Topics in theoretical physics I

self-organization on scale hierarchy

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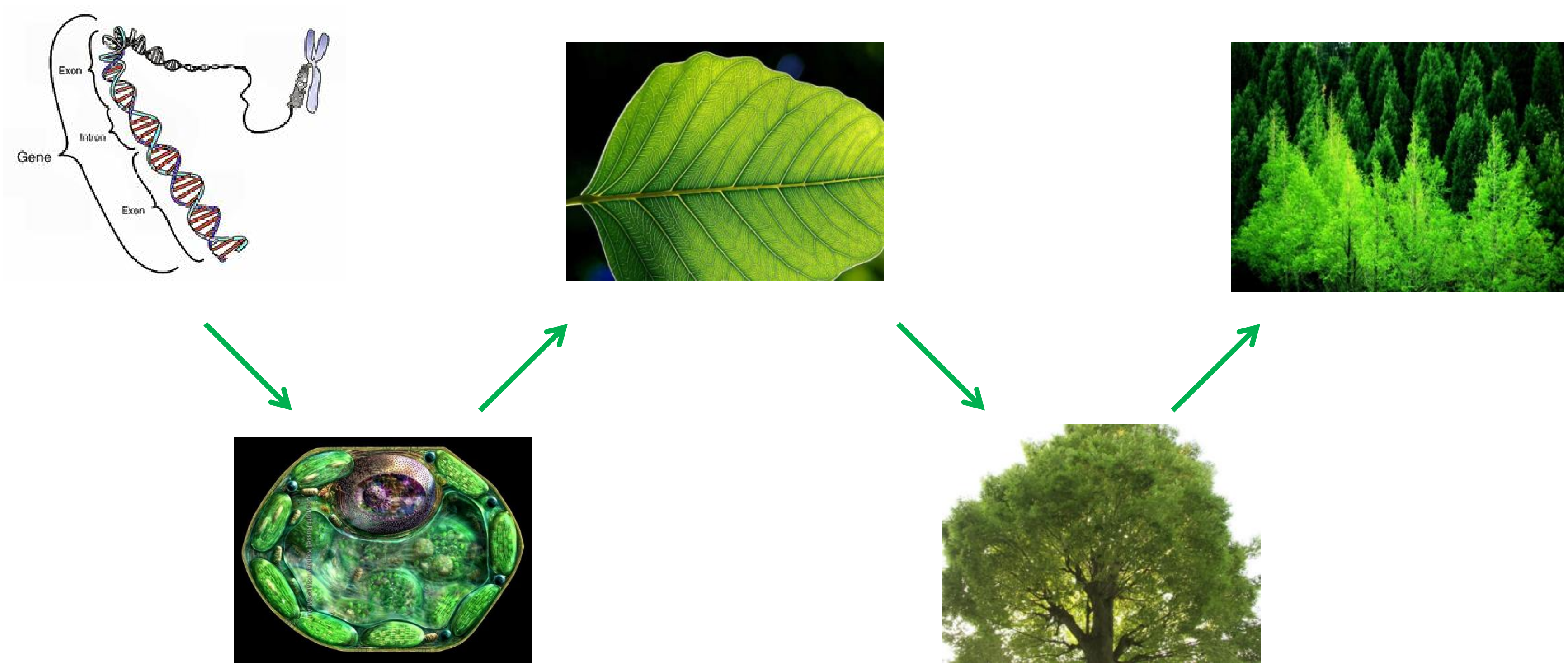
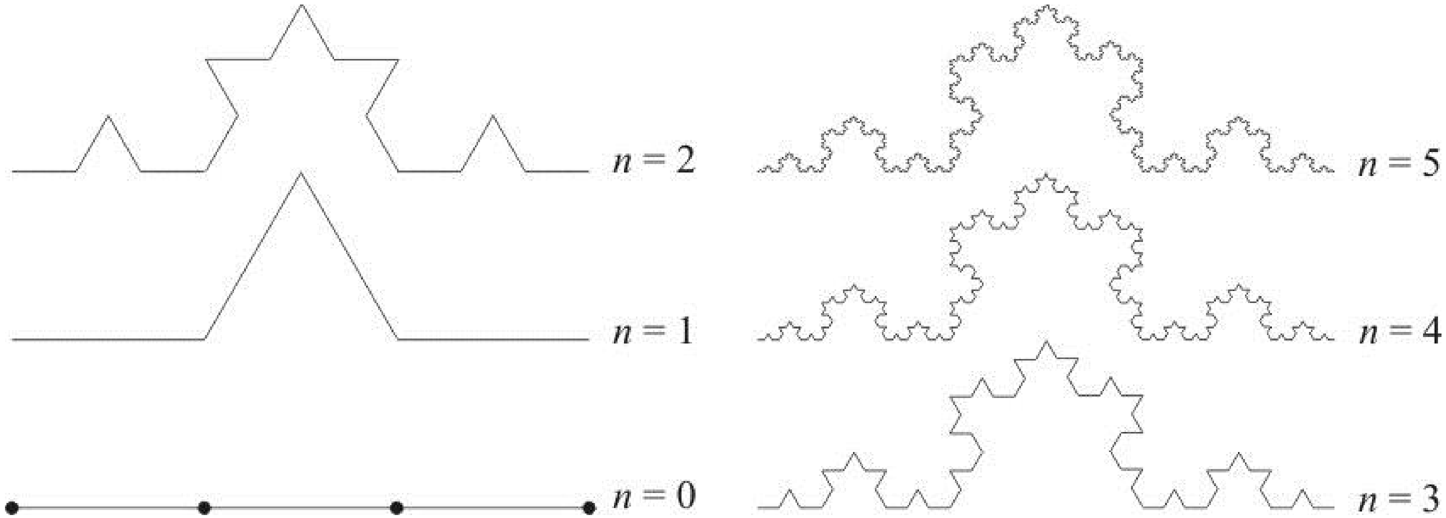
Scale hierarchy



- Surprisingly diverse structures and mechanisms on different scales
- How "macro" can be different from "micro" ?
- Physics: What is the mechanism that works without blueprints ?

subject defines a scale

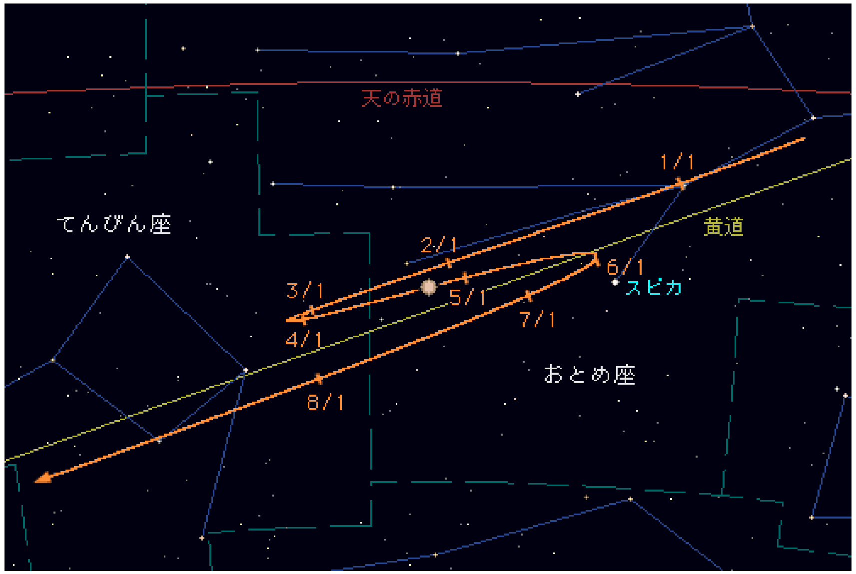


The definition (representation) of matter depends on the subject=scale.

Space-time: *a priori* of theory (?)

- *Kopernikanische Wendung*

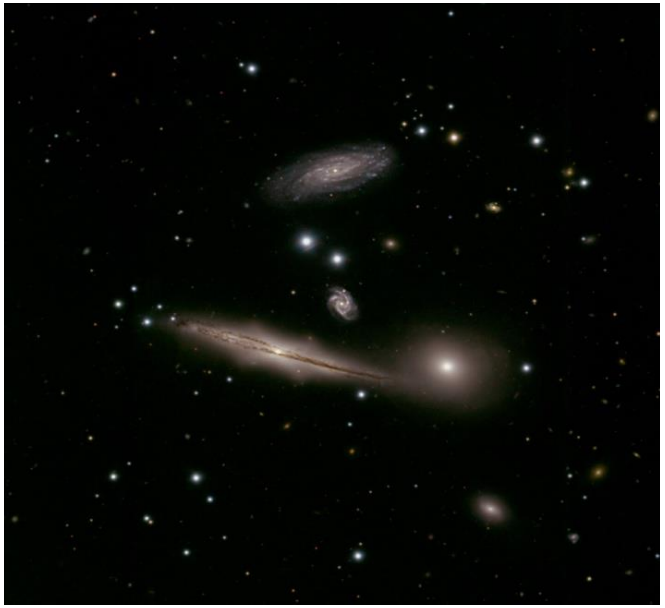
  a priori of recognition

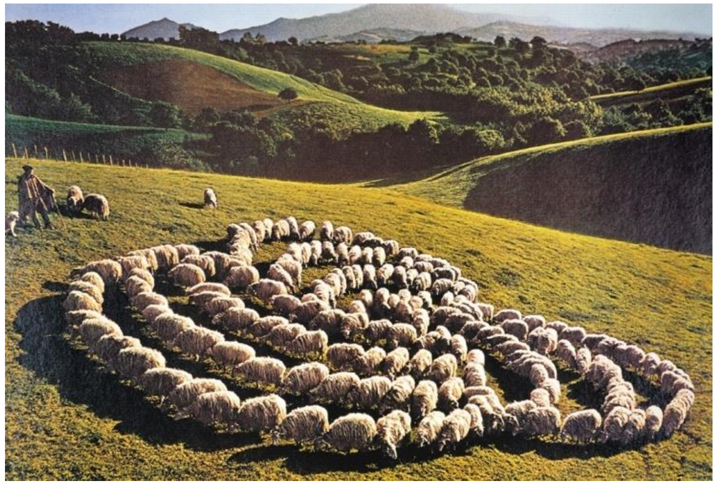


Immanuel Kant
(1724-1804)

phenomena (event) → framework of recognition (space-time)

Vortex = space-time distortion





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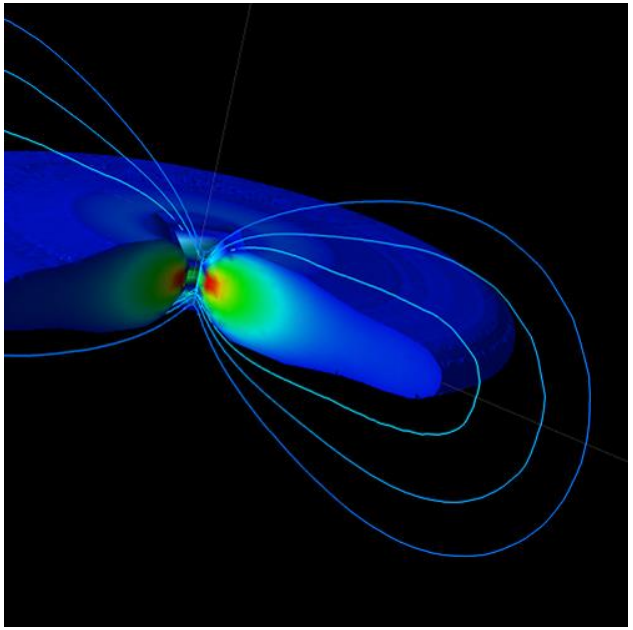
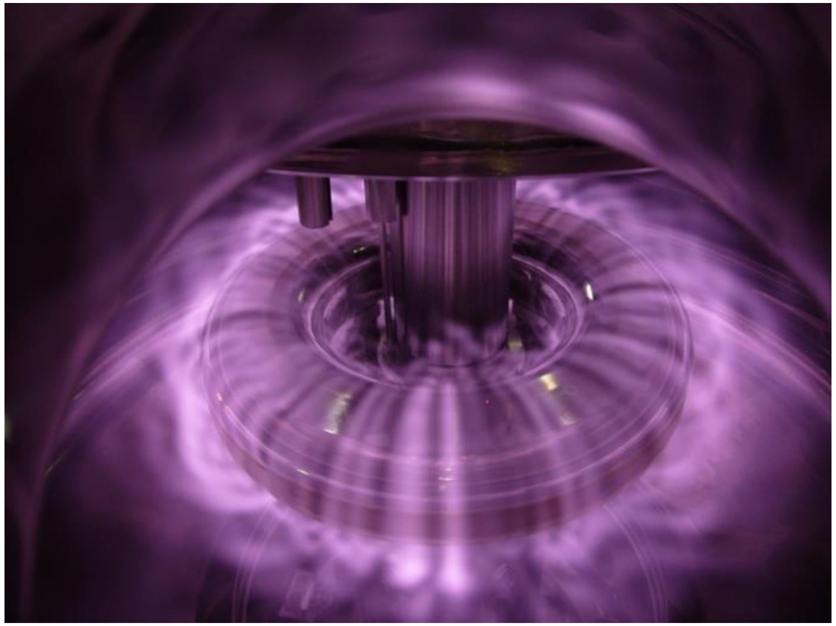
vortex → *self-organized confinement*

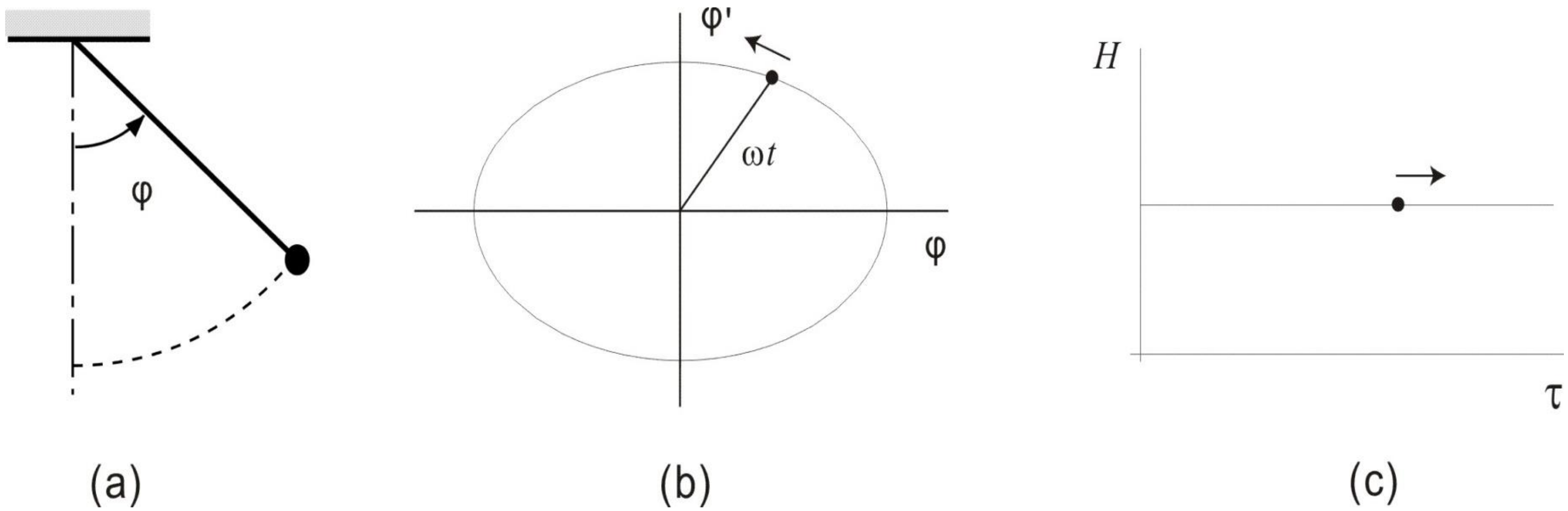


Jovian magnetosphere



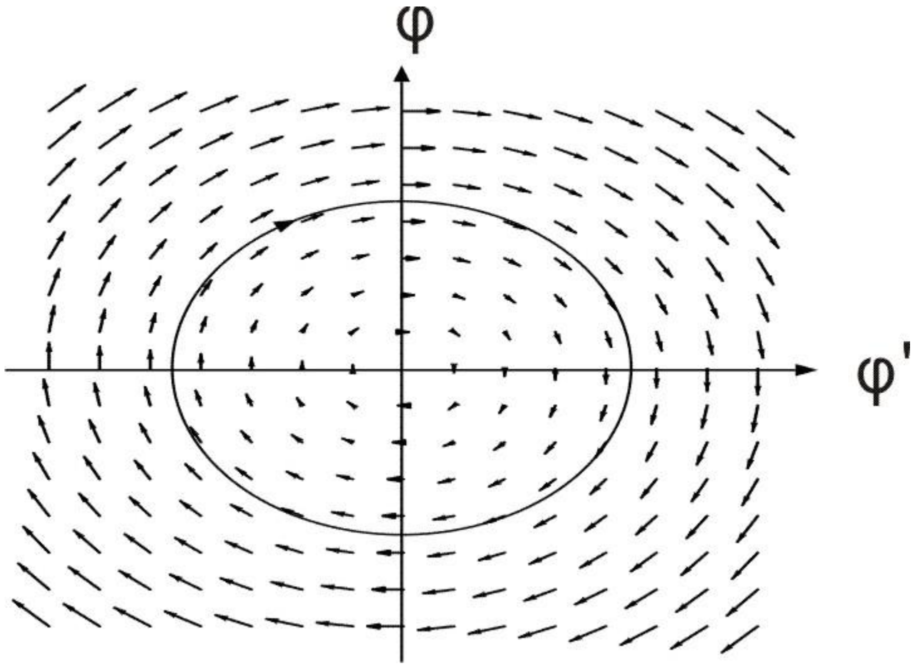
RT-1 magnetospheric plasma

“canonical vortex” = micro scale



repetition circulation continuation

Hamiltonian formalism



$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Symplectic group

→ canonical vortex

matter = energy = Hamiltonian H

space-time = symplectic geometry J

$$\frac{d}{dt}F = \{H, F\}$$

$$\{H, F\} = \langle \partial_z H, J\partial_z F \rangle$$

What is *macroscopic* space-time ?

- How can magnetic field "confine" a plasma, despite the fact that $f = \exp(-\beta E)$?

→ This is possible because of "scale separation".

- What is scale hierarchy?

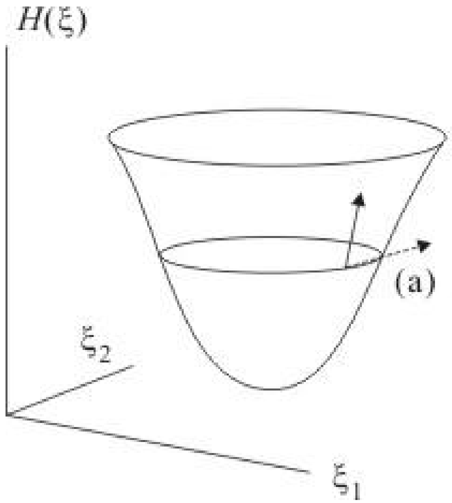
- What is "state" and what is "space"?

micro = canonical / macro = non-canonical

General form of Hamiltonian systems

- Hamiltonian mechanics is dictated by *J* (Poisson operator) and *H* (Hamiltonian)

$$\frac{d}{dt} z = J \partial_z H(z)$$



Poisson bracket : $\{G,F\} = \langle \partial_z G(z), J \partial_z F(z) \rangle$

$$\frac{d}{dt} F(z) = \{H,F\}$$

$$\{\{F,G\},H\} + \{\{G,H\},F\} + \{\{H,F\},G\} = 0$$

Examples of Hamiltonian systems

classical mechanics:

$$\frac{d}{dt}\mathbf{z} = \frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix} = J\partial_z H$$

quantum mechanics:

$$\partial_t \psi = -i\partial_\psi \langle \mathcal{H}\psi, \psi \rangle / 2 = J\partial_\psi H$$

These examples are "canonical" because *J* are regular operators.

Non-canonical Hamiltonian mechanics

- Non-canonicality : Ker(J)

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & -I & 0 \end{pmatrix}$$

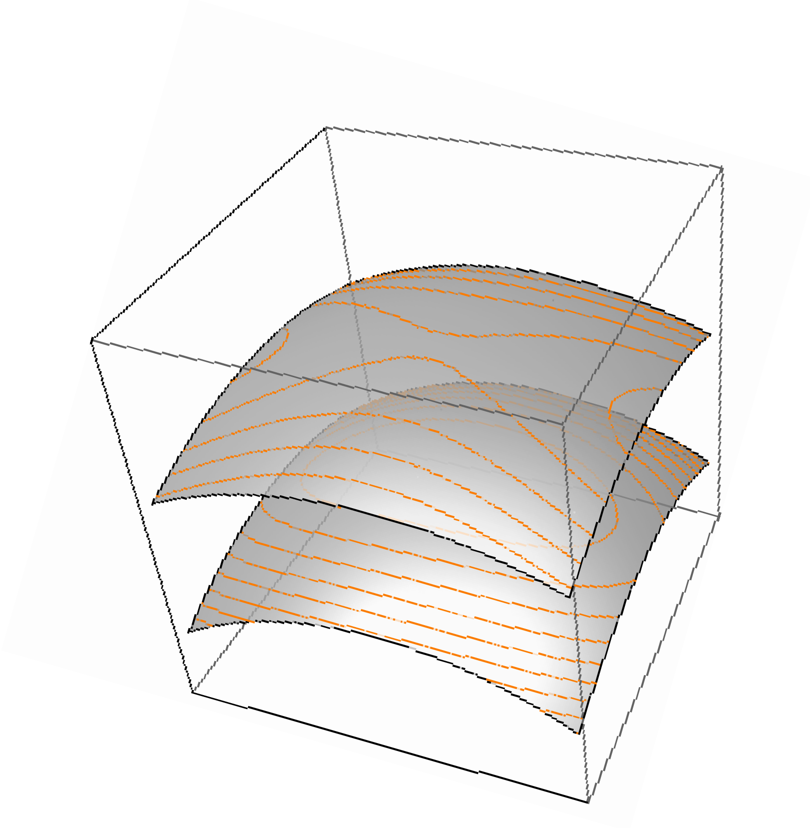
Effective phase space

- Ker(J) = Coker(J) $\rightarrow$ "topological defect"

non-canonical Hamiltonian mechanics

- Ker (J) = Coker(J) → “topological constraint”
- Foliation of Ker (J) → Casimir elements

$$\exists C \ s.t. \ \{G, C\} = 0 \ (\forall G)$$
$$i.e. \quad \partial_z C \in \mathrm{Ker}(J)$$



Scale hierarchy of magnetized particles

$$H = \frac{m}{2}\left(V_c^2 + V_\parallel^2 + V_\perp^2\right) + q\phi$$

$$H_c = \mu\omega_c + \frac{m}{2}\left(V_\parallel^2 + V_\perp^2\right) + q\phi$$

$$= \mu\omega_c + \frac{\left(P_\theta - q\psi\right)^2}{2mr^2} + \frac{p_\parallel^2}{2m} + q\phi$$

$$\mathbf{z} = (\vartheta_c, \mu; \zeta, p_\parallel; \theta, P_\theta)$$

Coarse graining → macro-hierarchy

$$z = (\vartheta_c, p_c; \zeta, p_\parallel; \theta, P_\theta) \to (\cancel{\vartheta_c}, \mu; \zeta, p_\parallel; \theta, P_\theta)$$

Coarse-graining → non-canonicalizetion

$$J = \begin{pmatrix} J_c & 0 & 0 \\ 0 & J_c & 0 \\ 0 & 0 & J_c \end{pmatrix} \to J_{nc} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & J_c & 0 \\ 0 & 0 & J_c \end{pmatrix}$$

Boltzmann distribution on Casimir leaf

$$\boxed{\delta\left(S-\alpha N-\beta E-\gamma M\right)=0}$$

$$S=-\int f\log f\,d^6z$$

$$N=\int f\,d^6z$$

$$E=\int H_c f\,d^6z$$

$$M=\int \mu f\,d^6z$$

$$\boxed{f=c\exp\left(-\beta H_c-\gamma\mu\right)}$$

Chemical potential Quasi-particle number

Embedding into the lab-frame space

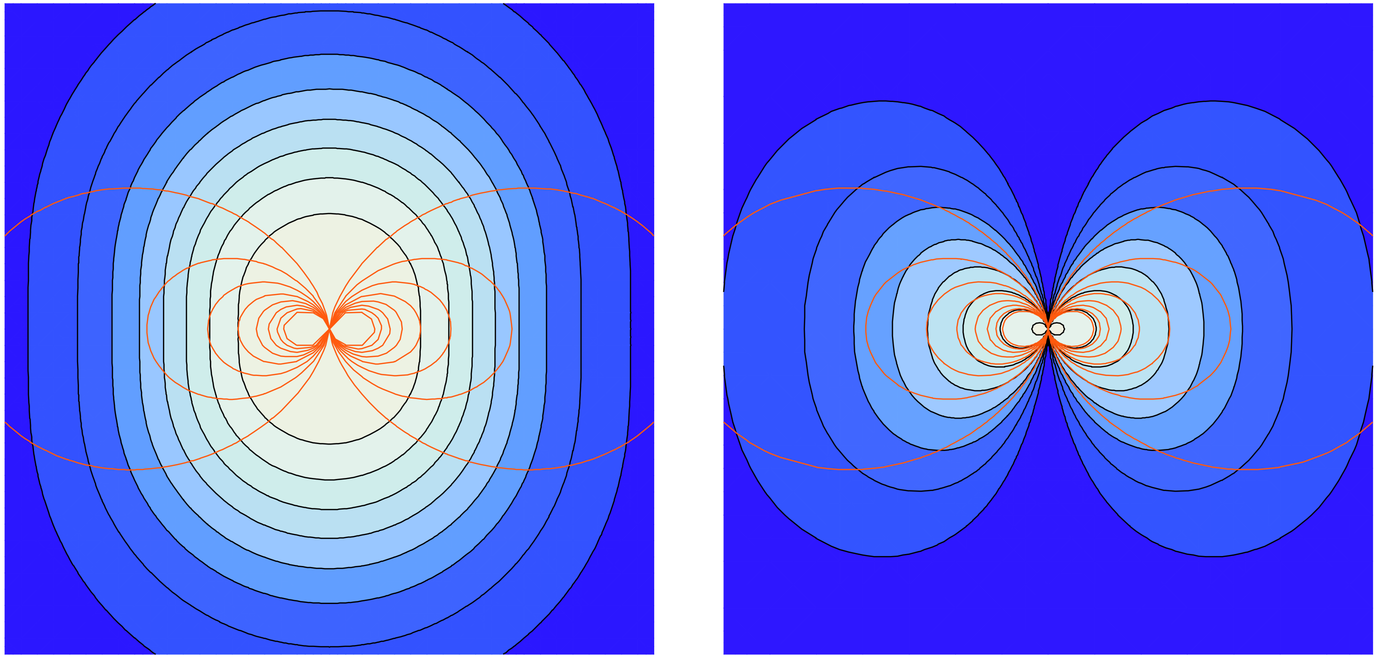
$$\rho(x) = \int f\, d^3 v = \int f\,(2\pi\omega_c / m)\, d\mu\, dv_\parallel\, dv_\theta$$

$$= c\int \exp\left(-\beta H_c - \gamma\mu\right)\frac{2\pi\omega_c\, d\mu}{m}\, dv_\parallel\, dv_\theta$$

$$= \frac{\omega_c(x)}{\omega_c(x) + \gamma}$$

Density clump in lab-frame space

Adiabatic invariants = *number* of quasi-particles
→ thermodynamic distribution on a Casimir leaf



(A) (B)

Topics in theoretical physics II
self-organization in foliated phasespace

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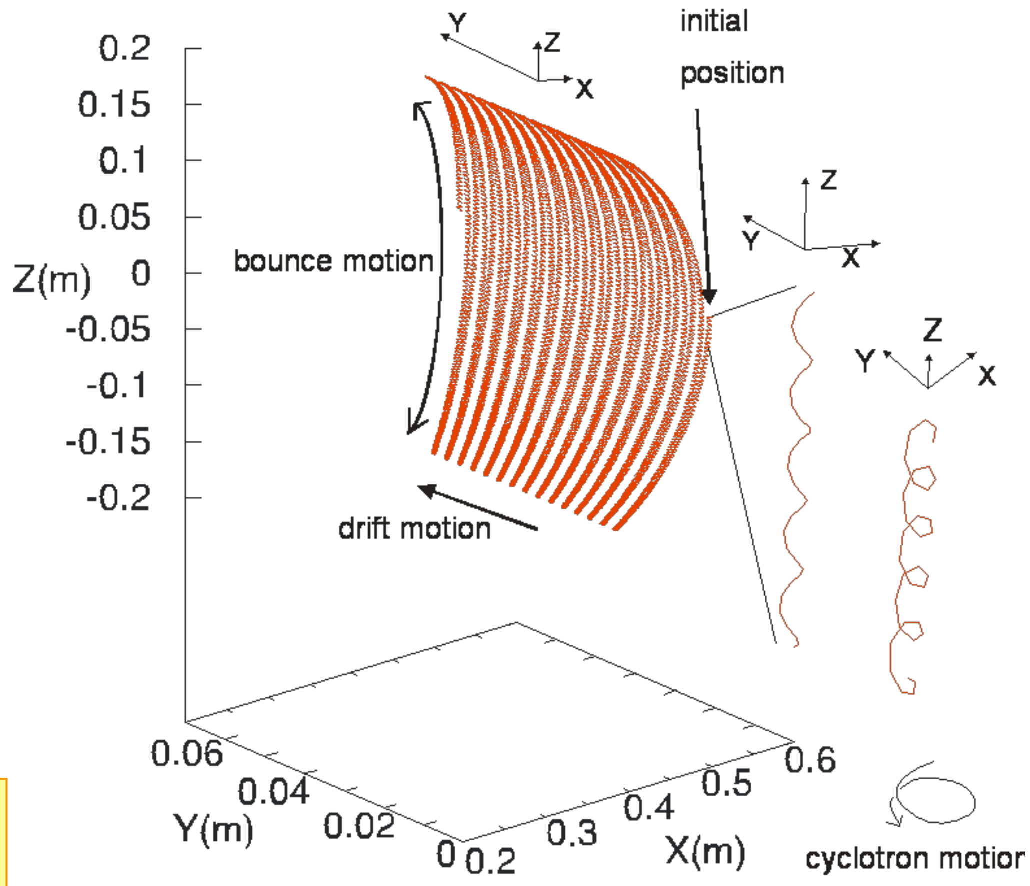
Scale hierarchy of magnetized particles

$$H = \frac{m}{2}\left(V_\perp^2 + V_\parallel^2\right) + q\phi$$

Guiding center = quasi-particle

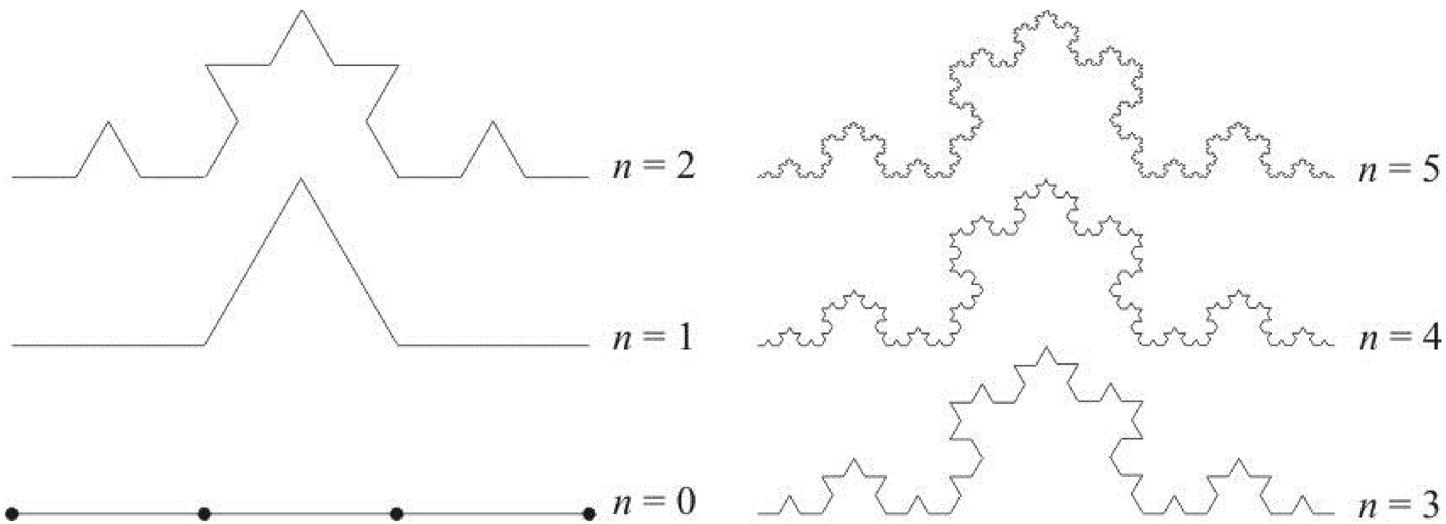
$$H_{gc} = \omega_c \mu + \omega_b J_\parallel + q\phi$$



subject defines a scale

$$z = (\vartheta_c, p_c; \zeta, p_\parallel; \theta, P_\theta) \rightarrow (\cancel{\vartheta_c}, \mu; \zeta, p_\parallel; \theta, P_\theta)$$

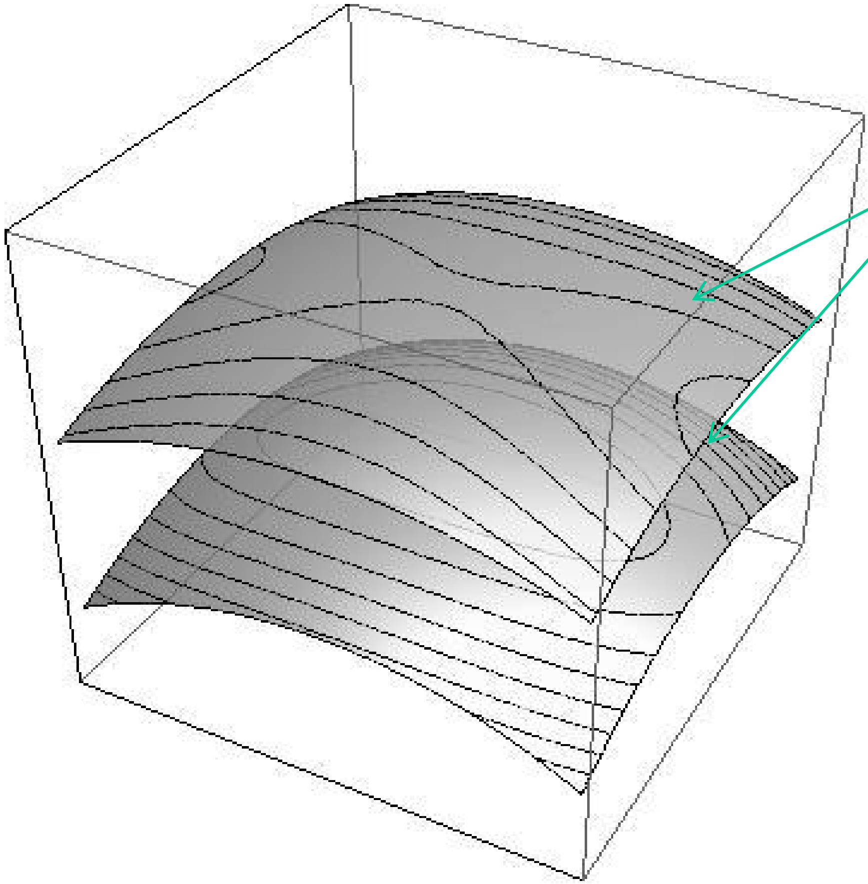
The definition (representation) of matter depends on the subject=scale.



Foliated phase space of *non-canonical* Hamiltonian system

- $\mathrm{Ker}(J) = \{\ \mathrm{grad}\, C\, (z)\ \}$

Casimir leaf



The “effective” phase space of constrained dynamics

•Equilibrium points?
•Stability?
•Thermal equilibrium?

Grand-canonical distribution on Casimir leaf

$$\boxed{\delta\left(S-\alpha N-\beta E-\gamma M\right)=0}$$

$$S=-\int f\log f\,d^6 z$$

$$N=\int f\,d^6 z$$

$$E=\int H_c f\,d^6 z$$

$$M=\int \mu f\,d^6 z$$

$$\boxed{f=c\exp\left[-\beta\left(H_c+\gamma'\mu\right)\right]}$$

Chemical potential

Quasi-particle number

Embedding into the lab-frame space

$$\rho(x) = \int f\, d^3 v = \int f \ \frac{2\pi\omega_c d\mu}{m} dv_{\parallel}$$

$$= c\int \exp\left(-\beta H_c - \gamma\mu\right)\frac{2\pi\omega_c d\mu}{m} dv_{\parallel}$$

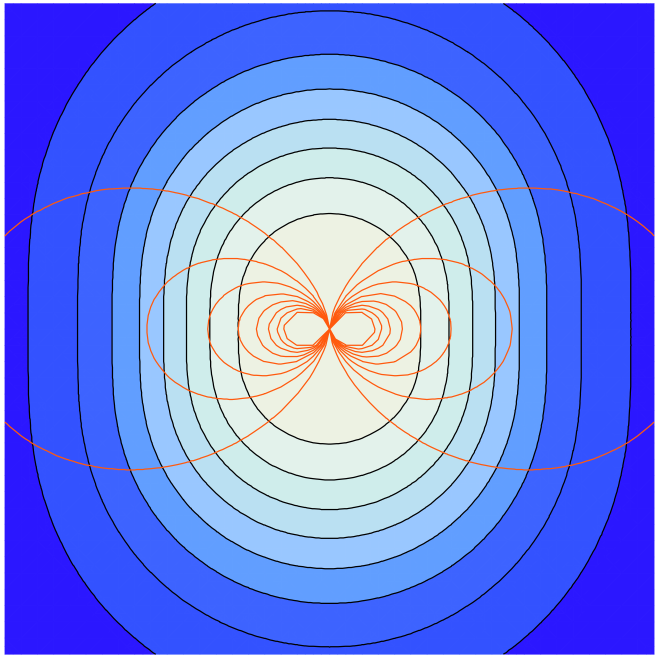
$$= \frac{\omega_c(x)}{\omega_c(x)+\gamma}$$

Density clump in lab-frame space

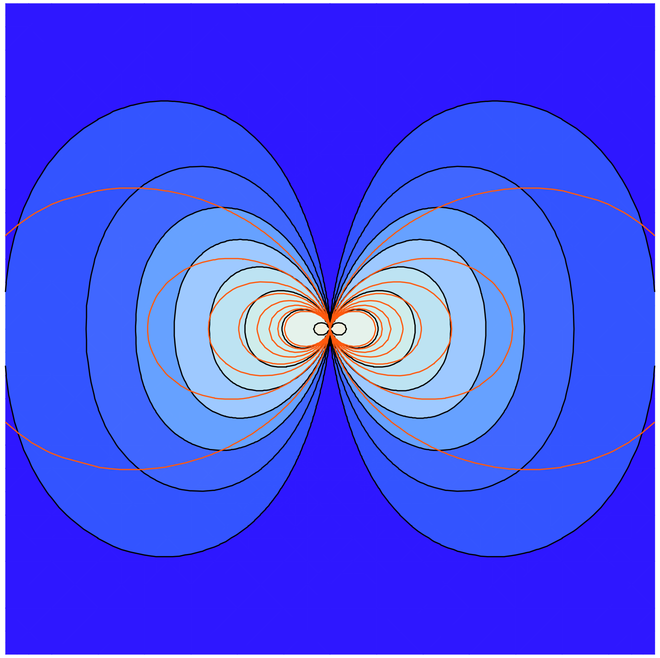
Adiabatic invariants

→ *phase space* of "quasi-particles"

→ thermodynamic distribution on a Casimir leaf



(A)



(B)

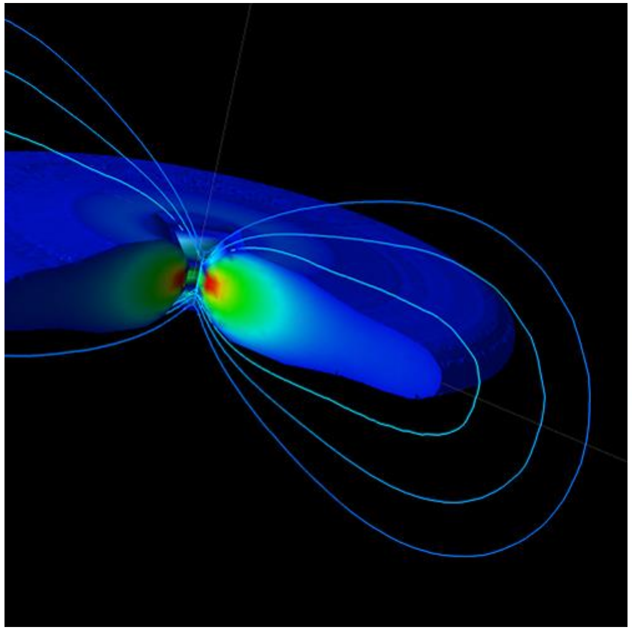
Conclusion (I)

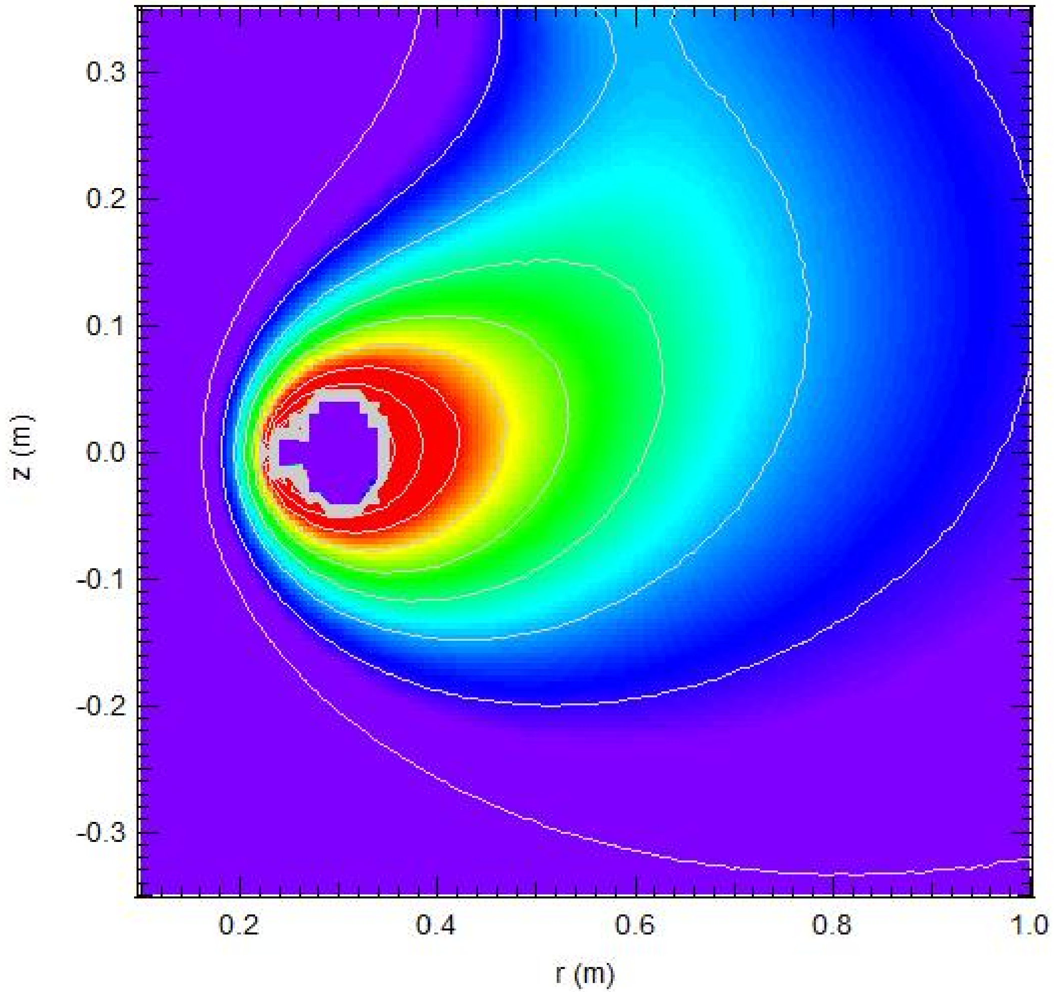
- Scale hierarchy = foliated phase space
- Adiabatic invariant = *Casimir element*
  → foliation
- Distorted metric on a leaf → structure

Z.Yoshida & S.M. Mahajan, *Self-organization in foliated phase space: construction of a scale hierarchy by adiabatic invariants of magnetized particles*, Prog. Theor. Exp. Phys. **2014** (2014), 073J01

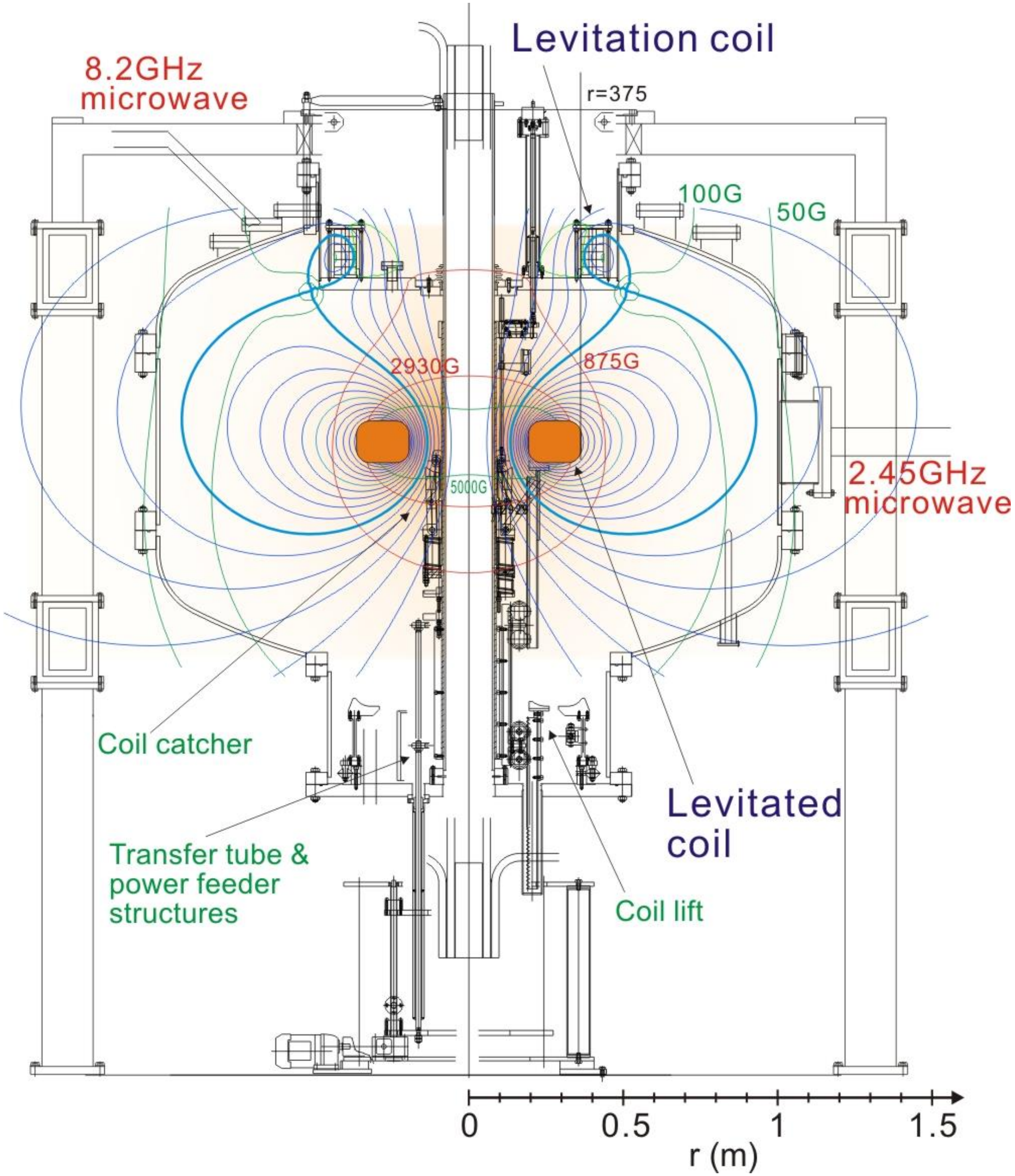
A magnetosphere on the Earth

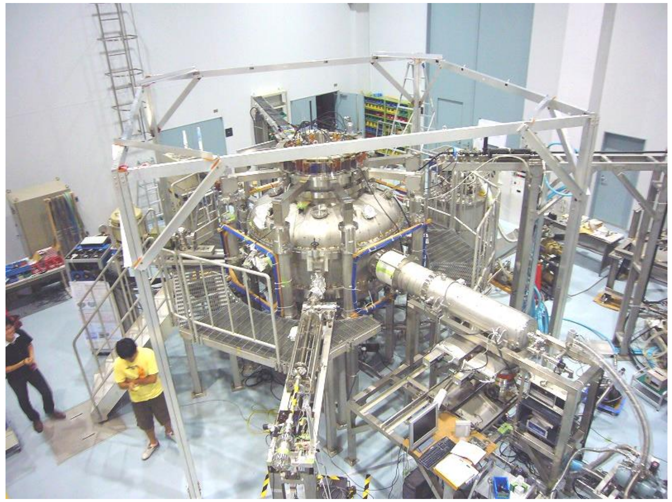
RT-1 project



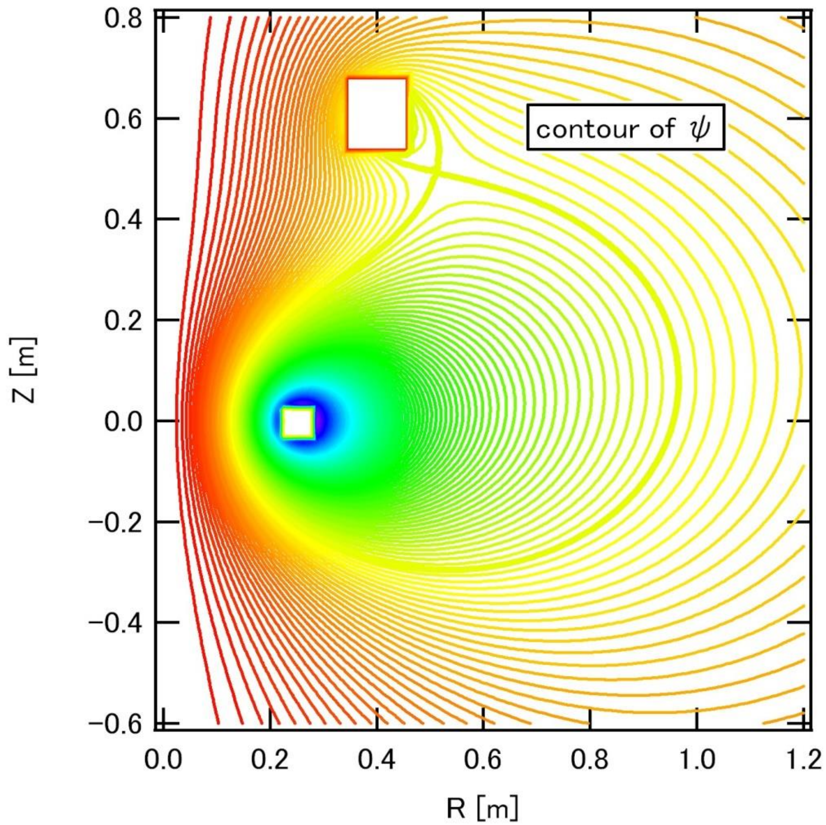


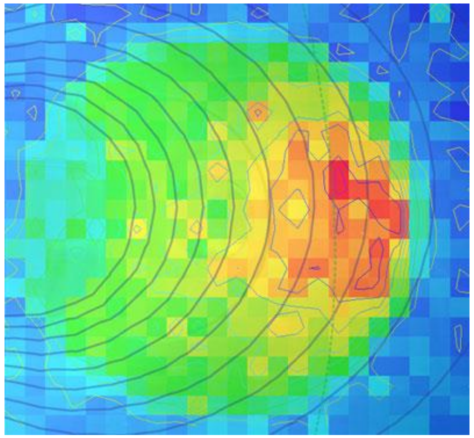
Levitating HTC superconducting magnet system





High-beta plasma confinement

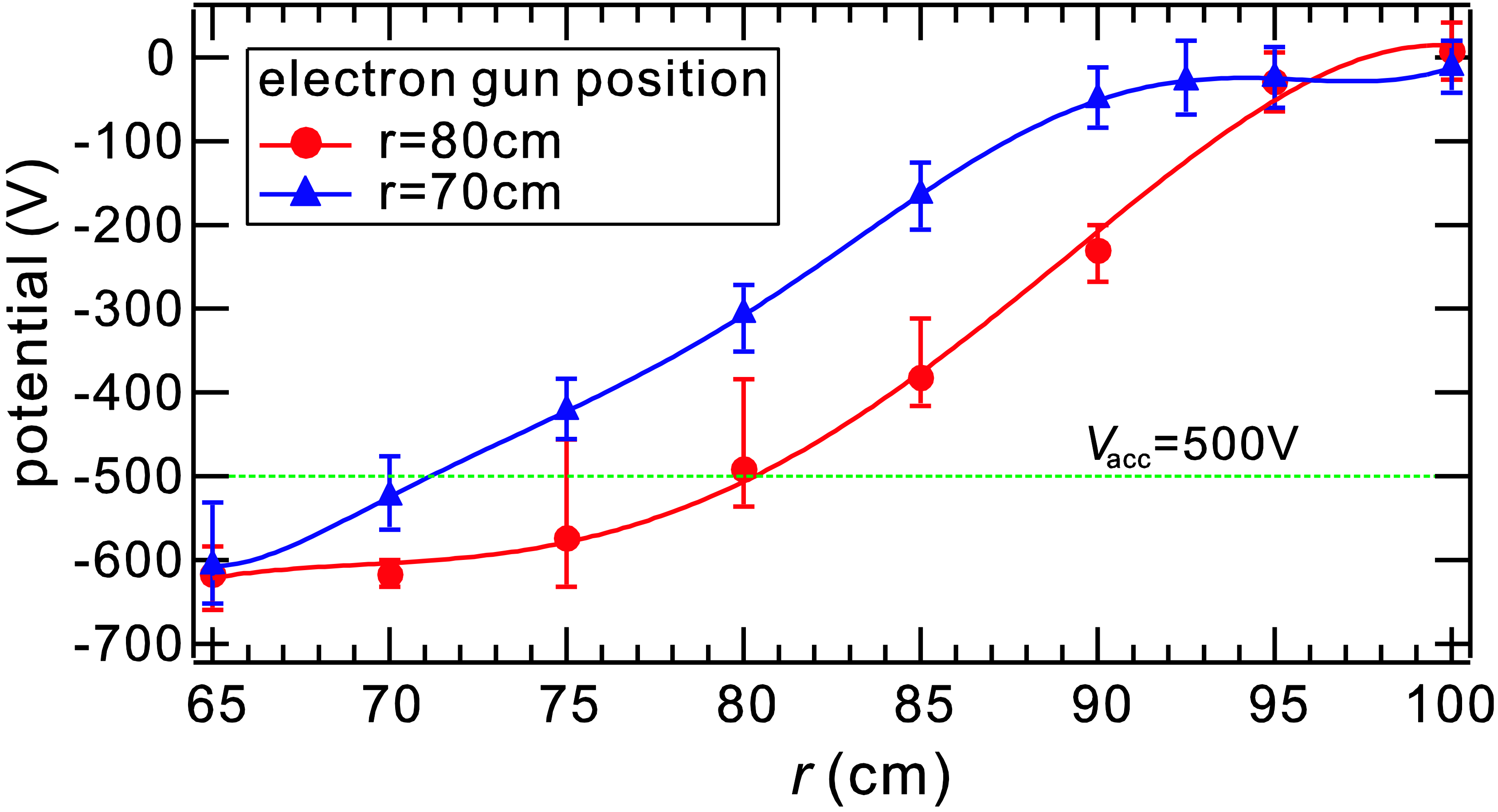




$$T_e \approx 10 \sim 30\mathrm{keV}, \quad n_e = 10^{16} - 10^{17}\ \mathrm{m}^{-3}$$

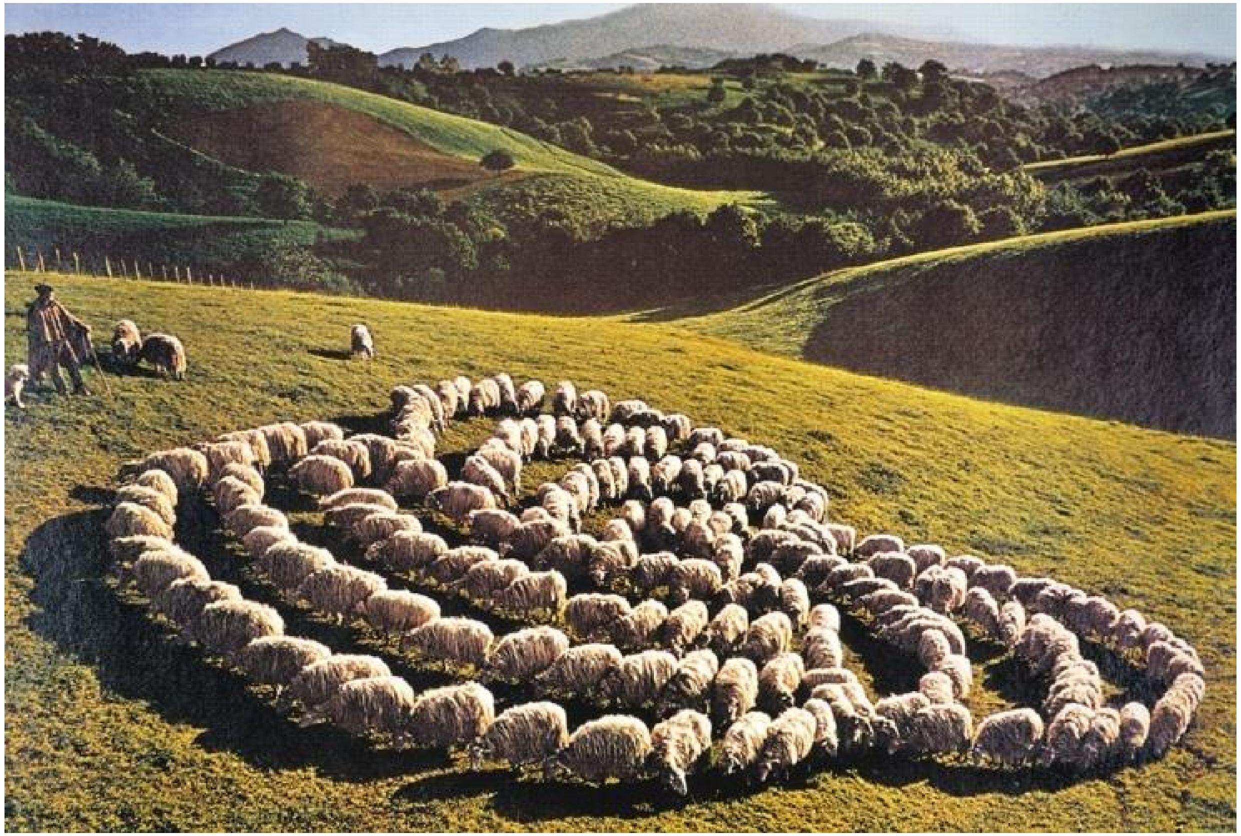
$$\beta \approx \beta_e \approx 0.7, \quad \tau_E \approx 0.5\,\mathrm{sec}$$

Inward (up-hill) diffusion



ZY *et al.*, Phys. Rev. Lett. **104** (2010), 235004 .

space for “confinement”



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Conclusion (II)

- Experimental proof of *self-organized confinement*.
- Experimental evidence of *inward diffusion*.

Topics in theoretical physics III
self-organization in MHD plasma

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MHD (incompressible)

- Naïve form:

$$\boxed{\begin{aligned}&\partial_t \boldsymbol{V} = -P_\sigma\left(\nabla \times \boldsymbol{V}\right)\times \boldsymbol{V} + P_\sigma\left(\nabla \times \boldsymbol{B}\right)\times \boldsymbol{B},\\ &\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}).\end{aligned}}$$

- State vector:

$$\boxed{u = (\boldsymbol{V}, \boldsymbol{B}) \in L^2_\sigma(\Omega) \times L^2_\sigma(\Omega)}$$

$$L^2_\sigma(\Omega) := \left\{ \boldsymbol{v} \in L^2_\sigma(\Omega); \nabla \cdot \boldsymbol{v} = 0, \boldsymbol{n} \cdot \boldsymbol{v} = 0 \right\}$$

$$P_\sigma : L^2(\Omega) \to L^2_\sigma(\Omega)$$

Hamiltonian form of MHD

- Hamiltonian

$$H(u) = \frac{1}{2}\int_\Omega \left(V^2 + B^2\right) dx = \frac{1}{2}\|u\|^2$$

- Poisson operator

$$J(u) = \begin{pmatrix} -P_\sigma(\nabla \times \boldsymbol{V}) \times \circ & P_\sigma(\nabla \times \circ) \times \boldsymbol{B} \\ \nabla \times (\circ \times \boldsymbol{B}) & 0 \end{pmatrix}$$

- *Casimir elements:*

$$C_1(u) = \frac{1}{2}\langle \boldsymbol{A}, \boldsymbol{B} \rangle, \quad C_2(u) = \langle \boldsymbol{V}, \boldsymbol{B} \rangle$$

Beltrami equilibria on helicity leaves

- Beltrami equilibrium:

$$\boxed{\partial_u H_\mu(u) = 0 \quad (H_\mu = H - \mu C_1).}$$

$$\rightarrow \boxed{\nabla \times \boldsymbol{B} = \mu \boldsymbol{B}, \quad \boldsymbol{B} \in L^2_\sigma(\Omega).}$$

- Two classes of Beltrami eigenvalues:

(1) Self-adjoint curl operator S

$\rightarrow$ discrete real eigenvalues $\mu \in \{\lambda_1, \lambda_2, \cdots\}$

(2) Non-self-adjoint extension T

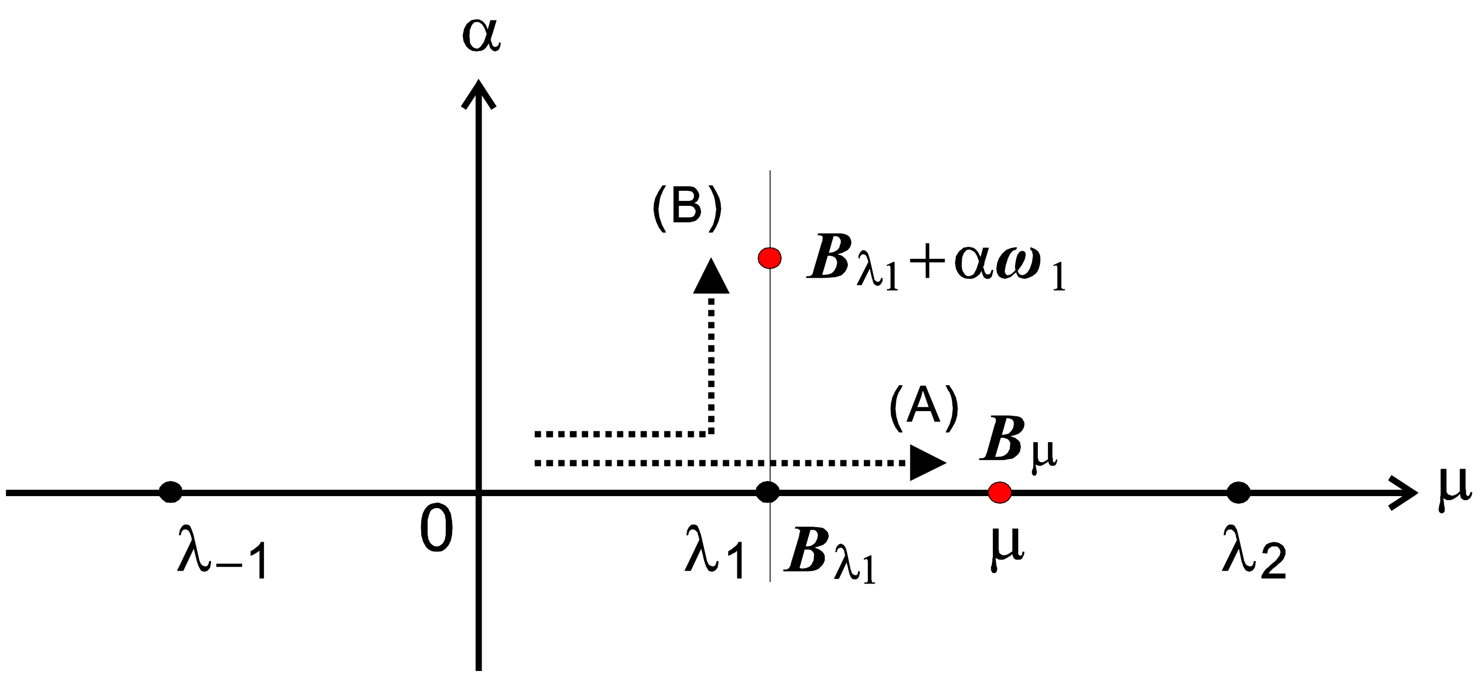
(if Ω is multiply connected) $\rightarrow \forall \mu \in \mathbf{C}$

ZY & Y. Giga, Math. Z. **204** (1990) 235.

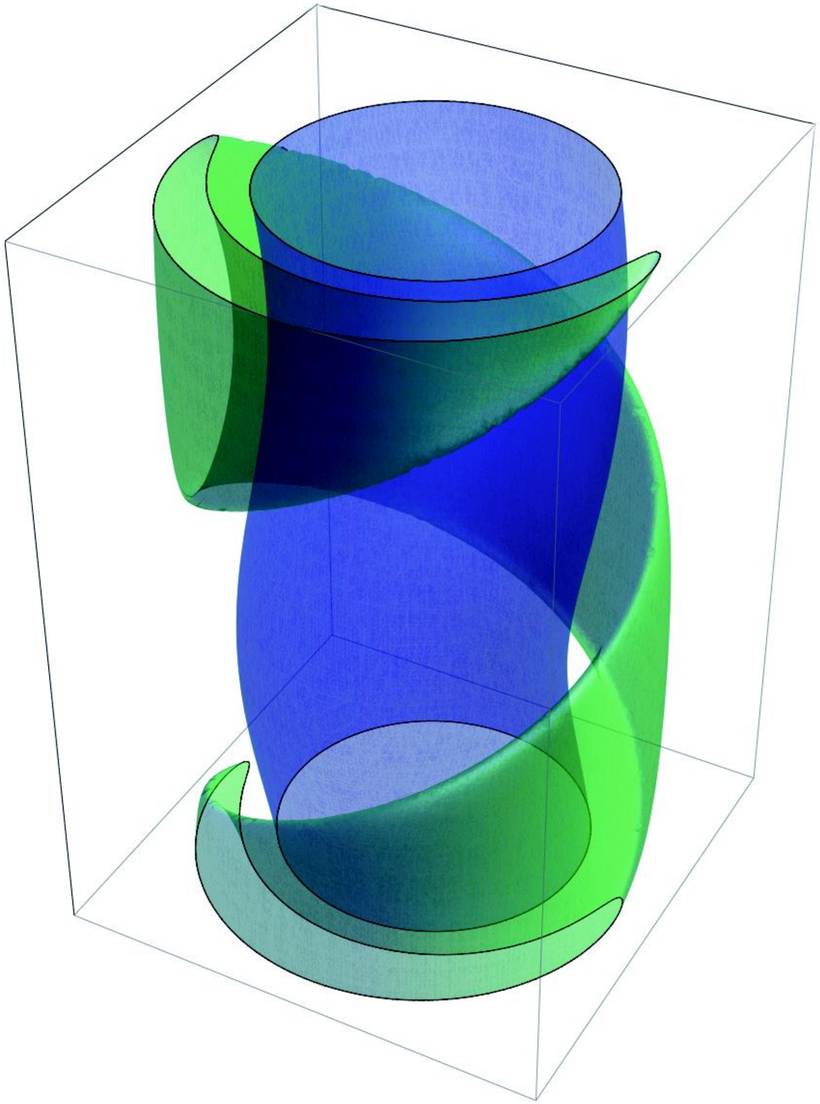
Bifurcation theorem

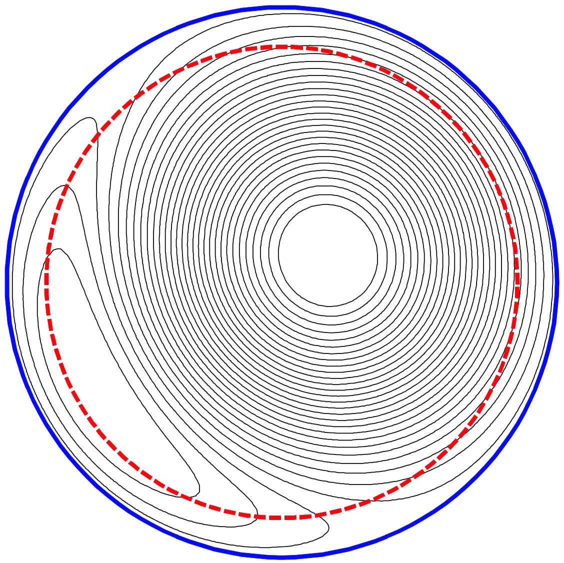
Theorem 1

Let $\boldsymbol{A}_{\mathrm{H}}$ *be the vector potential of* $\boldsymbol{B}_{\mathrm{H}}$ *(cohomology).*
If $\langle \boldsymbol{A}_{\mathrm{H}}, \boldsymbol{\omega}_j \rangle = 0$, *then branch-(B) bifurcates at* $\mu = \lambda_j \in \sigma_p(S)$.



Bifurcated Beltrami equilibrium





More Casimirs: Linearized MHD

- Linearize near a Beltrami equilibrium $\boldsymbol{B}_\mu$.
- Hamilton's operator

$$\boxed{L_\mu = \begin{pmatrix} 1 & 0 \\ 0 & 1-\mu S^{-1} \end{pmatrix}}$$

- Poisson operator

$$\boxed{J_\mu = \begin{pmatrix} 0 & P_\sigma(\nabla\times\circ)\times\boldsymbol{B}_\mu \\ \nabla\times(\circ\times\boldsymbol{B}_\mu) & 0 \end{pmatrix}}$$

Resonance singularity

- Ker (J_μ) consists of singular eigenfunctions.

$$\boldsymbol{v} \quad \text{such that } \nabla\times(\boldsymbol{B}_\mu\times\boldsymbol{v})=0,$$
$$\boldsymbol{b} \quad \text{such that } \boldsymbol{B}_\mu\times(\nabla\times\boldsymbol{b})=0$$

- In slab geometry: $\boldsymbol{b}$ obeys

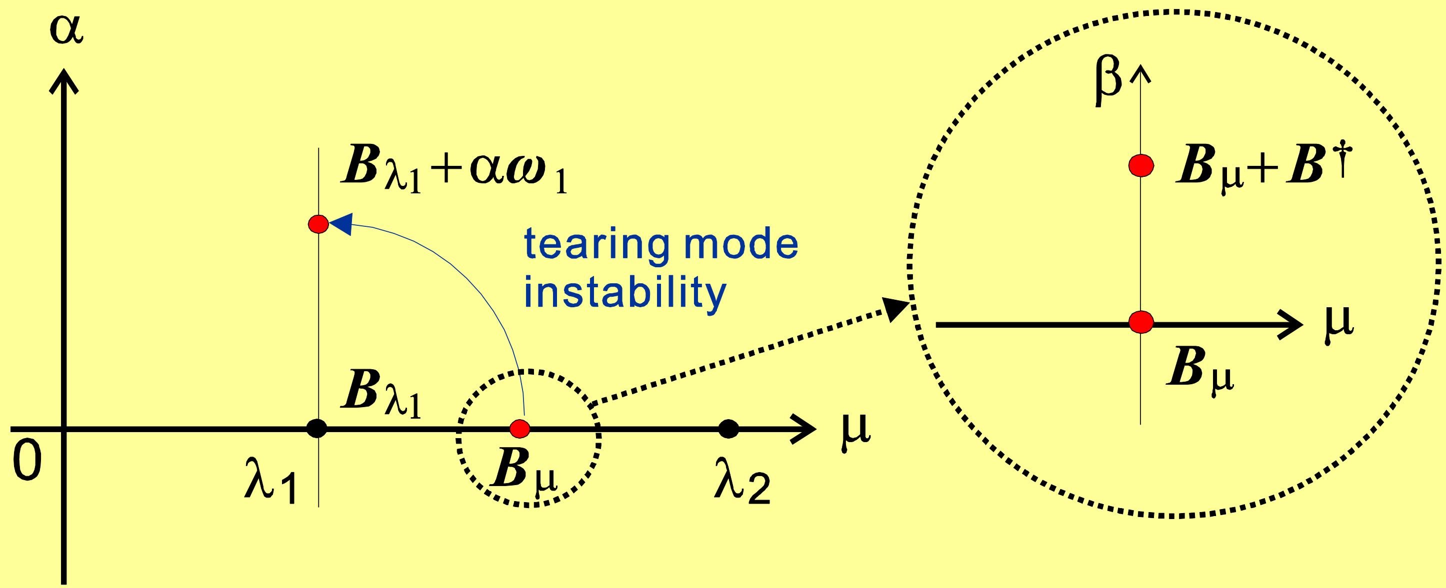
$$\boldsymbol{b}=(0,b_y(x),b_z(x))e^{i(k_y y+k_z z)}$$
$$b_y(x)=ik_y\theta(x),\ b_z(x)=ik_z\theta(x)$$
$$\Rightarrow\left[B_y(x)k_y+B_z(x)k_z\right]\partial_x\theta(x)=0$$
$$\Rightarrow\theta(x)=cY(x-x^+)$$

$$\Rightarrow C_b=(\widetilde{\boldsymbol{B}},\boldsymbol{b})$$

Tearing mode (singular equilibria)

Theorem 2

When Ω has a symmetry, J_μ has helical flux Casimirs $C_b(\tilde{\boldsymbol{u}}) = (\tilde{\boldsymbol{B}}, \boldsymbol{b})$. The stationary point of the energy-Casimir functional gives a tearing-mode singular eigenfunction.



Excess energy of Beltrami equilibrium

- *We may estimate*

$$(\tilde{u}, L_\mu \tilde{u})/2 \geq (1 - \mu/\lambda_1)(\tilde{\boldsymbol{B}}, \boldsymbol{\omega}_1)^2/2$$

- *Under the constraint of the helical-flux Casimir,*

$$\min_{C_b(\tilde{u}) = c_b} (\tilde{u}, L_\mu \tilde{u}) = (1 - \mu/\lambda_1)[c_b/(\boldsymbol{b}, \boldsymbol{\omega}_1)]^2/2$$

Conclusion (III)

- The *helicity* is a Casimir element that foliates the phase space.
- We find bifurcated equilibrium points on helicity leaves, because helicity leaves are distorted with respect to the energy norm.
- A tearing mode is an equilibrium on a leaf of singular (resonant) Casimir element

Z. Yoshida & R. L. Dewar; J. Phys. A **45** (2012), 365502 1-36.

Unfreezing Casimir → Canonicalization

$$J_{nc} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & -I & 0 \end{pmatrix} \Rightarrow \tilde{J} = \begin{pmatrix} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \end{pmatrix}$$

Effective (macro) phase space

"recovered" angle variable

Canonicalization of MHD

- Given a Casimir $C_b(\tilde{\boldsymbol{u}}) = (\tilde{\boldsymbol{B}}, \boldsymbol{b})$, we put

$$\boxed{P_\perp \tilde{\boldsymbol{B}} = (\tilde{\boldsymbol{B}}, \boldsymbol{b})\boldsymbol{b} \in \mathrm{Ker}(J_\mu), \quad P_\parallel \tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}} - P_\perp \tilde{\boldsymbol{B}}}$$

- Separating the kernel, we write

$$\boxed{J_\mu = \begin{pmatrix} 0 & P_\sigma(\nabla \times P_\parallel \circ) \times \boldsymbol{B}_\mu & 0 \\ P_\parallel \nabla \times (\circ \times \boldsymbol{B}_\mu) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

- Canonicalize

$$\boxed{\tilde{J}_\mu = \left(\begin{array}{ccc|c} 0 & P_\sigma(\nabla \times P_\parallel \circ) \times \boldsymbol{B}_\mu & 0 & 0 \\ P_\parallel \nabla \times (\circ \times \boldsymbol{B}_\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ \hline 0 & 0 & -I & 0 \end{array}\right)}$$

Singular perturbation

- Original Hamiltonian is independent of θ (Casimir → Action); denoting $K=1-\mu S^{-1}$

$$L_\mu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & P_\parallel K & P_\parallel K \\ 0 & P_\perp K & P_\perp K \end{pmatrix}$$

- perturbation

$$\tilde{L}_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & P_\parallel K & P_\parallel K & 0 \\ 0 & P_\perp K & P_\perp K & 0 \\ 0 & 0 & 0 & \boxed{D} \end{pmatrix}$$

Enlarged Hamiltonian system

- Denoting $p = C_b(\widetilde{\boldsymbol{u}}) = (\widetilde{\boldsymbol{B}}, \boldsymbol{b})$

$$\boxed{\begin{cases} \dot{p} = -D\theta \\ \dot{\theta} = (K_\mu \widetilde{\boldsymbol{B}}, b) \end{cases}}$$

- For $\widetilde{\boldsymbol{B}} = p\boldsymbol{\omega}_1$,

$$\boxed{(K_\mu \widetilde{\boldsymbol{B}}, \boldsymbol{b}) = (1 - \mu / \lambda_1)(\boldsymbol{\omega}_1, \boldsymbol{b})\, p}$$

- Beyond the bifurcation point, *negative mass*:

$$\boxed{(1 - \mu / \lambda_1)(\boldsymbol{\omega}_1, \boldsymbol{b}) < 0}$$

$\rightarrow$ *instability*

Conclusion (IV)

- Perturbing the Hamiltonian system by adding (recovering) an angle variable, we may unfreeze the Casimir invariant, allowing the tearing mode to emerge.

Z. Yoshida and P. J. Morrison, Unfreezing Casimir invariants: singular perturbations giving rise to forbidden instabilities, in *Nonlinear physical systems: spectral analysis, stability and bifurcation*, (ISTE and John Wiley and Sons, 2014), Chap. 18; arXiv:1303.0887

Remark

- Our Universe is not simply foliated: singularities yield complex bifurcation of leaves.

1. S.M. Mahajan and Z. Yoshida, PRL **105** (2010), 095005.
2. Z. Yoshida, P. J. Morrison and F. Dobarro, J. Math. Fluid Mech. **16** (2014), 41—57; arXiv:1107.5118
3. Z. Yoshida and P. J. Morrison, Fluid Dyn. Res. **46** (2014), 031412; arXiv:1401.7698
4. Z. Yoshida, Y. Kawazura, and T. Yokoyama, J. Math. Phys. **55** (2014), 043101