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Joint ICTP-TWAS School on Coherent State Transforms, Time-Frequency and Time-Scale Analysis, Applications

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Group-theoretical methods for the design and analysis of higher-dimensional wavelet systems III Sparse signals and function spaces

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Sparse signals and function spaces

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Agenda for this week

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I Wavelet transforms associated to groups of affine mappings (Monday)

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II Wavelet inversion, admissibility and the Plancherel formula (Tuesday)

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II Wavelet inversion, admissibility and the Plancherel formula (Tuesday)

III Sparse signals and function spaces (Wednesday)

IV Wavelet approximation theory over general dilation groups (Thursday, Friday)

Outline

1. Motivation for sparse signals

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2. ℓ^1-sparse vectors over general ONB's

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3. Nonlinear approximation

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2. ℓ^1-sparse vectors over general ONB's
3. Nonlinear approximation
4. Homogeneous Besov spaces

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2. ℓ^1-sparse vectors over general ONB's
3. Nonlinear approximation
4. Homogeneous Besov spaces
5. ℓ^1-sparsity w.r.t. wavelet ONB's

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Overview

Why are wavelets useful?

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- Underlying useful structures: Groups, multiresolution analysis, etc. (Useful both for mathematics and applications)
- Very important: Wavelet description of realistic signals is efficient, i.e., a realistic signal is well-described by a few building blocks.

Aim of this talk: Make the last statement precise.

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Recall: ONB's and norm preservation

Theorem 1

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is unitary.

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(d) *For all $y \in \mathcal{H}$: If $\langle y, x_i \rangle = 0$, for all $i \in I$, then $y = 0$.*

If the equivalent conditions hold, then the system is an orthonormal basis (ONB).

ℓ^1-sparse vectors

Note

Given any infinite set I, one has $\ell^1(I) \subset \ell^2(I)$ properly, i.e., the condition

$$\sum_{i \in I} |c_i| < \infty \Rightarrow \sum_{i \in I} |c_i|^2 < \infty \quad .$$

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$$y = \sum_{i \in I} c_i x_i \quad , \quad \sum_{i \in I} |c_i| < \infty \ .$$

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- ℓ^0-"norm", i.e, number of nonzero coefficients. See compressive sensing!
- Weighted versions, i.e. multiply coefficients by a suitable unbounded sequence of numbers before taking the norm.

We concentrate on ℓ^1 in the following for simplicity.

Overview

Nonlinear n-term approximation

Nonlinear *n*-term approximation

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$$\sigma_n(y)_X = \inf \left\{ \left\| y - \sum_{i \in J} c_i x_i \right\| : J \subset I, c_i \in \mathbb{C}, |J| \leq n \right\}$$

denote the (nonlinear) n-term approximation error of y with respect to the system X.

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Note: The optimal set J will depend on y (hence nonlinear approximation).

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$$y_n = \sum_{i \in J_n} \langle y, x_i \rangle x_i$$

is the optimal n-term approximation of y, and we have

$$\sigma_n(y)_X^2 = \sum_{j \notin J_n} |\langle y, x_j \rangle|^2 \quad .$$

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Alternatively, let $(c_n)_{n \in \mathbb{N}}$ denote the sequence of expansion coefficients of y, sorted by decreasing modulus. Then

$$\sigma_n(y)_X^2 = \sum_{k > n} |c_k|^2 \quad .$$

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- Can be extended to other sparsity measures.
  E.g., $p < 1$ will lead to faster decay.

For later use: Frame version

Theorem 6

Let $X = (x_i)_{i \in I} \subset \mathcal{H}$ denote a frame, and $y \in \mathcal{H}$. Assume that

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- Only one direction of the equivalence holds in the frame case.
- The ℓ^1-coefficients are not assumed to come from a dual frame.

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Modulus of continuity

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Definition 7

For $h \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{C}$, the (forward) difference operator of step h is given by

$$\Delta_h f(x) = f(x+h) - f(x),$$

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The $r-$th order modulus of smoothness of f in $L^2(\mathbb{R})$ is defined as the mapping

$$\omega_2^r(f,t) = \sup_{h \in \mathbb{R}, |h|<t} \|\Delta_h^r f(\cdot)\|_2.$$

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For $\alpha > 0, 1 \leq p, q < \infty$, $r = \lfloor \alpha \rfloor + 1$, a function f defined on $\mathbb{R}$ is said to be in the homogeneous Besov space $\dot{B}_{p,q}^{\alpha}(\mathbb{R})$, if

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Note

This definition extends the scale of homogeneous Sobolev spaces:
$\dot{W}^k_2(\mathbb{R}) = \dot{B}^k_{2,2}(\mathbb{R})$.

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- Answers: Good wavelets will be described in terms of smoothness, decay and vanishing moments;

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- Question 2: Different wavelets may have different sparse vectors! Can we control that, i.e., can we isolate a class of good wavelets which all have the same sparse vectors?
- Answers: Good wavelets will be described in terms of smoothness, decay and vanishing moments; the associated space of sparse vectors will be a suitable homogeneous Besov space.

Nice wavelets

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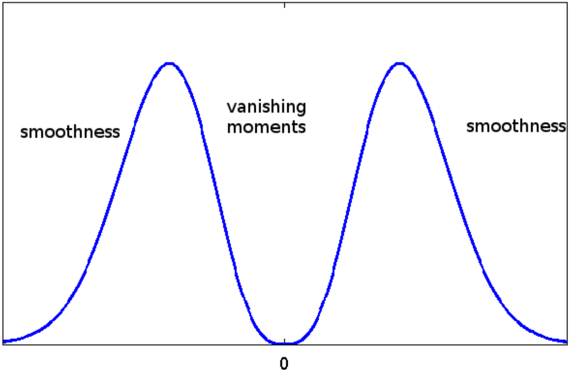
Note: Frequency-side localization is understood away from zero.

Cartoon: Fourier side decay of wavelets



Plot of $|\widehat{\psi}|$.

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Theorem 9 (DeVore/Jawerth/Popov)

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Let $f \in \mathrm{L}^2(\mathbb{R})$. Assume that the wavelet ψ has suitably many vanishing moments, degrees of smoothness and decay, and let $\Psi = (\psi_{j,k})_{j,k}$ denote the associated wavelet ONB.

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Observations

- Characterization of sparse vectors is independent of the wavelet.
- Characterization extends to arbitrary Besov spaces $\dot{B}^{\alpha}_{p,q} \rightsquigarrow$ weighted mixed summability condition on the coefficients.
- Wavelet basis is a joint unconditional basis of a whole scale of Besov spaces.

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In applications

Important source of heuristics for signal processing applications, most prominently

- compression;
- denoising.

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  Challenge: The notion will have to depend on the dilation group!

Overview

Ronald A. DeVore.
Nonlinear approximation.
In *Acta numerica, 1998*, volume 7 of *Acta Numer.*, pages 51–150. Cambridge Univ. Press, Cambridge, 1998.

Ronald A. DeVore, Björn Jawerth, and Bradley J. Lucier.
Image compression through wavelet transform coding.
IEEE Trans. Inform. Theory, 38(2, part 2):719–746, 1992.

Ronald A. DeVore, Björn Jawerth, and Vasil Popov.
Compression of wavelet decompositions.
Amer. J. Math., 114(4):737–785, 1992.

David L. Donoho and Iain M. Johnstone.
Adapting to unknown smoothness via wavelet shrinkage.
J. Amer. Statist. Assoc., 90(432):1200–1224, 1995.

Michael Frazier and Björn Jawerth.
Decomposition of Besov spaces.
Indiana Univ. Math. J., 34(4):777–799, 1985.

Michael Frazier, Björn Jawerth, and Guido Weiss.
Littlewood-Paley theory and the study of function spaces, volume 79 of *CBMS Regional Conference Series in Mathematics*.
Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1991.

Yves Meyer.
Wavelets and operators, volume 37 of *Cambridge Studies in Advanced Mathematics*.
Cambridge University Press, Cambridge, 1992.
Translated from the 1990 French original by D. H. Salinger.

P. Wojtaszczyk.
A mathematical introduction to wavelets, volume 37 of *London Mathematical Society Student Texts*.
Cambridge University Press, Cambridge, 1997.