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Joint ICTP-TWAS School on Coherent State Transforms, Time-Frequency and Time-Scale Analysis, Applications

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Group-theoretical methods for the design and analysis of higher-dimensional wavelet systems IV Wavelet approximation theory over general dilation groups

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Lehrstuhl A für Mathematik, RWTH Rheinisch-Westfälische Technische Hochschule Aachen

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- make objects of coorbit theory explicit and accessible.

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$\rightsquigarrow$ introduce sparsity on continuous wavelet transforms, and sample!

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The quasiregular representation is v_0-integrable: If $\psi \in \mathcal{F}^{-1}C_c(\mathcal{O})$, then $\mathcal{W}_\psi \psi \in L^1_{v_0}(G)$.

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- Since v_0 only depends on h-variable,

$$\|\mathcal{W}_\psi \psi\|_{\mathrm{L}^1_{v_0}} \le \int_H \left\| \left(\widehat{\psi} \cdot D_h \overline{\widehat{\psi}} \right)^\vee \right\|_{\mathrm{L}^1} v_0(h) |\det(h)|^{-1/2} dh \ .$$

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- The vector-valued mapping
$$H \ni h \mapsto \widehat{\psi} \cdot D_h\widehat{\psi} \in C_c(\mathcal{O})$$
is continuous with respect to the Schwartz topology.

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$$h \mapsto \|(\widehat{\psi} \cdot D_h\overline{\widehat{\psi}})^\vee\|_{\mathrm{L}^1} v_0(h) |\det(h)|^{-1/2}$$

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- In summary: $\|\mathcal{W}_\psi \psi\|_{\mathrm{L}^1_{v_0}} < \infty$.

Proof concluded.

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Essentially by the same proof:

Corollary 3

$\mathcal{F}^{-1}C_c(\mathcal{O}) \subset Co(L^p(G))$.

Overview

Frame atoms

Definition 4

Let $v : G \to \mathbb{R}^+$ be continuous and submultiplicative. We call $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ *v-frame atom* if $\mathcal{W}_\psi \psi \in W^R(L^\infty, \mathrm{L}^1_v)$,

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Note that $\mathcal{B}_v \subset \mathcal{A}_v$.

Use of $\mathcal{B}_v$

If the weight v is a *control weight* for the Banach function space Y, then choosing analyzing vectors from $\mathcal{B}_v$ guarantees (consistency and) *discretization*.

Sampling sets

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Lemma 6

For any compact neighborhood U there exists a separated, U-dense family $Z \subset G$.

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(b) $(\langle f, \pi(z)\psi \rangle)_{z \in Z} \in \ell^p(Z)$, *for some (equivalently: any)* $0 \neq \psi \in \mathcal{B}_v$ *and all separated subsets* $Z \subset G$.

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In addition, the ℓ^p*-norms of the coefficient sequences in (b) and (c) are equivalent to the* $Co(L^p)$*-norm of* f.

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- Still open: Is $\mathcal{B}_v$ nonempty?

Band-limited Schwartz functions are atoms

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Theorem 8 (HF, '12)

For all control weights v satisfying $v(x,h) \le (1+|x|)^t w(h)$, with suitable $t > 0$ and continuous weights w on H, we have

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Remaining challenge

Find simple criteria for compactly supported functions to be in $\mathcal{B}_v$.
Can one explicitly construct these functions?

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- Applicable to all discrete series representations in higher dimensions. So far only studied for a handfull of dilation groups.
- Obstacles:
  - Sampling rate is not easy to compute, and it quite possibly too conservative.
  - No easily checked criteria for nice wavelets (so far).

Overview

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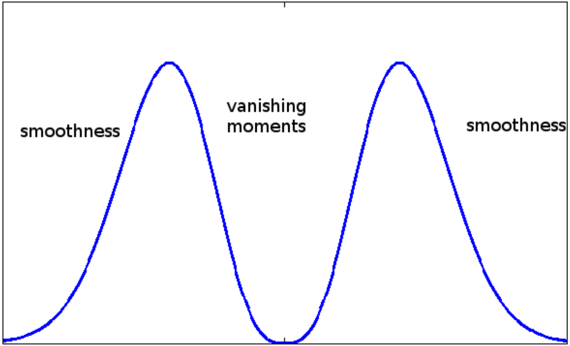
- Aims of the following: Develop sufficient criteria in terms of smoothness, decay, and vanishing moments. The last condition uses the blind spot of the wavelet transform.

Cartoon: Fourier side decay of wavelets

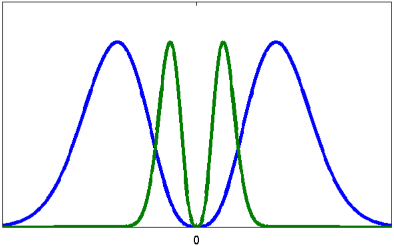


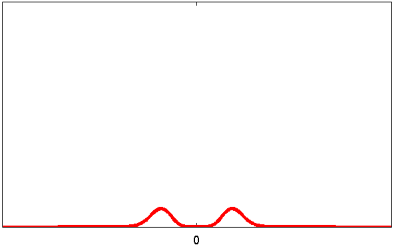
Plot of $|\widehat{\psi}|$.

Vanishing moments and wavelet coefficient decay

Assumptions on nice wavelet ψ guarantee fast decay of $\mathcal{W}_\psi \psi$:

$$|\mathcal{W}_\psi \psi(x,s)| \le \left\| \partial^\ell \left(\widehat{\psi} \cdot \overline{\widehat{\psi}(s^{-1} \cdot)} \right) \right\|_1 |s|^{-1/2} (1+|x|)^{-\ell}$$





Plot of $\widehat{\psi}$ and $\widehat{\psi}(3\cdot)$

Overlap $\widehat{\psi} \cdot \widehat{\psi}(3\cdot)$

$\Rightarrow$ vanishing moments, smoothness govern decay of overlap, as $|s| \to 0, \infty$

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- Wavelet coefficient decay can be measured employing suitably defined auxiliary functions.
- Still needed: Compatibility condition for Haar measure on H and Lebesgue measure on $\mathcal{O}$ ($\rightsquigarrow$ strong temperate embeddedness)

Vanishing moment conditions

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Definition 9

Let $r \in \mathbb{N}$ be given. $f \in \mathrm{L}^1(\mathbb{R}^d)$ has vanishing moments in $\mathcal{O}^c$ of order r if all distributional derivatives $\partial^\alpha \widehat{f}$ with $|\alpha| < r$ are continuous functions, identically vanishing on $\mathcal{O}^c$.

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Want to establish results of the form:

Theorem

A function ψ with suitably many degrees of smoothness, decay and vanishing moments is in $\mathcal{B}_v$.

This will depend on an additional technical assumption, involving auxiliary functions.

Fourier envelope

Fourier envelope

Definition 10

Let $\mathcal{O} \subset \mathbb{R}^d$ denote the dual orbit. Given $\xi \in \mathcal{O}$, let $\mathrm{dist}(\xi, \mathcal{O}^c)$ denote the euclidean distance of ξ to $\mathcal{O}^c$. Let

$$A(\xi) = \min\left(\frac{\mathrm{dist}(\xi, \mathcal{O}^c)}{1 + \sqrt{|\xi|^2 - \mathrm{dist}(\xi, \mathcal{O}^c)^2}}, \frac{1}{1 + |\xi|} \right) .$$

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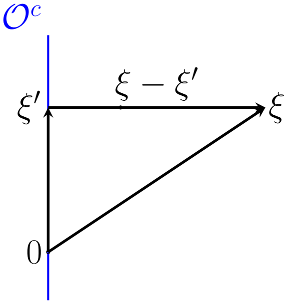

$$A(\xi) = \min \left(\frac{|\xi - \xi'|}{1 + |\xi'|}, \frac{1}{1 + |\xi|} \right)$$

with ξ' = point in $\mathcal{O}^c$ closest to ξ

Purpose of Fourier envelope

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Vanishing moments and Fourier envelope

If ψ has ℓ vanishing moments, then

$$|\widehat{\psi}(\xi)| \preceq |\widehat{\psi}|_{\ell,\ell} A(\xi)^{\ell} \ .$$

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Let $\Phi_\ell : H \to \mathbb{R}^+ \cup \{\infty\}$ via

$$\Phi_\ell(h) = \int_{\mathbb{R}^d} A(\xi)^\ell A(h^T \xi)^\ell d\xi$$

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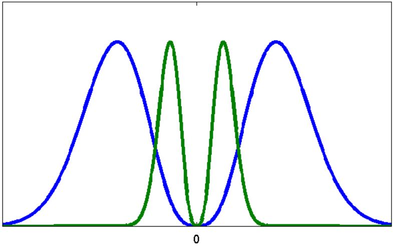
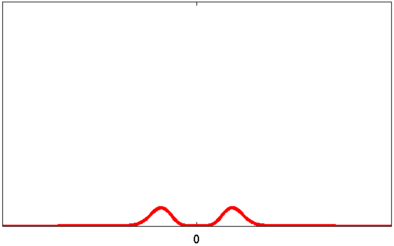
Informal meaning of Φ_ℓ

Φ_ℓ measures the overlap of two dilated copies of the wavelet with smoothness, decay and vanishing moments of order ℓ; compare one-dimensional case.

Overlap and vanishing moment decay



Sketch of $\widehat{\psi}$ and $\widehat{\psi}(h^T \cdot)$



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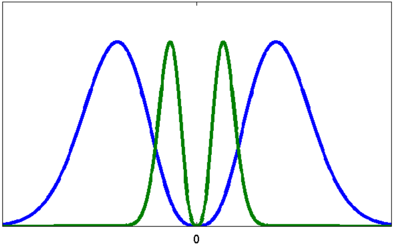

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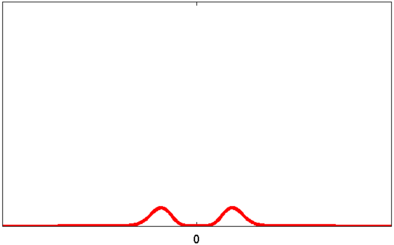

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If ψ has ℓ vanishing moments,
$|\mathcal{W}_\psi \psi(x,h)| \preceq |\widehat{\psi}|^2_{\ell,\ell} (1+|x|)^{-\ell} |\det(h)|^{1/2} (1+\|h\|_\infty)^\ell \Phi_\ell(h)$.

Technical condition for vanishing moment criteria

Definition 12

Let $w : H \to \mathbb{R}^+$ denote a weight, $s \geq 0$. We call $\mathcal{O}$ strongly (s,w)-temperately embedded (with index $\ell \in \mathbb{N}$) if $\Phi_\ell \in W(L^\infty, \mathrm{L}^1{}_m)$, where the weight $m : H \to \mathbb{R}^+$ is defined by

$$m(h) = w(h)|\mathrm{det}(h)|^{-1/2}(1 + \|h\|)^{2(s+d+1)} \ .$$

Vanishing moment criteria for atoms

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Theorem 13 (HF '13)

Assume that $\mathcal{O}$ is strongly temperately (s, w_0)-embedded with index ℓ. Then any function $\psi \in \mathrm{L}^1(\mathbb{R}^d) \cap C^{\ell+d+1}(\mathbb{R}^d)$ with vanishing moments in $\mathcal{O}^c$ of order $t > \ell + s + d$ and $|\widehat{\psi}|_{t,t} < \infty$ is contained in $\mathcal{B}_{v_0}$, for any weight v_0 satisfying $v_0(x,h) \leq (1+|x|)^s w_0(h)$.

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There exists a partial differential operator D with constant coefficients such that $\psi = D^t \rho$ has vanishing moments in $\mathcal{O}^c$ of order t, for every function ρ with sufficient smoothness and decay.

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Theorem 14 (HF '13)

There exists a partial differential operator D with constant coefficients such that $\psi = D^t \rho$ has vanishing moments in $\mathcal{O}^c$ of order t, for every function ρ with sufficient smoothness and decay.
In particular, if $\mathcal{O}$ is strongly temperately (s, w_0)-embedded, there exist compactly supported $\psi \in \mathcal{B}_{v_0}$, for any weight v_0 satisfying $v_0(x, h) \leq (1 + |x|)^s w_0(h)$.

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Strong temperate embeddness conditions have been checked for

- all dilation groups in dimension 2 (check list of representatives modulo conjugacy);
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In fact, so far no examples are known where the dual orbit is not strongly temperately embedded.

A simplified criterion for strong temperate embeddedness

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Theorem 15 (HF, R. Raissi-Toussi, '14)

Let $s > 0$, and suppose that for suitable $e_1, \ldots, e_4 \geq 0$:

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\begin{aligned}
w(h^{\pm 1}) A_H(h)^{e_1} &\preceq 1 && (1)\\
\|h^{\pm 1}\| A_H(h)^{e_2} &\preceq 1 && (2)\\
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\end{aligned}
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Then $\mathcal{O}$ is strongly (s, w)-temperately embedded, with index

$$\ell = \lfloor e_1 + e_2(2s + 2d + 2) + \frac{3}{2} e_3 + e_4 \rfloor + d + 1\,.$$

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Lemma 16 (HF, R. Raissi-Toussi, '14)

Condition (2) implies (3) and (4), with constants $e_3 = d e_2$ and $e_4 = 2 e_2 \mathrm{dim}(H)$.

Sample class: Shearlet groups in arbitrary dimensions

(i) Classical shearlet group (Dahlke/Kutyniok/Maass/Sagiv/Teschke):

$$H = \left\{ \begin{pmatrix} a & s_1 & \dots & s_{d-1} \\ & a^{\alpha_2} & & \\ & & \ddots & \\ & & & a^{\alpha_d} \end{pmatrix} : a > 0, s_1, \dots, s_{d-1} \in \mathbb{R} \right\} .$$

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(ii) Toeplitz shearing subgroup (Dahlke,Teschke, Häuser)

$$H = \left\{ \begin{pmatrix} a & s_1 & s_2 & \dots & \dots & s_{d-1} \\ & a & s_1 & s_2 & \dots & s_{d-2} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & s_2 \\ & & & & & s_1 \\ & & & & & a \end{pmatrix} : a > 0, s_1, \dots, s_{d-1} \in \mathbb{R} \right\} .$$

We let Y denote the infinitesimal generator of the diagonal subgroup in H, with first entry normalized to one.

Unified criteria for admissible vectors and atoms

Theorem 17

Let $H < \mathrm{GL}(\mathbb{R}^d)$ denote a generalized shearlet dilation group, and let Y denote the infinitesimal generator of the diagonal part.

(a) *The open dual orbit is $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{d-1}$.*

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(c) *H fulfills the estimates (2)-(4) from Theorem 15, with exponents*

$$e_2 = d - 1 + 2\|Y\|_\infty \ , \ e_3 = |\mathrm{trace}(Y)| \ , \ e_4 = |d - \mathrm{trace}(Y)| \ .$$

In particular, the associated dual orbit is strongly (s, w)-temperately embedded.

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4. If ρ was chosen compactly supported, then ψ is compactly supported.
5. For the classical two-dimensional shearlets with hyperbolic scaling, we have $\|Y\| = 1$ and $\mathrm{Trace}(Y) = 3/2$, resulting in $r = 28$.

Overview

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Decomposition space

$$\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) := \left\{ f \in \mathcal{D}'\left(\mathcal{O}\right) : \|f\|_{\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)} < \infty \right\},$$

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where $(\varphi_i)_{i \in I}$ is a suitable partition of unity on $\mathcal{O}$ subordinate to $\mathcal{Q}$ and

$$\|f\|_{\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)} = \left\| \left(\left\| \mathcal{F}^{-1}\left(\varphi_i f\right) \right\|_p \right)_{i \in I} \right\|_{\ell^q_u} = \left\| \left(u_i \cdot \left\| \mathcal{F}^{-1}\left(\varphi_i f\right) \right\|_p \right)_{i \in I} \right\|_{\ell^q} .$$

From coorbit spaces to decomposition spaces

Group H

Dual orbit $\mathcal{O}$

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$\downarrow$ $p_{\xi_0} : H \to \mathcal{O}, h \mapsto h^T \xi_0$
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$\mathrm{Co}\left(L^{p,q}_v\right)$ Group H

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$\mathcal{F} \downarrow \wr$

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Example: Covering induced by the shearlet group

We consider the **Shearlet type group** with parameter $c \in \mathbb{R}$,

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Identifying wavelet coorbit spaces as decomposition spaces

Theorem 18 (Felix Voigtlaender, HF)

Let $p, q \in [1, \infty]$ and let $\mathcal{Q} = \left(h_i^{-T} Q \right)_{i \in I}$ be a decomposition covering induced by H.

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The set of sparse signals only depends on the way in which the dual action partitions the frequency space.

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Informal interpretation

The set of sparse signals only depends on the way in which the dual action partitions the frequency space.
Different dilation groups may have the same sparse signals.

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- Common to all problems: Crucial role of the dual action.

Overview

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