The Abdus Salam International Centre for Theoretical Physics
50th Anniversary 1964-2014

2585-5

Joint ICTP-TWAS School on Coherent State Transforms, Time-Frequency and Time-Scale Analysis, Applications

2 - 20 June 2014

Group-theoretical methods for the design and analysis of higher-dimensional wavelet systems contd.

H. Fuhr
*RWTH, Aachen
Germany*

Group-theoretical methods for the design and analysis of higher-dimensional wavelet systems II
Wavelet inversion, admissibility and the Plancherel formula

Hartmut Führ
fuehr@matha.rwth-aachen.de

Trieste, June 2014

Lehrstuhl A für Mathematik, RWTH Rheinisch-Westfälische Technische Hochschule Aachen

Agenda for this week

Agenda for this week

I Wavelet transforms associated to groups of affine mappings (Monday)

Agenda for this week

I Wavelet transforms associated to groups of affine mappings (Monday)

II Wavelet inversion, admissibility and the Plancherel formula (Tuesday)

Agenda for this week

I Wavelet transforms associated to groups of affine mappings (Monday)

II Wavelet inversion, admissibility and the Plancherel formula (Tuesday)

III Sparse signals and function spaces (Wednesday)

Agenda for this week

I Wavelet transforms associated to groups of affine mappings (Monday)

II Wavelet inversion, admissibility and the Plancherel formula (Tuesday)

III Sparse signals and function spaces (Wednesday)

IV Wavelet approximation theory over general dilation groups (Thursday, Friday)

Outline

1. Some examples

Outline

1. Some examples
2. Admissibility for reducible actions

Outline

1. Some examples
2. Admissibility for reducible actions
3. A toy example

Outline

Outline

Outline

Overview

Recall general setup: d-dimensional CWT

- $H < \mathrm{GL}(d, \mathbb{R})$, a closed matrix group
- $G = \mathbb{R}^d \rtimes H$, the affine group generated by H and translations. As a set, $G = \mathbb{R}^d \times H$, with group law

$$(x, h)(y, g) = (x + hy, hg) \ .$$

- Modular function: $\Delta_G(x, h) = \Delta_G(h) = \Delta_H(h)/|\det h|$.
- Define the translation and dilation operators via

$$(T_x f)(y) = f(y - x) \ , \ (D_h f)(y) = |\mathrm{det}(h)|^{-1/2} f(h^{-1}y) \ .$$

- Quasi-regular representation of G acts on $\mathrm{L}^2(\mathbb{R}^d)$ via

$$(\pi(x, h)f)(y) = |\det(h)|^{-1/2} f(h^{-1}(y - x)) \ .$$

- Continuous wavelet transform: Given $f, \psi \in \mathrm{L}^2(\mathbb{R}^d)$, we let

$$\mathcal{W}_\psi f : G \to \mathbb{C} \ , \ \mathcal{W}_\psi f(x, h) = \langle f, \pi(x, h)\psi \rangle \ .$$

- Dual orbit space $\mathbb{R}^d / H^T$

Admissibility condition

Lemma 1 (Recall from Talk I)

Let $H < \mathrm{GL}(\mathbb{R}^d)$ be a closed matrix group, and $\psi, f \in \mathrm{L}^2(\mathbb{R}^d)$. Then

$$\|W_\psi f\|_2^2 = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \int_H |\widehat{\psi}(h^T \xi)|^2 dh \, d\xi \; .$$

In particular, letting

$$\Phi : \mathbb{R}^d \to \mathbb{R}^+ \cup \infty \; , \; \xi \mapsto \int_H |\widehat{\psi}(h^T \xi)|^2 dh$$

we have that ψ is

- *weakly admissible iff Φ is bounded and almost nowhere vanishing;*
- *admissible iff Φ is a constant map.*

Admissibility condition

Lemma 1 (Recall from Talk I)

Let $H < \mathrm{GL}(\mathbb{R}^d)$ *be a closed matrix group, and* $\psi, f \in \mathrm{L}^2(\mathbb{R}^d)$*. Then*

$$\|W_\psi f\|_2^2 = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \int_H |\widehat{\psi}(h^T \xi)|^2 dh \, d\xi \ .$$

In particular, letting

$$\Phi : \mathbb{R}^d \to \mathbb{R}^+ \cup \infty \ , \ \xi \mapsto \int_H |\widehat{\psi}(h^T \xi)|^2 dh$$

we have that ψ *is*

- *weakly admissible iff* Φ *is bounded and almost nowhere vanishing;*
- *admissible iff* Φ *is a constant map.*

Note: This is also applicable to reducible group actions.

Semidiscrete wavelet transform (Mallat/Zhong)

Semidiscrete wavelet transform (Mallat/Zhong)

- Consider $H = 2^{\mathbb{Z}} < \mathbb{R}'$, and the semidirect product $G = \mathbb{R} \rtimes H$, acting on $\mathrm{L}^2(\mathbb{R})$ through the quasi-regular representation.

Semidiscrete wavelet transform (Mallat/Zhong)

- Consider $H = 2^{\mathbb{Z}} < \mathbb{R}'$, and the semidirect product $G = \mathbb{R} \rtimes H$, acting on $\mathrm{L}^2(\mathbb{R})$ through the quasi-regular representation.
- Pick $\psi \in \mathrm{L}^2(\mathbb{R})$ such that

$$\forall_{a.e.} \xi \in \mathbb{R} \ : \ \sum_{k \in \mathbb{Z}} |\widehat{\psi}(2^k \xi)|^2 = 1$$

Semidiscrete wavelet transform (Mallat/Zhong)

- Consider $H = 2^{\mathbb{Z}} < \mathbb{R}'$, and the semidirect product $G = \mathbb{R} \rtimes H$, acting on $\mathrm{L}^2(\mathbb{R})$ through the quasi-regular representation.
- Pick $\psi \in \mathrm{L}^2(\mathbb{R})$ such that

$$\forall_{a.e.} \xi \in \mathbb{R} \ : \ \sum_{k \in \mathbb{Z}} |\widehat{\psi}(2^k \xi)|^2 = 1$$

- Then ψ is admissible for the quasiregular representation of G, yielding

$$\forall f \in \mathrm{L}^2(\mathbb{R}) \ : \ \|f\|_2^2 = \sum_{k \in \mathbb{Z}} 2^{-k} \int_{\mathbb{R}} |W_\psi f(x, 2^k)|^2 dx \ .$$

Semidiscrete wavelet transform (Mallat/Zhong)

- Consider $H = 2^{\mathbb{Z}} < \mathbb{R}'$, and the semidirect product $G = \mathbb{R} \rtimes H$, acting on $\mathrm{L}^2(\mathbb{R})$ through the quasi-regular representation.
- Pick $\psi \in \mathrm{L}^2(\mathbb{R})$ such that

$$\forall_{a.e.} \xi \in \mathbb{R} \ : \ \sum_{k \in \mathbb{Z}} |\widehat{\psi}(2^k \xi)|^2 = 1$$

- Then ψ is admissible for the quasiregular representation of G, yielding

$$\forall f \in \mathrm{L}^2(\mathbb{R}) \ : \ \|f\|_2^2 = \sum_{k \in \mathbb{Z}} 2^{-k} \int_{\mathbb{R}} |W_\psi f(x, 2^k)|^2 dx \ .$$

- Note: π does not have irreducible subrepresentations.

Wavelets over $E(2)$ (Isham/Klauder)

- Consider the dilation group $H = \mathrm{SO}(2)$, and $G = E(2) = \mathbb{R}^2 \rtimes H$.

Wavelets over $E(2)$ (Isham/Klauder)

- Consider the dilation group $H = \mathrm{SO}(2)$, and $G = E(2) = \mathbb{R}^2 \rtimes H$.
- Fix measurable $R \subset \mathbb{R}^+$, and let

$$U = \{\xi \in \mathbb{R}^2 \ : \ |\xi| \in R\} \ , \mathcal{H}_U = \{f \in \mathrm{L}^2(\mathbb{R}^d) : \widehat{f} \cdot \mathbf{1}_U = \widehat{f}\}$$

Wavelets over $E(2)$ (Isham/Klauder)

- Consider the dilation group $H = \mathrm{SO}(2)$, and $G = E(2) = \mathbb{R}^2 \rtimes H$.
- Fix measurable $R \subset \mathbb{R}^+$, and let

$$U = \{\xi \in \mathbb{R}^2 \ : \ |\xi| \in R\} \ , \mathcal{H}_U = \{f \in \mathrm{L}^2(\mathbb{R}^d) : \widehat{f} \cdot \mathbf{1}_U = \widehat{f}\}$$

- Let $\xi_0 = (1,0)^T$, and suppose that $\psi \in \mathcal{H}$ fulfills

$$\forall_{a.e.} r \in R \ : \ \int_0^{2\pi} \left| \widehat{\psi}(r h_\theta \xi_0) \right|^2 d\theta = 1 \ .$$

$h_\theta =$ rotation by angle θ.

Wavelets over $E(2)$ (Isham/Klauder)

- Consider the dilation group $H = \mathrm{SO}(2)$, and $G = E(2) = \mathbb{R}^2 \rtimes H$.
- Fix measurable $R \subset \mathbb{R}^+$, and let

$$U = \{\xi \in \mathbb{R}^2 \ : \ |\xi| \in R\} \ , \mathcal{H}_U = \{f \in \mathrm{L}^2(\mathbb{R}^d) : \widehat{f} \cdot \mathbf{1}_U = \widehat{f}\}$$

- Let $\xi_0 = (1,0)^T$, and suppose that $\psi \in \mathcal{H}$ fulfills

$$\forall_{a.e.} r \in R \ : \ \int_0^{2\pi} \left| \widehat{\psi}(r h_\theta \xi_0) \right|^2 d\theta = 1 \ .$$

$h_\theta =$ rotation by angle θ.

- Then ψ is admissible for $\mathcal{H}_U$, yielding

$$\forall f \in \mathcal{H}_U \ : \ \|f\|_2^2 = \int_0^{2\pi} \int_{\mathbb{R}} |W_\psi f(x,\theta)|^2 dx d\theta \ .$$

Wavelets over $E(2)$ (Isham/Klauder)

- Consider the dilation group $H = \mathrm{SO}(2)$, and $G = E(2) = \mathbb{R}^2 \rtimes H$.
- Fix measurable $R \subset \mathbb{R}^+$, and let

$$U = \{\xi \in \mathbb{R}^2 \ : \ |\xi| \in R\} \ , \mathcal{H}_U = \{f \in \mathrm{L}^2(\mathbb{R}^d) : \widehat{f} \cdot \mathbf{1}_U = \widehat{f}\}$$

- Let $\xi_0 = (1,0)^T$, and suppose that $\psi \in \mathcal{H}$ fulfills

$$\forall_{a.e.} r \in R \ : \ \int_0^{2\pi} \left| \widehat{\psi}(r h_\theta \xi_0) \right|^2 d\theta = 1 \ .$$

  $h_\theta =$ rotation by angle θ.
- Then ψ is admissible for $\mathcal{H}_U$, yielding

$$\forall f \in \mathcal{H}_U \ : \ \|f\|_2^2 = \int_0^{2\pi} \int_{\mathbb{R}} |W_\psi f(x,\theta)|^2 dx d\theta \ .$$

- Note:

Wavelets over $E(2)$ (Isham/Klauder)

- Consider the dilation group $H = \mathrm{SO}(2)$, and $G = E(2) = \mathbb{R}^2 \rtimes H$.
- Fix measurable $R \subset \mathbb{R}^+$, and let

$$U = \{\xi \in \mathbb{R}^2 \ : \ |\xi| \in R\} \ , \mathcal{H}_U = \{f \in \mathrm{L}^2(\mathbb{R}^d) : \widehat{f} \cdot \mathbf{1}_U = \widehat{f}\}$$

- Let $\xi_0 = (1,0)^T$, and suppose that $\psi \in \mathcal{H}$ fulfills

$$\forall_{a.e.} r \in R \ : \ \int_0^{2\pi} \left| \widehat{\psi}(r h_\theta \xi_0) \right|^2 d\theta = 1 \ .$$

$h_\theta =$ rotation by angle θ.
- Then ψ is admissible for $\mathcal{H}_U$, yielding

$$\forall f \in \mathcal{H}_U \ : \ \|f\|_2^2 = \int_0^{2\pi} \int_{\mathbb{R}} |W_\psi f(x,\theta)|^2 dx d\theta \ .$$

- Note:
  - Any subset $R' \subset R$ gives rise to a smaller subspace $\mathcal{H}_{U'}$, hence π_U does not have irreducible subrepresentations.

Wavelets over $E(2)$ (Isham/Klauder)

- Consider the dilation group $H = \mathrm{SO}(2)$, and $G = E(2) = \mathbb{R}^2 \rtimes H$.
- Fix measurable $R \subset \mathbb{R}^+$, and let

$$U = \{\xi \in \mathbb{R}^2 \ : \ |\xi| \in R\} \ , \mathcal{H}_U = \{f \in \mathrm{L}^2(\mathbb{R}^d) : \widehat{f} \cdot \mathbf{1}_U = \widehat{f}\}$$

- Let $\xi_0 = (1,0)^T$, and suppose that $\psi \in \mathcal{H}$ fulfills

$$\forall_{a.e.} r \in R \ : \ \int_0^{2\pi} \left| \widehat{\psi}(r h_\theta \xi_0) \right|^2 d\theta = 1 \ .$$

  $h_\theta =$ rotation by angle θ.
- Then ψ is admissible for $\mathcal{H}_U$, yielding

$$\forall f \in \mathcal{H}_U \ : \ \|f\|_2^2 = \int_0^{2\pi} \int_{\mathbb{R}} |W_\psi f(x,\theta)|^2 dx d\theta \ .$$

- Note:
  - Any subset $R' \subset R$ gives rise to a smaller subspace $\mathcal{H}_{U'}$, hence π_U does not have irreducible subrepresentations.
  - The admissibility criterion is only fulfillable if $\int_R r dr < \infty$.

Overview

A general question

General problem

A general question

General problem

- Given H, decide whether H is (weakly) admissible.

A general question

General problem

- Given H, decide whether H is (weakly) admissible.
- Provide representation-theoretic explanation. (Note: Duflo/Moore is no longer applicable.)

A general question

General problem

- Given H, decide whether H is (weakly) admissible.
- Provide representation-theoretic explanation. (Note: Duflo/Moore is no longer applicable.)

Starting point

Assume that H is weakly admissible. Then there exists an integrable function $\varphi : \mathbb{R}^d \to \mathbb{R}^+$ such that

$$\forall_{a.e.} \xi \in \mathbb{R}^d \ : \ 0 < \int_H \varphi(h^T \xi) d\xi < \infty \ .$$

A general question

General problem

- Given H, decide whether H is (weakly) admissible.
- Provide representation-theoretic explanation. (Note: Duflo/Moore is no longer applicable.)

Starting point

Assume that H is weakly admissible. Then there exists an integrable function $\varphi : \mathbb{R}^d \to \mathbb{R}^+$ such that

$$\forall_{a.e.} \xi \in \mathbb{R}^d \ : \ 0 < \int_H \varphi(h^T \xi) d\xi < \infty \ .$$

(Recall admissibility criterion from last talk, with $\varphi = |\widehat{\psi}|^2$.)
What does that tell us about H?

A general question

General problem

- Given H, decide whether H is (weakly) admissible.
- Provide representation-theoretic explanation. (Note: Duflo/Moore is no longer applicable.)

Starting point

Assume that H is weakly admissible. Then there exists an integrable function $\varphi : \mathbb{R}^d \to \mathbb{R}^+$ such that

$$\forall_{a.e.} \xi \in \mathbb{R}^d \ : \ 0 < \int_H \varphi(h^T \xi) d\xi < \infty \ .$$

(Recall admissibility criterion from last talk, with $\varphi = |\widehat{\psi}|^2$.)
What does that tell us about H?
($\rightsquigarrow$ Regularity properties of the dual action!)

Regularity of the orbit space

Definition 2

We let $\mathbb{R}^d/H^T$ denote the space of orbits under the dual action, endowed with the quotient Borel structure.

Regularity of the orbit space

Definition 2

We let $\mathbb{R}^d/H^T$ denote the space of orbits under the dual action, endowed with the quotient Borel structure.
$\mathbb{R}^d/H^T$ admits a λ-transversal if there exists an H^T-invariant λ-conull Borel set $Y \subset \mathbb{R}^d$ and a Borel set $C \subset Y$ meeting each orbit in Y in precisely one point.

Regularity of the orbit space

Definition 2

We let $\mathbb{R}^d/H^T$ denote the space of orbits under the dual action, endowed with the quotient Borel structure.
$\mathbb{R}^d/H^T$ admits a λ-transversal if there exists an H^T-invariant λ-conull Borel set $Y \subset \mathbb{R}^d$ and a Borel set $C \subset Y$ meeting each orbit in Y in precisely one point.

Chief purpose of this condition

Exclude pathological behaviour (proper ergodicity etc.)
If a transversal T of the orbit exists, we can identify T with the orbit space $\mathbb{R}^d/H^T$.

Measure decompositions

Definition 3

Measure decompositions

Definition 3

(a) A measurable family of measures is a family $(\beta_{\mathcal{O}})_{\mathcal{O} \in \mathbb{R}^d/H^T}$, such that for all Borel sets $B \subset X$, the map $\mathcal{O} \mapsto \beta_{\mathcal{O}}(B)$ is Borel on X/H.

Measure decompositions

Definition 3

(a) A measurable family of measures is a family $(\beta_{\mathcal{O}})_{\mathcal{O}\in\mathbb{R}^d/H^T}$, such that for all Borel sets $B \subset X$, the map $\mathcal{O} \mapsto \beta_{\mathcal{O}}(B)$ is Borel on X/H.

(b) A measure decomposition of λ is a pair $\left(\overline{\lambda}, (\beta_{\mathcal{O}})_{\mathcal{O}\subset\mathbb{R}^d}\right)$, with $\overline{\lambda}$ a suitable measure on $\mathbb{R}^d/H^T$, and a family $(\beta_{\mathcal{O}})_{\mathcal{O}\in\mathbb{R}^d/H^T}$ of measures such that for all $B \subset \mathbb{R}^d$ Borel,

$$\lambda(B) = \int_{\mathbb{R}^d/H^T} \beta_{\mathcal{O}}(B) d\overline{\lambda}(\mathcal{O}) \ .$$

Measure decompositions

Definition 3

(a) A measurable family of measures is a family $(\beta_{\mathcal{O}})_{\mathcal{O}\in\mathbb{R}^d/H^T}$, such that for all Borel sets $B \subset X$, the map $\mathcal{O} \mapsto \beta_{\mathcal{O}}(B)$ is Borel on X/H.

(b) A measure decomposition of λ is a pair $\left(\overline{\lambda}, (\beta_{\mathcal{O}})_{\mathcal{O}\subset\mathbb{R}^d}\right)$, with $\overline{\lambda}$ a suitable measure on $\mathbb{R}^d/H^T$, and a family $(\beta_{\mathcal{O}})_{\mathcal{O}\in\mathbb{R}^d/H^T}$ of measures such that for all $B \subset \mathbb{R}^d$ Borel,

$$\lambda(B) = \int_{\mathbb{R}^d/H^T} \beta_{\mathcal{O}}(B) d\overline{\lambda}(\mathcal{O}) \ .$$

(c) Lebesgue measure decomposes over the orbits if there exists a measure decomposition such that, for $\overline{\lambda}$-almost every $\mathcal{O} \in \mathbb{R}^d/H^T$, the measure $\beta_{\mathcal{O}}$ is supported in $\mathcal{O}$.

Measure decomposition and transversals: Example

Measure decomposition and transversals: Example

- Consider the group $H = 2^{\mathbb{Z}}$.

Measure decomposition and transversals: Example

- Consider the group $H = 2^{\mathbb{Z}}$.
- The H^T-orbit of $\xi \in \mathbb{R}$ is given by $\mathcal{O} = \{2^k \xi : k \in \mathbb{Z}\}$.

Measure decomposition and transversals: Example

- Consider the group $H = 2^{\mathbb{Z}}$.
- The H^T-orbit of $\xi \in \mathbb{R}$ is given by $\mathcal{O} = \{2^k \xi : k \in \mathbb{Z}\}$.
- Transversal for the orbit space: $T = [1,2) \cup (-2,-1]$.

Measure decomposition and transversals: Example

- Consider the group $H = 2^{\mathbb{Z}}$.
- The H^T-orbit of $\xi \in \mathbb{R}$ is given by $\mathcal{O} = \{2^k \xi : k \in \mathbb{Z}\}$.
- Transversal for the orbit space: $T = [1,2) \cup (-2,-1]$.
- We let $\beta_{\mathcal{O}}$ denote counting measure, weighted by 2^k, and let $\overline{\lambda}$ be Lebesgue measure on T

Measure decomposition and transversals: Example

- Consider the group $H = 2^{\mathbb{Z}}$.
- The H^T-orbit of $\xi \in \mathbb{R}$ is given by $\mathcal{O} = \{2^k \xi : k \in \mathbb{Z}\}$.
- Transversal for the orbit space: $T = [1,2) \cup (-2,-1]$.
- We let $\beta_{\mathcal{O}}$ denote counting measure, weighted by 2^k, and let $\overline{\lambda}$ be Lebesgue measure on T
- Then, for all Borel sets $B \subset \mathbb{R}$:

$$\lambda(B) \ = \ \sum_{k \in \mathbb{Z}} \lambda(B \cap 2^k T)$$

Measure decomposition and transversals: Example

- Consider the group $H = 2^{\mathbb{Z}}$.
- The H^T-orbit of $\xi \in \mathbb{R}$ is given by $\mathcal{O} = \{2^k \xi : k \in \mathbb{Z}\}$.
- Transversal for the orbit space: $T = [1,2) \cup (-2,-1]$.
- We let $\beta_{\mathcal{O}}$ denote counting measure, weighted by 2^k, and let $\overline{\lambda}$ be Lebesgue measure on T
- Then, for all Borel sets $B \subset \mathbb{R}$:

$$\begin{aligned} \lambda(B) &= \sum_{k \in \mathbb{Z}} \lambda(B \cap 2^k T) \\ &= \sum_{k \in \mathbb{Z}} 2^k \int_{\pm[1,2)} \mathbf{1}_B(2^k \xi) d\xi \end{aligned}$$

Measure decomposition and transversals: Example

- Consider the group $H = 2^{\mathbb{Z}}$.
- The H^T-orbit of $\xi \in \mathbb{R}$ is given by $\mathcal{O} = \{2^k \xi : k \in \mathbb{Z}\}$.
- Transversal for the orbit space: $T = [1,2) \cup (-2,-1]$.
- We let $\beta_{\mathcal{O}}$ denote counting measure, weighted by 2^k, and let $\overline{\lambda}$ be Lebesgue measure on T
- Then, for all Borel sets $B \subset \mathbb{R}$:

$$\begin{aligned}
\lambda(B) &= \sum_{k \in \mathbb{Z}} \lambda(B \cap 2^k T) \\
&= \sum_{k \in \mathbb{Z}} 2^k \int_{\pm[1,2)} \mathbf{1}_B(2^k \xi) d\xi \\
&= \int_{\mathbb{R}/H^T} \beta_{\mathcal{O}}(B) d\overline{\lambda}(\mathcal{O}) \ .
\end{aligned}$$

Measure decomposition and transversals: Second example

Measure decomposition and transversals: Second example

- Consider the group $H = \mathrm{SO}(2)$.

Measure decomposition and transversals: Second example

- Consider the group $H = \mathrm{SO}(2)$.
- The H^T-orbit $\mathcal{O}$ of $\xi \in \mathbb{R}^2$ is given by the circle with center 0 and radius $|\xi|$.

Measure decomposition and transversals: Second example

- Consider the group $H = \mathrm{SO}(2)$.
- The H^T-orbit $\mathcal{O}$ of $\xi \in \mathbb{R}^2$ is given by the circle with center 0 and radius $|\xi|$.
- Transversal to the orbit space: $\mathbb{R}^+ \cdot (1,0)^T$.

Measure decomposition and transversals: Second example

- Consider the group $H = \mathrm{SO}(2)$.
- The H^T-orbit $\mathcal{O}$ of $\xi \in \mathbb{R}^2$ is given by the circle with center 0 and radius $|\xi|$.
- Transversal to the orbit space: $\mathbb{R}^+ \cdot (1,0)^T$.
- We let $\beta_{\mathcal{O}}$ denote the rotation invariant measure on the sphere, and let $\overline{\lambda}$ be given by rdr.

Measure decomposition and transversals: Second example

- Consider the group $H = \mathrm{SO}(2)$.
- The H^T-orbit $\mathcal{O}$ of $\xi \in \mathbb{R}^2$ is given by the circle with center 0 and radius $|\xi|$.
- Transversal to the orbit space: $\mathbb{R}^+ \cdot (1,0)^T$.
- We let $\beta_{\mathcal{O}}$ denote the rotation invariant measure on the sphere, and let $\overline{\lambda}$ be given by rdr.
- Then, for all Borel sets $B \subset \mathbb{R}^2$:

$$\lambda(B) = \int_{\mathbb{R}^+} \int_{|\xi|=r} \mathbf{1}_B(\xi) d\xi \; rdr$$

Measure decomposition and transversals: Second example

- Consider the group $H = \mathrm{SO}(2)$.
- The H^T-orbit $\mathcal{O}$ of $\xi \in \mathbb{R}^2$ is given by the circle with center 0 and radius $|\xi|$.
- Transversal to the orbit space: $\mathbb{R}^+ \cdot (1,0)^T$.
- We let $\beta_{\mathcal{O}}$ denote the rotation invariant measure on the sphere, and let $\overline{\lambda}$ be given by rdr.
- Then, for all Borel sets $B \subset \mathbb{R}^2$:

$$\begin{aligned} \lambda(B) &= \int_{\mathbb{R}^+} \int_{|\xi|=r} \mathbf{1}_B(\xi) d\xi \; rdr \\ &= \int_{\mathbb{R}^2/H^T} \beta_{\mathcal{O}}(B) d\overline{\lambda}(\mathcal{O}) \; . \end{aligned}$$

A characterization of admissibility

Theorem 4 (HF, '10)

Let $H < \mathrm{GL}(d, \mathbb{R})$ be closed. Then H is weakly admissible iff only almost every stabilizer is compact, and in addition, one of the following equivalent conditions hold:

(a) *λ decomposes over the orbits.*

(b) *$\mathbb{R}^d / H^T$ admits a λ-transversal.*

A characterization of admissibility

Theorem 4 (HF, '10)

Let $H < \mathrm{GL}(d, \mathbb{R})$ be closed. Then H is weakly admissible iff only almost every stabilizer is compact, and in addition, one of the following equivalent conditions hold:

(a) *λ decomposes over the orbits.*

(b) *$\mathbb{R}^d / H^T$ admits a λ-transversal.*

Theorem 5 (HF, '10)

Let $H < \mathrm{GL}(d, \mathbb{R})$ be closed. Then H is admissible iff H is weakly admissible and in addition, G is nonunimodular.

Summary: Wavelet inversion without irreducibility

Summary: Wavelet inversion without irreducibility

- Admissibility condition is useful also in the reducible case.

Summary: Wavelet inversion without irreducibility

- Admissibility condition is useful also in the reducible case.
- Usually, instead of a single integral condition infinitely many (usually uncountably many) such conditions are required.

Summary: Wavelet inversion without irreducibility

- Admissibility condition is useful also in the reducible case.
- Usually, instead of a single integral condition infinitely many (usually uncountably many) such conditions are required.
- Weak admissibility $\Leftrightarrow$ well-behaved dual orbit space

Summary: Wavelet inversion without irreducibility

- Admissibility condition is useful also in the reducible case.
- Usually, instead of a single integral condition infinitely many (usually uncountably many) such conditions are required.
- Weak admissibility $\Leftrightarrow$ well-behaved dual orbit space
- Strong admissibility $\Leftrightarrow$ well-behaved dual orbit space and non-unimodularity

Summary: Wavelet inversion without irreducibility

- Admissibility condition is useful also in the reducible case.
- Usually, instead of a single integral condition infinitely many (usually uncountably many) such conditions are required.
- Weak admissibility $\Leftrightarrow$ well-behaved dual orbit space
- Strong admissibility $\Leftrightarrow$ well-behaved dual orbit space and non-unimodularity
- Abstract admissibility criteria $\rightsquigarrow$ Plancherel theory, see remainder of this set of slides.

Overview

Towards a representation-theoretic view

Abstract harmonic approach to admissibility

Towards a representation-theoretic view

Abstract harmonic approach to admissibility

- Recall: If a representation has a weakly admissible vector, it is contained in the regular representation.

Towards a representation-theoretic view

Abstract harmonic approach to admissibility

- Recall: If a representation has a weakly admissible vector, it is contained in the regular representation.
- Thus, we need to understand
  - leftinvariant subspaces of $\mathrm{L}^2(G)$,
  - and their admissible vectors.

Towards a representation-theoretic view

Abstract harmonic approach to admissibility

- Recall: If a representation has a weakly admissible vector, it is contained in the regular representation.
- Thus, we need to understand
  - leftinvariant subspaces of $\mathrm{L}^2(G)$,
  - and their admissible vectors.
- For well-behaved groups, representations and their invariant subspaces are best understood in terms of their decomposition into irreducibles.

Towards a representation-theoretic view

Abstract harmonic approach to admissibility

- Recall: If a representation has a weakly admissible vector, it is contained in the regular representation.
- Thus, we need to understand
  - leftinvariant subspaces of $\mathrm{L}^2(G)$,
  - and their admissible vectors.
- For well-behaved groups, representations and their invariant subspaces are best understood in terms of their decomposition into irreducibles.
- $\rightsquigarrow$ Plancherel theory!

Toy example $G = \mathbb{R}$: Invariant subspaces

- Let $G = \mathbb{R}$, $\mathcal{H} \subset \mathrm{L}^2(\mathbb{R})$ translation-invariant closed subspace.
- **Theorem:** $T : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$ bounded, translationinvariant $\Longrightarrow \exists$ unique $\widehat{T} \in \mathrm{L}^\infty(\widehat{\mathbb{R}})$ such that $(Tf)^\wedge(\omega) = \widehat{T}(\omega)\widehat{f}(\omega)$.
- Applied to projection P onto $\mathcal{H}$: There exists a measurable $U \subset \widehat{\mathbb{R}}$, unique up to nullsets, such that $(Pf)^\wedge(\omega) = \chi_U(\omega)\widehat{f}(\omega)$, or

$$\mathcal{H} = \mathcal{H}_U = \{f \in \mathrm{L}^2(\mathbb{R}) : \mathrm{supp}(\widehat{f}) \subset U\}$$

Toy example $G = \mathbb{R}$: Admissible vectors

- $\mathcal{H} = \mathcal{H}_U$ as on the previous slide, $\eta \in \mathcal{H}$. Then $V_\eta \phi = \phi * \eta^*$, and the convolution theorem yields

$$(V_\eta \phi)^\wedge(\omega) = \widehat{\phi}(\omega)\overline{\widehat{\eta}(\omega)} \quad .$$

- In particular,

$$\begin{aligned} \eta \text{ admissible} \quad &\Leftrightarrow \quad \widehat{\phi} \mapsto \widehat{\phi}\overline{\widehat{\eta}} \text{ is an isometry on } \mathrm{L}^2(U) \\ &\Leftrightarrow \quad |\eta(\omega)| = 1 \text{ a.e. on } U \quad . \end{aligned}$$

- $\Longrightarrow \mathcal{H}_U$ has admissible vectors iff $|U| < \infty$

Toy example $G = \mathbb{R}$: CWT and Plancherel inversion

- For $U \subset \mathbb{R}$ measurable, with $|U| < \infty$, let π_U be the restriction of left regular representation to $\mathcal{H}_U$.
- Under the Plancherel transform, π is equivalent to the representation $\widehat{\pi}_U$ acting on $\mathrm{L}^2(U)$ via

$$(\widehat{\pi}(x)f)(\omega) = e^{-2\pi i \omega x} f(\omega) \quad .$$

- Then, given $\eta \in \mathrm{L}^2(U)$,

$$V_\eta \phi = \int_U \phi(\omega)\overline{\eta(\omega)} e^{2\pi i \omega x} d\omega = (\phi\overline{\eta})^\vee(x) \quad .$$

$\leadsto$ wavelet transform coincides with Plancherel inversion.

Overview

Operator-valued Fourier transform

Loose description:

Decompose regular representation by integrating functions against irreducible representations. Compare to the reals:

$$\widehat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-2\pi i \omega x} dx \quad ,$$

uses the characters $x \mapsto e^{-2\pi i \omega x}$.

Definition 6

Let G be a locally compact group. From now on: G is assumed to be type I.

(a) $\widehat{G}$ denotes the unitary dual, the set of (equivalence classes of) irreducible representations.

(b) Given $f \in \mathrm{L}^1(G)$, $\sigma \in \widehat{G}$, let

$$\mathcal{F}(f)(\sigma) := \widehat{f}(\sigma) := \int f(x) \sigma(x) dx \ .$$

Plancherel Theorem

Theorem 7 (Duflo/Moore, '76)

There exist

(i) ν_G, *positive* σ*-finite measure on* $\widehat{G}$,

(ii) C_σ, C_σ^{-1} ($\sigma \in \widehat{G}$), *densely defined, positive operators,*

such that

(a) $\forall f \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G) \quad : \ \sigma(f) \circ C_\sigma^{-1} \in \mathcal{B}_2(\mathcal{H}_\sigma)$, ν_G*-a.e.*

(b) *The mapping* $\mathrm{L}^1(G) \cap \mathrm{L}^2(G) \ni f \mapsto (\sigma(f) \circ C_\sigma^{-1})_{\sigma \in \widehat{G}}$ *extends to a unitary equivalence*

$$\mathcal{P} : \mathrm{L}^2(G) \to \mathcal{B}_2^{\oplus} := \int HS(\mathcal{H}_\sigma) d\nu_G(\sigma) .$$

(c) G *unimodular iff* C_σ *scalar* ν_G*-almost everywhere. In this case picking* $C_\sigma = \mathrm{Id}_{\mathcal{H}_\sigma}$ *determines* ν_G *uniquely.*

Invariant subspaces

Recall toy example: Invariant subspace of $\mathrm{L}^2(\mathbb{R})$ correspond to Borel subsets $U \subset \widehat{\mathbb{R}}$.

Theorem 8 (Characterization of invariant subspaces)

$P : \mathrm{L}^2(G) \to \mathrm{L}^2(G)$ left-invariant projection operator iff $\exists$ measurable family $(\widehat{P}_\sigma)_{\sigma \in \widehat{G}}$ of projections, such that

$$(Pf)^\wedge(\sigma) = \widehat{f}(\sigma) \circ \widehat{P}_\sigma \quad \text{for } \nu_G - \text{almost every } \sigma$$

This is a generalisation:

Think of the characteristic function χ_U in the real case as a field of projection operators, each acting on a one-dimensional space.

Existence of admissible vectors

Theorem 9 (Admissibility criterion, HF '00)

Let $\mathcal{H} \subset \mathrm{L}^2(G)$ be left-invariant, with associated family $(\widehat{P}_\sigma)_{\sigma \in \widehat{G}}$ of projection operators.
Let $a \in \mathcal{H}$ with Plancherel transform $(A_\sigma)_{\sigma \in \widehat{G}}$ fulfilling ν_G-a.e.

- $A_\sigma^* C_\sigma$ *extends to a bounded operator* $[A_\sigma^* C_\sigma]$
- $[A_\sigma^* C_\sigma]^* = C_\sigma A_\sigma$ *is an isometry on range of* $\widehat{P}_\sigma$

Then a is admissible

Existence of admissible vectors

Theorem 9 (Admissibility criterion, HF '00)

Let $\mathcal{H} \subset \mathrm{L}^2(G)$ be left-invariant, with associated family $(\widehat{P}_\sigma)_{\sigma \in \widehat{G}}$ of projection operators.
Let $a \in \mathcal{H}$ with Plancherel transform $(A_\sigma)_{\sigma \in \widehat{G}}$ fulfilling ν_G-a.e.

- $A_\sigma^* C_\sigma$ *extends to a bounded operator* $[A_\sigma^* C_\sigma]$
- $[A_\sigma^* C_\sigma]^* = C_\sigma A_\sigma$ *is an isometry on range of* $\widehat{P}_\sigma$

Then a is admissible

Theorem 10 (HF'00)

$\mathcal{H}$ as in Theorem 9 has admissible vectors iff either

(a) *G is unimodular, and* $\int_{\widehat{G}} \mathrm{rank}(\widehat{P}_\sigma) d\nu_G(\sigma) < \infty$

(b) *G is nonunimodular*

Overview

A second look at measure decompositions

A second look at measure decompositions

- Assume that Lebesgue measure decomposes over the H^T-orbits, via the measures $\overline{\lambda}$ and $(\beta_{\mathcal{O}})_{\mathcal{O} \in \mathbb{R}^d / H^T}$.

A second look at measure decompositions

- Assume that Lebesgue measure decomposes over the H^T-orbits, via the measures $\overline{\lambda}$ and $(\beta_{\mathcal{O}})_{\mathcal{O}\in\mathbb{R}^d/H^T}$.
- We have that $f \in \mathrm{L}^2(\mathbb{R}^d)$ corresponds to a family of functions on the dual orbits, via

$$f \mapsto \left(\widehat{f_{\mathcal{O}}}\right)_{\mathcal{O}\in\mathbb{R}^d/H^T}$$

where each $\widehat{f_{\mathcal{O}}}$ is the restriction of $\widehat{f}$ to $\mathcal{O}$.

A second look at measure decompositions

- Assume that Lebesgue measure decomposes over the H^T-orbits, via the measures $\overline{\lambda}$ and $(\beta_{\mathcal{O}})_{\mathcal{O}\in\mathbb{R}^d/H^T}$.
- We have that $f \in \mathrm{L}^2(\mathbb{R}^d)$ corresponds to a family of functions on the dual orbits, via
$$f \mapsto \left(\widehat{f}_{\mathcal{O}}\right)_{\mathcal{O}\in\mathbb{R}^d/H^T}$$
where each $\widehat{f}_{\mathcal{O}}$ is the restriction of $\widehat{f}$ to $\mathcal{O}$.
- The measure decomposition formula entails that $\widehat{f}_{\mathcal{O}} \in \mathrm{L}^2(\mathcal{O}, d\beta_{\mathcal{O}})$, for almost every $\mathcal{O} \in \mathbb{R}^d/H^T$, and that
$$\|f\|_2^2 = \int_{\mathbb{R}^d/H^T} \left\|\widehat{f}_{\mathcal{O}}\right\|_2^2 d\overline{\lambda}(\mathcal{O}) \ .$$

A second look at measure decompositions

- Assume that Lebesgue measure decomposes over the H^T-orbits, via the measures $\overline{\lambda}$ and $(\beta_{\mathcal{O}})_{\mathcal{O}\in\mathbb{R}^d/H^T}$.
- We have that $f \in \mathrm{L}^2(\mathbb{R}^d)$ corresponds to a family of functions on the dual orbits, via
$$f \mapsto \left(\widehat{f_{\mathcal{O}}}\right)_{\mathcal{O}\in\mathbb{R}^d/H^T}$$
where each $\widehat{f_{\mathcal{O}}}$ is the restriction of $\widehat{f}$ to $\mathcal{O}$.
- The measure decomposition formula entails that $\widehat{f_{\mathcal{O}}} \in \mathrm{L}^2(\mathcal{O}, d\beta_{\mathcal{O}})$, for almost every $\mathcal{O} \in \mathbb{R}^d/H^T$, and that
$$\|f\|_2^2 = \int_{\mathbb{R}^d/H^T} \left\|\widehat{f_{\mathcal{O}}}\right\|_2^2 d\overline{\lambda}(\mathcal{O}) \ .$$
- Thus $\mathrm{L}^2(\mathbb{R}^d)$ decomposes as a direct integral
$$\int_{\mathbb{R}^d/H^T}^{\oplus} \mathrm{L}^2(\mathcal{O}, d\beta_{\mathcal{O}}) d\overline{\lambda}(\mathcal{O}).$$

Measure decompositions and direct integrals

Measure decompositions and direct integrals

- In addition, we have representations $\pi_{\mathcal{O}}$ acting on $\mathrm{L}^2(\mathcal{O}, d\beta_{\mathcal{O}})$ via

$$(\pi_{\mathcal{O}}(x,h)f_{\mathcal{O}})(\xi) = |\mathrm{det}(h)|^{1/2} \exp(2\pi i \langle \xi, x \rangle) f_{\mathcal{O}}(h^T \xi) \ .$$

Measure decompositions and direct integrals

- In addition, we have representations $\pi_{\mathcal{O}}$ acting on $\mathrm{L}^2(\mathcal{O}, d\beta_{\mathcal{O}})$ via
$$(\pi_{\mathcal{O}}(x,h)f_{\mathcal{O}})(\xi) = |\det(h)|^{1/2} \exp(2\pi i \langle \xi, x \rangle) f_{\mathcal{O}}(h^T \xi) \ .$$
- Each $\pi_{\mathcal{O}}$ is irreducible.

Measure decompositions and direct integrals

- In addition, we have representations $\pi_{\mathcal{O}}$ acting on $\mathrm{L}^2(\mathcal{O}, d\beta_{\mathcal{O}})$ via

$$(\pi_{\mathcal{O}}(x,h)f_{\mathcal{O}})(\xi) = |\mathrm{det}(h)|^{1/2} \exp(2\pi i \langle \xi, x \rangle) f_{\mathcal{O}}(h^T \xi) \ .$$

- Each $\pi_{\mathcal{O}}$ is irreducible.
- In summary:
  - The Fourier transform effects a unitary equivalence

$$\pi \simeq \int_{\mathbb{R}^d/H^T}^{\oplus} \pi_{\mathcal{O}} d\overline{\lambda}(\mathcal{O}) \ ,$$

    decomposing π into irreducibles.

Measure decompositions and direct integrals

- In addition, we have representations $\pi_{\mathcal{O}}$ acting on $\mathrm{L}^2(\mathcal{O}, d\beta_{\mathcal{O}})$ via

$$(\pi_{\mathcal{O}}(x,h)f_{\mathcal{O}})(\xi) = |\mathrm{det}(h)|^{1/2} \exp(2\pi i \langle \xi, x \rangle) f_{\mathcal{O}}(h^T \xi) \ .$$

- Each $\pi_{\mathcal{O}}$ is irreducible.
- In summary:
  - The Fourier transform effects a unitary equivalence

$$\pi \simeq \int_{\mathbb{R}^d/H^T}^{\oplus} \pi_{\mathcal{O}} d\overline{\lambda}(\mathcal{O}) \ ,$$

    decomposing π into irreducibles.
  - Weak admissibility of H can now be related to absolute continuity of measures, and is essentially equivalent to almost everywhere compactness of the stabilizers.

Measure decompositions and direct integrals

- In addition, we have representations $\pi_{\mathcal{O}}$ acting on $\mathrm{L}^2(\mathcal{O}, d\beta_{\mathcal{O}})$ via

$$(\pi_{\mathcal{O}}(x,h)f_{\mathcal{O}})(\xi) = |\mathrm{det}(h)|^{1/2}\exp(2\pi i\langle \xi, x\rangle) f_{\mathcal{O}}(h^T\xi) \ .$$

- Each $\pi_{\mathcal{O}}$ is irreducible.
- In summary:
  - The Fourier transform effects a unitary equivalence

$$\pi \simeq \int_{\mathbb{R}^d/H^T}^{\oplus} \pi_{\mathcal{O}} d\overline{\lambda}(\mathcal{O}) \ ,$$

    decomposing π into irreducibles.
  - Weak admissibility of H can now be related to absolute continuity of measures, and is essentially equivalent to almost everywhere compactness of the stabilizers.
  - Concrete admissibility condition in Lemma 1 and abstract version in Theorem 9 coincide.

Measure decompositions and direct integrals

- In addition, we have representations $\pi_{\mathcal{O}}$ acting on $\mathrm{L}^2(\mathcal{O}, d\beta_{\mathcal{O}})$ via

$$(\pi_{\mathcal{O}}(x,h)f_{\mathcal{O}})(\xi) = |\mathrm{det}(h)|^{1/2} \exp(2\pi i \langle \xi, x \rangle) f_{\mathcal{O}}(h^T \xi) \ .$$

- Each $\pi_{\mathcal{O}}$ is irreducible.
- In summary:
  - The Fourier transform effects a unitary equivalence

$$\pi \simeq \int_{\mathbb{R}^d / H^T}^{\oplus} \pi_{\mathcal{O}} d\overline{\lambda}(\mathcal{O}) \ ,$$

    decomposing π into irreducibles.
  - Weak admissibility of H can now be related to absolute continuity of measures, and is essentially equivalent to almost everywhere compactness of the stabilizers.
  - Concrete admissibility condition in Lemma 1 and abstract version in Theorem 9 coincide.
  - Non-unimodularity condition in Theorem 4 is related to the same condition in Theorem 10.

Measure decompositions and direct integrals

- In addition, we have representations $\pi_{\mathcal{O}}$ acting on $\mathrm{L}^2(\mathcal{O}, d\beta_{\mathcal{O}})$ via

$$(\pi_{\mathcal{O}}(x,h)f_{\mathcal{O}})(\xi) = |\mathrm{det}(h)|^{1/2} \exp(2\pi i \langle \xi, x \rangle) f_{\mathcal{O}}(h^T \xi) \ .$$

- Each $\pi_{\mathcal{O}}$ is irreducible.
- In summary:
  - The Fourier transform effects a unitary equivalence

$$\pi \simeq \int_{\mathbb{R}^d/H^T}^{\oplus} \pi_{\mathcal{O}} d\overline{\lambda}(\mathcal{O}) \ ,$$

    decomposing π into irreducibles.
  - Weak admissibility of H can now be related to absolute continuity of measures, and is essentially equivalent to almost everywhere compactness of the stabilizers.
  - Concrete admissibility condition in Lemma 1 and abstract version in Theorem 9 coincide.
  - Non-unimodularity condition in Theorem 4 is related to the same condition in Theorem 10.
  - All Plancherel-theoretic objects can be made explicit.

Overview

M. Duflo and Calvin C. Moore.
On the regular representation of a nonunimodular locally compact group.
J. Functional Analysis, 21(2):209–243, 1976.

Hartmut Führ.
Wavelet frames and admissibility in higher dimensions.
J. Math. Phys., 37(12):6353–6366, 1996.

Hartmut Führ.
Continuous wavelet transforms with abelian dilation groups.
J. Math. Phys., 39(8):3974–3986, 1998.

Hartmut Führ.
Admissible vectors for the regular representation.
Proc. Amer. Math. Soc., 130(10):2959–2970 (electronic), 2002.

Hartmut Führ.
Abstract harmonic analysis of continuous wavelet transforms, volume 1863 of *Lecture Notes in Mathematics*.
Springer-Verlag, Berlin, 2005.

Hartmut Führ.
Generalized Calderón conditions and regular orbit spaces.
Colloq. Math., 120(1):103–126, 2010.

Hartmut Führ and Matthias Mayer.
Continuous wavelet transforms from semidirect products: cyclic representations and Plancherel measure.
J. Fourier Anal. Appl., 8(4):375–397, 2002.

C. J. Isham and J. R. Klauder.
Coherent states for n-dimensional Euclidean groups $E(n)$ and their application.
J. Math. Phys., 32(3):607–620, 1991.

J. R. Klauder and R. F. Streater.
A wavelet transform for the Poincaré group.
J. Math. Phys., 32(6):1609–1611, 1991.

Richard S. Laugesen, Nik Weaver, Guido L. Weiss, and Edward N. Wilson.
A characterization of the higher dimensional groups associated with continuous wavelets.
J. Geom. Anal., 12(1):89–102, 2002.

Stephane Mallat and Sifen Zhong.
Wavelet transform maxima and multiscale edges.
In *Wavelets and their applications*, pages 67–104. Jones and Bartlett, Boston, MA, 1992.

S. Twareque Ali, Hartmut Führ, and Anna E. Krasowska.
Plancherel inversion as unified approach to wavelet transforms and Wigner functions.
Ann. Henri Poincaré, 4(6):1015–1050, 2003.