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Group-theoretical methods for the design and analysis of higher-dimensional wavelet systems

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Group-theoretical methods for the design and analysis of higher-dimensional wavelet systems I
Wavelet transforms associated to groups of affine mappings

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Lehrstuhl A für Mathematik, RWTH Rheinisch-Westfälische Technische Hochschule Aachen

Agenda for this week

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- Wavelet transforms associated to groups of affine mappings (Monday)

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II. Wavelet inversion, admissibility and the Plancherel formula (Tuesday)
III. Sparse signals and function spaces (Wednesday)
IV. Wavelet approximation theory over general dilation groups (Thursday, Friday)

Outline

1. 1D-CWT and the affine group

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2. Representations and wavelet transforms

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2. Representations and wavelet transforms
3. CWT in higher dimensions

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Continuous wavelet transform of one-dimensional signals

Definition 1 (Translation and dilation)

Let $\psi \in \mathrm{L}^2(\mathbb{R})$. Given $a \neq 0, b \in \mathbb{R}$, define

$$T_b : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R}) \ , \ (T_b f)(x) = f(x-b)$$

and

$$D_a : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R}) \ , \ (D_a f)(x) = |a|^{-1/2} f(a^{-1}x) \ .$$

Definition 2

Given $\psi \in \mathrm{L}^2(\mathbb{R})$, we let

$$\psi_{b,a} : \mathbb{R} \to \mathbb{C} \ , \ \psi_{b,a}(x) = T_b D_a \psi(x) = |a|^{-1/2} \psi\left(\mathfrak{x} - \mathfrak{b} a\right) \ .$$

Given $f \in \mathrm{L}^2(\mathbb{R})$, we define

$$W_\psi f : \mathbb{R} \times \mathbb{R}' \to \mathbb{C} \ , \ (b,a) \mapsto \langle f, \psi_{b,a} \rangle \ .$$

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Design systems of building blocks indexed by position and additional features (such as scale, orientation, aspect ratio etc.)

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Challenge: What are "good" choices of dilations?

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$$C_\psi = \int_{\mathbb{R}} |\widehat{\psi}(\xi)|^2 |\xi|^{-1} d\xi < \infty$$

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Then for $f \in \mathrm{L}^2(\mathbb{R})$, we have

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Informally: f is decomposed into details of varying positions and scales.

Wavelets and the affine group

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- The wavelet transform is a matrix coefficient associated to the representation,

$$W_\psi f(b,a) = \langle f, \pi(b,a)\psi \rangle\ .$$

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(b) $\psi \in \mathcal{H}_\pi$ is called weakly admissible if $V_\psi : \mathcal{H}_\pi \hookrightarrow \mathrm{L}^2(G)$ is bounded injective map, and admissible if $V_\psi : \mathcal{H}_\pi \to \mathrm{L}^2(G)$ is a nonzero scalar multiple of an isometry.

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- Note: V_ψ intertwines π with left translation. Hence, if π a weakly admissible vector, it is (equivalent to) a subrepresentation of the regular representation.

Admissibility for irreducible representations

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$$\langle C_\pi \eta', C_\pi \eta \rangle \langle \varphi, \varphi' \rangle = \langle V_\eta \varphi, V_{\eta'} \varphi' \rangle \quad , \tag{1}$$

for all $\varphi, \varphi' \in \mathcal{H}_\pi$ and $\eta, \eta' \in \mathrm{dom}(C_\pi)$.

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(d) *C_π is scalar iff G is unimodular, or equivalently, if every nonzero vector is admissible up to normalization.*

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Theorem 7

The quasi-regular representation π *of* $G = \mathbb{R} \rtimes \mathbb{R}'$ *is a discrete series representation. The associated Duflo-Moore operator is given by*

$$(C_\pi f)^\wedge(\xi) = |\xi|^{1/2}\widehat{f}(\xi) \ .$$

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- Find explicit criteria for H to be (weakly, irreducibly) admissible. Key question: Understand the dual action.
- Develop methods for the systematic construction of (weakly, irreducibly) admissible matrix groups.

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- Assume that the wavelet fulfills the Calderón condition
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Then, for all $f \in \mathrm{L}^2(\mathbb{R}^d)$, we have
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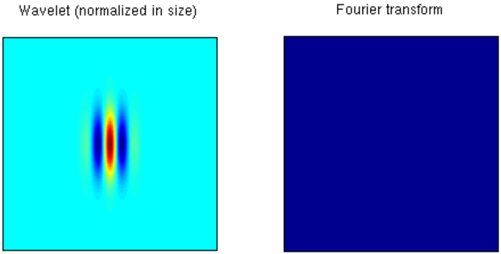
- The group is irreducibly admissible.

Similitude wavelets



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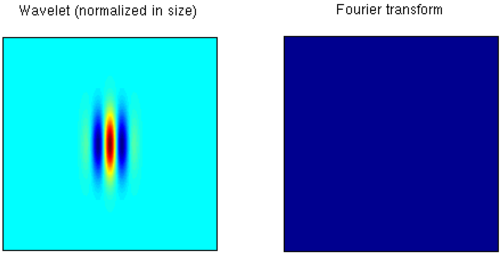

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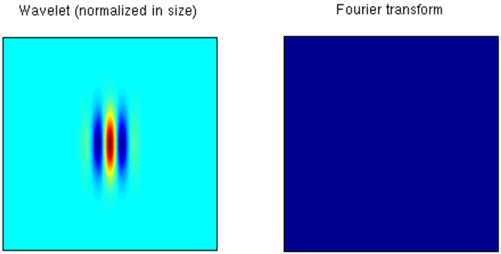

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Let $U \subset \mathbb{R}^d$ denote a Borel-measurable set. Define

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(b) *In particular: $\pi_U := \pi|_{\mathcal{H}_U}$ is an* **irreducible** *representation iff U cannot be decomposed into two H^T-invariant subsets of positive measure, i.e., iff H^T acts ergodically on U.*

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  independent of ξ.
- Ξ is continuous on $\mathcal{O}$, thus the integral becomes finite as soon as $\widehat{\psi}$ is bounded with compact support inside $\mathcal{O}$.

Sketch of proof

Proof of sufficiency part in (a), and of part (b):

- Assume there exists a conull open orbit with compact stabilizer. Then π is irreducible by Lemma 10.
- Let $\xi \in \mathcal{O}$ be arbitrary. Then the image map μ_ξ of left Haar measure on H under the projection map $h \mapsto h^T\xi$ is a well-defined Radon measure on the orbit $\mathcal{O}$, and independent of the choice of ξ, i.e. $\mu_\xi = \mu_{\xi_0}$.
- μ_{ξ_0} is Lebesgue-absolutely continuous, with Radon-Nikodym-derivative Ξ. Thus, for arbitrary $\xi \in \mathcal{O}$,

$$\begin{aligned} \int_H |\widehat{\psi}(h^T\xi)|^2 dh &= \int_{\mathcal{O}} |\widehat{\xi'}|^2 d\mu_\xi(\xi') \\ &= \int_{\mathcal{O}} |\widehat{\xi'}|^2 \Xi(\xi') d\xi' \ , \end{aligned}$$

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- Ξ is continuous on $\mathcal{O}$, thus the integral becomes finite as soon as $\widehat{\psi}$ is bounded with compact support inside $\mathcal{O}$. Then Lemma 11 entails admissibility for ψ.

Back to the examples

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- In all cases, the admissibility conditions can be obtained by applying Theorem 12.

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- Generally: If $H < \mathrm{GL}(d, \mathbb{R})$ is compact, it is not irreducibly admissible.

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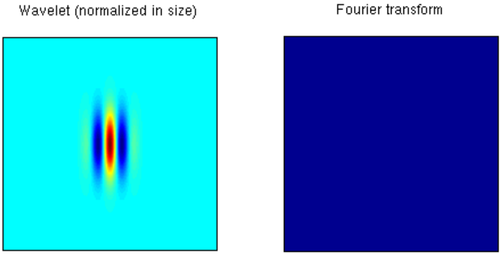
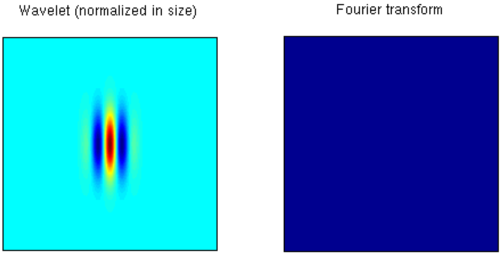
Note: Up to choice of coordinates, this list is (essentially) complete!

Similitude wavelets



Shearlets



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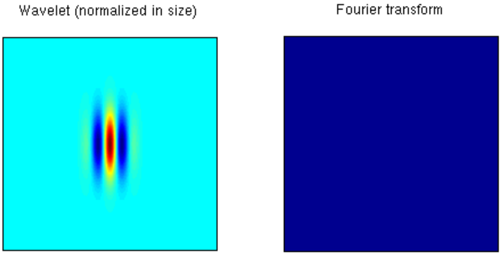
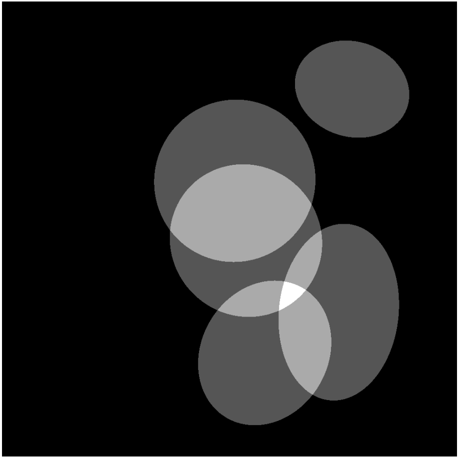


CWT: Test image



CWT over similitude group

Shearlet analysis

CWT over diagonal group

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- Main tool from representation theory (so far): Discrete series representations.
- Main tool from Fourier analysis: Plancherel theorem, and the dual action.
- For nonirreducible setting: Need a better understanding of representation theory, and of the dual action.

Outline

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