-D images

- Strategy for designing a continuous WT on manifold $\mathcal{M}$:
  - Identify operations to be applied to signals $s \in L^2(\mathcal{M}, d\mu)$
  - If these operations constitute a group G, find a unitary irreducible, square integrable representation U of G in $L^2(\mathcal{M}, d\mu)$ and write
    $$\psi_g(\zeta) := [U(g)\psi](\zeta) = \psi(g^{-1}\zeta), \; g \in G, \zeta \in \mathcal{M}$$
  - CWT of $f \in L^2(\mathcal{M}, d\mu)$ w.r. to (admissible) wavelet ψ is defined as
    $$W_\psi f(g) := \langle \psi_g | f \rangle = \int_{\mathcal{M}} \overline{\psi(g^{-1}\zeta)}\, f(\zeta)\, d\mu(\zeta), \; g \in G$$
- Geometric transformations in the plane $\mathbb{R}^2$:
  - (i) translation by $\vec{b} \in \mathbb{R}^2 : \vec{x} \mapsto \vec{x}' = \vec{x} + \vec{b}$
  - (ii) dilation by a factor $a > 0 : \vec{x} \mapsto \vec{x}' = a\vec{x}$
  - (iii) rotation by an angle $\theta : \vec{x} \mapsto \vec{x}' = r_\theta(\vec{x})$
  $$r_\theta \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \; 0 \leqslant \theta < 2\pi, \text{ rotation matrix}$$
- Action on finite energy signals
  $$\left[U(\vec{b}, a, \theta)s\right](\vec{x}) \equiv s_{\vec{b},a,\theta}(\vec{x}) = a^{-1} s(a^{-1}\, r_{-\theta}(\vec{x} - \vec{b}))$$

Wavelet analysis of 2-D images

- Basic formulas for CWT :
  $$\begin{aligned} S(\vec{b}, a, \theta) &= \langle \psi_{\vec{b},a,\theta} | s \rangle \\ &= a^{-1} \int_{\mathbb{R}^2} \overline{\psi(a^{-1}\, r_{-\theta}(\vec{x} - \vec{b}))}\, s(\vec{x})\, d^2\vec{x} \\ &= a \int_{\mathbb{R}^2} e^{i\vec{b}\cdot\vec{k}}\, \overline{\widehat{\psi}(a r_{-\theta}(\vec{k}))}\, \widehat{s}(\vec{k})\, d^2\vec{k} \end{aligned}$$
- Admissibility of wavelet ψ :
  $$c_\psi \equiv (2\pi)^2 \int_{\mathbb{R}^2} \frac{|\widehat{\psi}(\vec{k})|^2}{|\vec{k}|^2}\, d^2\vec{k} < \infty$$
- Necessary condition :
  $$\widehat{\psi}(\vec{0}) = 0 \iff \int_{\mathbb{R}^2} \psi(\vec{x})\, d^2\vec{x} = 0.$$
- Possible additional requirements :
  - restrictions on the support of ψ and of $\widehat{\psi}$
  - vanishing moments, up to order $N \geqslant 1$ ($N = 0$: admissibility) :
    $$\int d^2\vec{x}\, x^\alpha\, y^\beta\, \psi(\vec{x}) = 0, \quad \vec{x} = (x, y), \quad 0 \leqslant \alpha + \beta \leqslant N$$
    $\Rightarrow$ improved efficiency at detecting singularities in the signal : transform is blind to smoothest part of the signal, i.e., polynomial of degree up to N (less interesting, in general)

Group-theoretical justification

- In 1-D : dilation + translation = affine transformation of the line

$$y = (b,a)x \equiv ax + b, \quad a \neq 0,\ b \in \mathbb{R},\ x \in \mathbb{R}$$

Composition rule : $(b,a)(b',a') = (b + ab', aa')$
$\Rightarrow \{(b,a)\} \equiv G_{\rm aff} \simeq \mathbb{R}^2_* =$ affine group

- In 2-D : dilations + translations + rotations
$=$ similitude group of the plane : $\mathrm{SIM}(2) = \mathbb{R}^2 \rtimes (\mathbb{R}^+_* \times \mathrm{SO}(2))$

$$\vec{y} = (\vec{b}, a, \theta)\vec{x} \equiv a r_\theta \vec{x} + \vec{b},$$

- Action on finite energy signals

$$\left[U(\vec{b}, a, \theta)s\right](\vec{x}) = a^{-1} s(a^{-1}\, r_{-\theta}(\vec{x} - \vec{b}))$$

and $U =$ unitary irreducible representation of SIM(2) in $L^2(\mathbb{R}^2)$

- U is square integrable

$$\psi \text{ admissible} \iff \iiint_{\mathrm{SIM}(2)} \left|\langle U(\vec{b}, a, \theta)\psi|\psi\rangle\right|^2 \, d^2\vec{b}\, \frac{da}{a^3}\, d\theta < \infty$$

Mathematical properties of CWT

- Energy conservation

$$c_\psi^{-1} \iiint_{\mathrm{SIM}(2)} |S(\vec{b}, a, \theta)|^2 \, d^2\vec{b}\, \frac{da}{a^3}\, d\theta = \int_{\mathbb{R}^2} |s(\vec{x})|^2 \, d^2\vec{x}$$

i.e., isometry from space of signals $L^2(\mathbb{R}^2)$ onto closed subspace of $L^2(\mathrm{SIM}(2))$ = space of wavelet transforms

- Reconstruction formula
Inversion of CWT by adjoint map :

$$s(\vec{x}) = c_\psi^{-1} \iiint_{\mathrm{SIM}(2)} \psi_{\vec{b},a,\theta}(\vec{x})\, S(\vec{b}, a, \theta)\, d^2\vec{b}\, \frac{da}{a^3}\, d\theta$$

i.e., decomposition of the signal in terms of the analyzing wavelets $\psi_{\vec{b},a,\theta}$, with coefficients $S(\vec{b}, a, \theta)$

Mathematical properties of CWT

- Reproduction property (reproducing kernel)

$$S(\vec{b}', a', \theta') = c_\psi^{-1} \iiint_{\mathrm{SIM}(2)} \langle \psi_{\vec{b}',a',\theta'} | \psi_{\vec{b},a,\theta} \rangle \; S(\vec{b}, a, \theta)\, d^2\vec{b}\, \frac{da}{a^3}\, d\theta$$

- WT is covariant under translations, dilations and rotations : the correspondence $W_\psi : s(\vec{x}) \mapsto S(\vec{b}, a, \theta)$ implies the following ones

$$\begin{array}{rcl} s(\vec{x} - \vec{b}_o) & \mapsto & S(\vec{b} - \vec{b}_o, a, \theta) \\ a_o^{-1} s(a_o^{-1}\vec{x}) & \mapsto & S(a_o^{-1}\vec{b}, a_o^{-1} a\theta) \\ s(r_{\theta_o}(\vec{x})) & \mapsto & S(r_{-\theta_o}(\vec{b}), a, \theta - \theta_o) \end{array}$$

- Note: translation covariance ("shift invariance") is lost in the standard formulation of the discrete WT, based on multiresolution
$\Rightarrow$ problems in pattern recognition, e.g.

Choice of the analyzing wavelet

(i) *Isotropic wavelets*

- Pointwise analysis
- Directions irrelevant

$\Rightarrow$ rotation invariant wavelet

Examples :

- 2-D Mexican hat wavelet

$$\psi_H(\vec{x}) = (2 - |\vec{x}|^2)\, \exp(-\tfrac{1}{2}|\vec{x}|^2)$$

$$\widehat{\psi}_H(\vec{k}) = |\vec{k}|^2\, \exp(-\tfrac{1}{2}|\vec{k}|^2)$$

- Difference-of-Gaussians or DOG wavelet

$$\psi_D(\vec{x}) = \tfrac{1}{2\alpha^2}\, \exp(-\tfrac{1}{2\alpha^2}|\vec{x}|^2) - \exp(-\tfrac{1}{2}|\vec{x}|^2) \qquad (0 < \alpha < 1)$$

Choice of the analyzing wavelet - 1

(ii) *Directional wavelets*

- Detection of directional features
- Directional filtering

$\Rightarrow$ direction sensitive wavelet

Example :
directional wavelet $\Leftrightarrow$ num supp $\widehat{\psi} \subset$ convex cone, apex at 0

- 2-D Morlet wavelet

$$\widehat{\psi}_M(\vec{k}) = \sqrt{\epsilon}\left(\exp(-\tfrac{1}{2}|A^{-1}(\vec{k}-\vec{k}_0)|^2) - h(\vec{k})\right)$$

where $A = \mathrm{diag}[\epsilon^{-1/2}, 1], \epsilon \geqslant 1$, is a 2×2 anisotropy matrix and the correction term $h(\vec{k})$ is negligible in practice

  - The Morlet wavelet is directional, but has poor aperture selectivity
  - In addition, its angular selectivity increases with $\|\vec{k}_0\|$, since the support cone gets narrower, but at the same time the amplitude decreases as $\exp(-|\vec{k}_0|^2)$

Choice of the analyzing wavelet - 2

- Cauchy wavelet
  - Convex cone determined by the unit vectors $\vec{e}_{-\alpha}, \vec{e}_{\alpha}$

$$\mathcal{C} := \mathcal{C}(-\alpha,\alpha) = \{\vec{k} \in \mathbb{R}^2 : -\alpha \leqslant \arg \vec{k} \leqslant \alpha,\ \alpha < \pi/2\},$$

$\Rightarrow$ aperture of the cone $\mathcal{C} = 2\alpha$

  - Dual cone

$$\widetilde{\mathcal{C}} = \mathcal{C}(-\widetilde{\alpha},\widetilde{\alpha}) = \{\vec{k} \in \mathbb{R}^2 : \vec{k}\cdot\vec{k}' > 0,\ \forall \vec{k}' \in \mathcal{C}(-\alpha,\alpha)\},\ \widetilde{\alpha} = -\alpha + \pi/2$$

so that $\vec{e}_{-\alpha}\cdot\vec{e}_{\widetilde{\alpha}} = \vec{e}_{\alpha}\cdot\vec{e}_{-\widetilde{\alpha}} = 0,$

  - Cauchy wavelet

$$\widehat{\psi}^{C}_{lm}(\vec{k}) = \begin{cases} (\vec{k}\cdot\vec{e}_{\widetilde{\alpha}})^l\,(\vec{k}\cdot\vec{e}_{-\widetilde{\alpha}})^m\, e^{-\vec{k}\cdot\vec{\eta}}, & \vec{k} \in \mathcal{C}(-\alpha,\alpha) \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow$ Supp $\widehat{\psi}^{C}_{lm}$ = cone $\mathcal{C}(-\alpha,\alpha)$, $l, m \in \mathbb{N}^*, l, m \geqslant 1$, give the number of vanishing moments of $\widehat{\psi}$ on the edges of the cone

Choice of the analyzing wavelet - 3

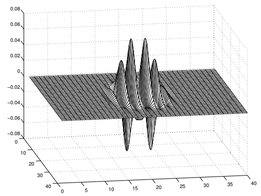
- Properties of Cauchy wavelet
  - Contrary to Morlet, opening angle $\widehat{\psi}^{C}_{lm}$ is totally controllable, independently of the amplitude
  - Good angular selectivity, but poor radial selectivity, since the exponential term decays slowly as $|\vec{k}| \to \infty$
- Alternative : Gaussian-Conical (GC) wavelet

$$\widehat{\psi}^{GC}_{lm}(\vec{k}) = \begin{cases} (\vec{k}\cdot\vec{e}_{-\widetilde{\alpha}})^l(\vec{k}\cdot\vec{e}_{\widetilde{\alpha}})^m e^{-\frac{\sigma}{2}(k_x-\chi(\sigma))^2}, & \vec{k} \in \mathcal{C}, \\ 0, \ \text{otherwise.} \end{cases}$$

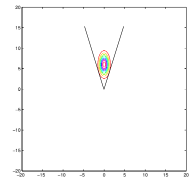
  - Central frequency $(\sqrt{l+m}, 0)$
  - $\sigma > 0$ controls the scale localization of the Gaussian
  - Center correction term, $\chi(\sigma) = \sqrt{l+m}\frac{\sigma-1}{\sigma}$, controls radial support of ψ

Choice of the analyzing wavelet - 4

- A directional wavelet : The 2-D Morlet wavelet

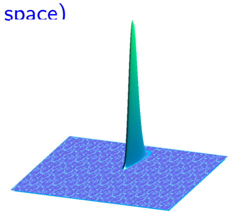


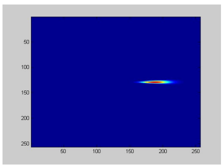
in position space



in spatial frequency space

- A very directional wavelet : The Gaussian conical wavelet (in spatial frequency space)





SPATIO-TEMPORAL WAVELETS

MOTION ANALYSIS

Introduction

- There exist many methods for motion estimation
  - Optical flow
  - Block matching
  - Phase difference
  - ...
- Alternative : motion-tuned continuous wavelet transform
  M. Duval-Destin (1991), R.Murenzi (1992), F. Mujica (1999)
- Idea : adapt to (2D+T) space-time the general formalism of the continuous wavelet transform on a manifold
  (coherent state formalism)

Extension to spatio-temporal wavelets - 1

Principle : build time dependent wavelet, separable in frequency space

$$\widehat{\psi}_{ST}(\vec{k},\omega) = \underbrace{\widehat{\psi}_S(k_x,k_y)}_{\text{2D wavelet}} \cdot \underbrace{\widehat{\psi}_T(\omega)}_{\text{1D wavelet}}$$

- Act on it by space-time (2D+T) group, containing space and time translations, space and time dilations, space rotations :
  $$G = \mathrm{SIM}(2) \times G^+_{\mathrm{aff}}$$
  via a square integrable representation
- Replace separate space and time dilations a_s, a_t by a global dilation a and a speed tuning parameter c
- The group G has a unitary irreducible representation in the space of signals (image sequences) $L^2(\mathbb{R}^2 \times \mathbb{R}, d\vec{x}\, dt)$, with norm
  $$\|s\|^2 = \iint_{\mathbb{R}^2\times\mathbb{R}} d\vec{x}\, dt\, |s(\vec{x},t)|^2 < \infty$$
  and Fourier transform
  $$\widehat{s}(\vec{k},\omega) = (2\pi)^{-3/2} \iint_{\mathbb{R}^2\times\mathbb{R}} d\vec{x}\, dt\, e^{-i(\vec{k}\cdot\vec{x}+\omega t)} s(\vec{x},t)$$

Extension to spatio-temporal wavelets - 2

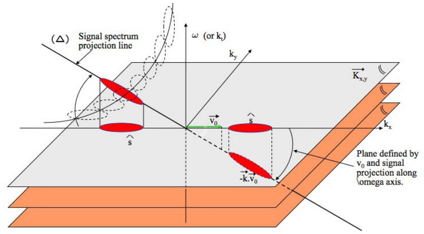
Principle : Speed detection and quantization is done in the Fourier space, because the wavelet measures the inclination of the spectrum

- Motion operators in Fourier space
  $$\begin{array}{lrcl} \text{Dilation :} & [\widehat{D}^a\, \widehat{\psi}](\vec{k},\omega) & = & a^{3/2}\widehat{\psi}(a\vec{k}, a\,\omega) \\ \text{Translation :} & [\widehat{T}^{\vec{b},\tau}\, \widehat{\psi}](\vec{k},\omega) & = & e^{-i(\vec{k}\cdot\vec{b}+\omega\tau)}\widehat{\psi}(\vec{k},\omega) \\ \text{Rotation :} & [\widehat{R}^{\theta}\, \widehat{\psi}](\vec{k},\omega) & = & \widehat{\psi}(r_{-\theta}\vec{k},\omega) \\ \text{Speed tuning :} & [\widehat{\Lambda}^{c}\, \widehat{\psi}](\vec{k},\omega) & = & \widehat{\psi}(c^q\vec{k}, c^{-p}\omega) \end{array}$$
- Constraints : $\widehat{\Lambda}^c$ must map the $\vec{v}_o$-plane, $\vec{k}\cdot\vec{v}_o + \omega = 0$, into the $c\vec{v}_o$-plane and must be unitary $\Rightarrow$ $p = 2/3$ and $q = 1/3$

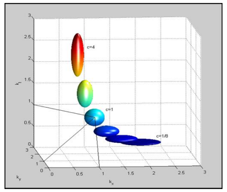
  This correspond to the psycho-visual effect :
  - Fast moving object : only large details can be detected
  - Slow moving object : small details can be detected
- Next step : choose adequate wavelets for space and time components

Principle of directional speed analysis with wavelets - 1

- Inclination of a signal spectrum with speed
  - Static object $\widehat{s}$ lives in the plane $\omega = 0$ of zero frequency
  - Object $\widehat{s}$ moving with constant speed $\vec{v}$ lives in the plane $\vec{k} \cdot \vec{v} + \omega = 0$



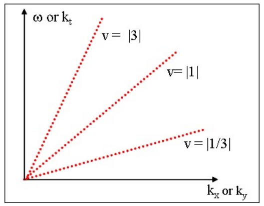
- A high speed object must be large to be "captured" and a low speed object can be small (psycho-visual effect)
  $\Rightarrow$ wavelets must be speed-tuned (distorted and elongated) to "capture" a moving objet
  $\Rightarrow$ wavelets move on hyperbola-like curve with increasing speed



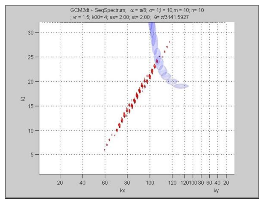
Principle of directional speed analysis with wavelets - 2

Speed analysis of an object moving at constant speed v

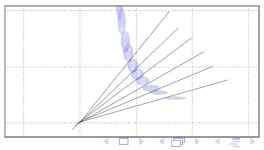
- The slope of the spectrum of the object increases for higher speed (the slope decreases in direct space $(\vec{x}, t)$)



- Capture is achieved when the signal spectrum (red) intersect the family of speed-tuned wavelets (blue)



- Example of exponential discretization of the speed parameter, chosen so as to avoid overlapping between successive wavelets along the hyperbola



The 2D+T GCM wavelet - 1

- Standard 2D+T wavelet : the Duval-Destin-Murenzi (DDM) wavelet

$$\widehat{\psi}_{DDM}(\vec{k},\omega) = \underbrace{\widehat{\psi}_M(k_x,k_y)}_{\text{2D Morlet wavelet}} \cdot \underbrace{\widehat{\psi}_M(\omega)}_{\text{1D Morlet}}$$

where the 1D Morlet wavelet is

$$\widehat{\psi}_M(\omega) = \exp(-\tfrac{1}{2}(\omega-\omega_0)^2) - h(\omega)$$

with the correction term $h(\omega)$ negligible in practice (for $\omega_0 \gtrsim 5.5$)

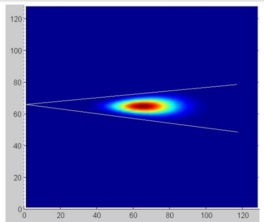
- The 2D Morlet wavelet has poor selectivity properties
  $\Rightarrow$ replace it by a Gaussian-conical wavelet and get a GCM 2D+T wavelet

$$\widehat{\psi}^{GCM}_{lm}(\vec{k},\omega) = \begin{cases} \underbrace{\widehat{\psi}^{GC}_{lm}(k_x,k_y)}_{\text{2D Gaussian-Conical}} \cdot \underbrace{\widehat{\psi}^{M}(\omega)}_{\text{1D Morlet}}, & \vec{k} \in \mathbb{C}(-\alpha,\alpha), \\ 0, & \text{otherwise,} \end{cases}$$

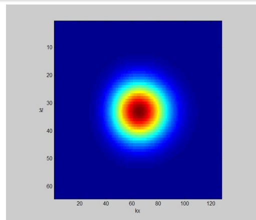
- Explicitly

$$\widehat{\psi}^{GCM}_{lm}(\vec{k},\omega) = \begin{cases} (\vec{k}\cdot\vec{e}_{-\tilde{\alpha}})^l\,(\vec{k}\cdot\vec{e}_{\tilde{\alpha}})^m\, e^{-\frac{\sigma}{2}(k_x-\chi(\sigma))^2}\, e^{-\frac{1}{2}(\omega-\omega_0)^2}, & \vec{k} \in \mathbb{C}(-\alpha,\alpha), \\ 0, & \text{otherwise} \end{cases}$$

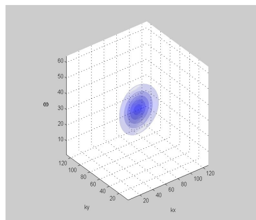
The 2D+T GCM wavelet - 2



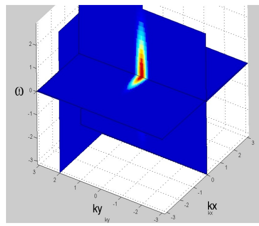
The GCM wavelet ($c = 1, \alpha = \pi/16$)



Lateral view in plane (k_x, ω)



3D view of GCM in (k_x, k_y, ω)
$c = 1$

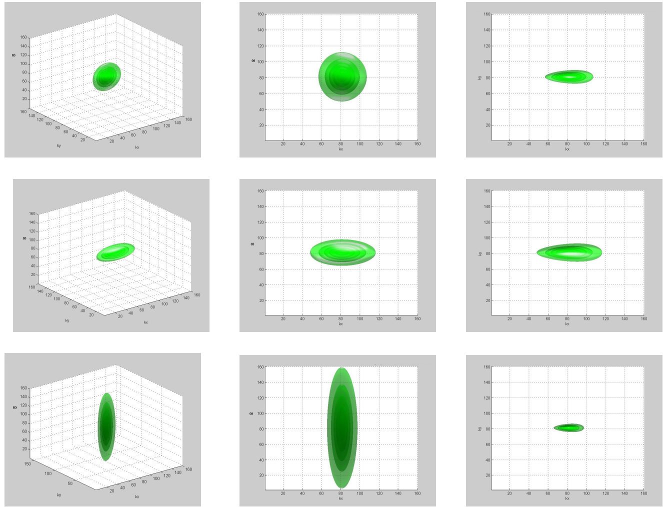


3D sectional views in (k_x, k_y, ω)
$c = 2$

The last panel shows the Gaussian behavior of the (vertical) Morlet part and the (horizontal) conical part

The 2D+T GCM wavelet - 3

The GCM wavelet tuned to different velocities c



$c = 1$, $c = 0.4$, $c = 4$

3D view | Side view in (k_x, ω) | Top view in (k_x, k_y)

Properties of the 2D+T GCM wavelet

- Central frequency and speed capture initialization
  - In order to capture the initial speed, independently of the object scale, it is useful to center the wavelet in the Fourier plane
  - i.e., translating the wave-vector $\vec{k}$ by its central frequency $\vec{k}_0$ and canceling the term ω_0 in the Morlet part

$$\vec{k}_0 = \frac{1}{a^2}\frac{\sqrt{l+m}}{c^q}(\cos\theta, \sin\theta)$$

$\Rightarrow$ Modified GCM is a simple filter (no oscillation), not a wavelet anymore !

- Angular resolving power
  - For Morlet : for $k_0 \gg 1, ARP(\psi^M) = 2\cot^{-1}(k_0\sqrt{\epsilon})$
  - For GCM : $ARP = 2\alpha$, opening angle of the cone
- Frame bounds

$$\text{Frame } \{\psi_n\} : \quad A||s||^2 \leqslant \sum_n |\langle\psi_n, s\rangle|^2 \leqslant B||s||^2, \text{ with } A > 0, B < \infty$$

Estimates for the frame bounds have been given by Murenzi for Morlet 2D, for DDM by Mujica, and these are valid for GCM as well (rather complicated!)

Analysis algorithm

We have to discretize the CWT :

- Given a sequence of N frames, corresponding to the time variables $\tau_i, i = 1, \ldots, N$, compute its discretized CWT for discrete speeds $c = c_j$: $\mathcal{W}_\psi(\vec{b}, \tau_i, \theta; a, c_j)$
- Compute the energy density of i^{th} frame $|\mathcal{W}_\psi(c_j, \tau_i)|^2$, taken as a function of speed c_j only
- For a group of N_0 frames among the N frames of the sequence, compute the total energy

$$E_{\text{tot}}(c_j) = \sum_{i\in N_0}\sum_{\vec{b}_{nm}} |\mathcal{W}_\psi(\vec{b}_{nm}, c_j, \tau_i)|^2$$

$\{\vec{b}_{nm}\}$ = discretized version of the $\vec{b}$-plane

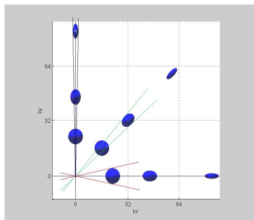
- Study the curve $f(c_j) := E_{\text{tot}}(c_j)$: maximum v_m when the speed of the tuned wavelet c_j matches the real speed v_r of the object (example of a traveling 2D Gaussian below)

Experimentation - 1

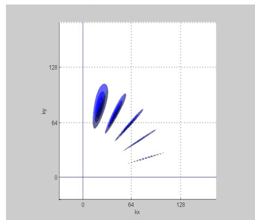
Comparative aperture adjustments : Morlet vs. GCM

- Morlet : for small k_0, ϵ, very large aperture; aperture decreases by increasing k_0, ϵ, but as k_0 increases, the Morlet wavelet moves away from the Fourier center (0,0), along its radius

  $\Rightarrow$ very difficult to adjust the spatial positioning of this wavelet w.r.t. a change in aperture selectivity



Morlet



GCM

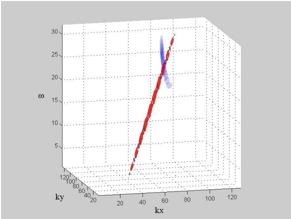
- GCM : the couple orientation/aperture is extremely simple to adjust; 5 orientations $\pi/12 \leqslant \theta_m \leqslant 5\pi/12\}$ with GCM tuned to aperture $\pi/256 \leqslant \alpha_m \leqslant \pi/16 \Rightarrow$ GCM is superior to 2D+T Morlet for object recognition and tracking by spectral signature

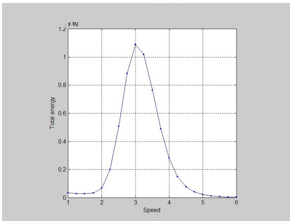
Experimentation - 2

Velocity capture comparison

- Test sequence of $128 \times 128 \times 32$, describing the motion, at constant speed $v_r = 3$ pixels/fr, of a 2D Gaussian; the angles of the Gaussian and of its trajectory are varied (along OX, at 45° and along OY)
- Isotropic Gaussian : speed capture equally good with GCM and with Morlet
- Anisotropic Gaussian, large σ_y, small σ_x (spectrum narrow along k_x) $\Rightarrow$ test the aperture selectivity of the wavelet, i.e. its accuracy in directional speed capture
- (Left) Spectrum of sequence "traveling Gaussian" (red) intersects speed-tuned GCM wavelets hyperbola-like family (blue):

  (Right) Total energy is maximum for $c = v_r = 3$

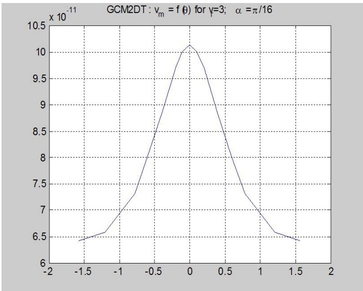




Experimentation - 3

Orientation accuracy in directional speed capture

Curve v_m vs. θ_{wav} for $-\pi/2 \leqslant \theta \leqslant +\pi/2$, with GCM



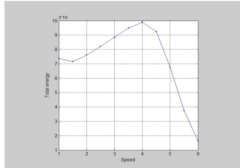
$\Rightarrow$ the correct speed is captured when the wavelet orientation ($\theta = 0$) exactly corresponds to the spectrum orientation (along k_x)

$\Rightarrow$ contrary to Morlet, GCM has good angular selectivity and is efficient at detecting the correct speed of the sequence in a very narrow angular aperture and not elsewhere

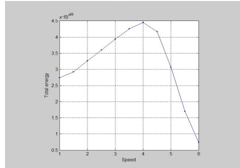
Experimentation - 4

Aperture accuracy in speed capture

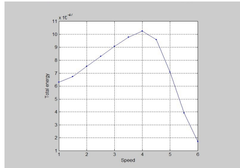
Speed capture with GCM at $v_r = 4$ and for increasingly narrow conical apertures : $\pi/8, \pi/16, \pi/64, \pi/256\,(0.70°)$



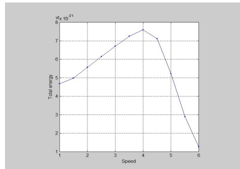
$\pi/8$



$\pi/16$



$\pi/64$



$\pi/256\,(0.70°)$

$\Rightarrow$ GCM captures the spectrum for extremely weak apertures without moving in the Fourier space

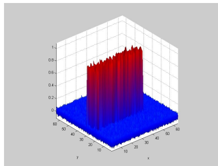
$\Rightarrow$ contrary to Morlet, GCM is very robust to scale initialization and object size detection and tracking

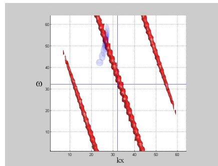
Experimentation - 5

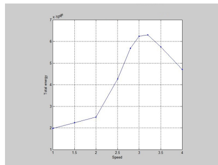
Stability with respect to noise

- Test sequence : rectangle of size 9×2 pixels in horizontal translation at a speed of 3 pixels/fr, plus white Gaussian noise
- Noise level : from an SNR of 50 dB up to 25 dB

$\Rightarrow$ correct speed captured up to a noise level of 32 dB







(Top) Noisy sequence and intersection of the sequence spectrum (red) with the hyperbola of GCM speed-tuned wavelets (blue)
(Bottom) Curve Energy $= f(c)$

Conclusion

- The GCM wavelet provides a highly directionally selective speed-tuned wavelet
- Much more powerful tool than the 2D+T Morlet wavelet for spectral signature recognition and tracking
  - Extreme efficiency in directional speed selectivity down to angle apertures of less than 1 degree ($\pi/256$)
  - Good capacity of radial stability and adjustment with respect to aperture variation
- Possible extension to curvelets and shearlets for motion analysis, but probably much more complex
- Future work : target tracking with GCM instead of DDM, speed detection/quantization in video sequences, explicit or estimated frame bounds for GCM, trajectory detection in direct space with GCM, . . .

REFERENCES

References

- I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelpia, 1992
- B. Torrésani, *Analyse continue par ondelettes*, InterÉditions/CNRS Éditions, Paris, 1995
- S.G. Mallat, *A Wavelet Tour of Signal Processing*, 2nd ed., Academic Press, San Diego, 1999
- J-P. Antoine, R. Murenzi, P. Vandergheynst and S.T. Ali, *Two-dimensional Wavelets and Their Relatives*, Cambridge Univ. Press, Cambridge (UK), 2004
- S.T. Ali, J-P. Antoine and J-P. Gazeau, *Coherent States, Wavelets and Their Generalizations*, Springer-Verlag, New York et al., 2nd ed. 2014
- Wavelet Toolbox YAWTb: `http://sites.uclouvain.be/ispgroup/yawtb/`
- P. Brault and J-P. Antoine, A spatio-temporal Gaussian-Conical wavelet with high aperture selectivity for motion and speed analysis, *Applied Comput. Harmon. Anal.* **34** (2013) 148–161 – ArXiv 1205.0437v1 [math-ph]