# Elementary Ideas- Advanced Understanding
# Conservation Laws-Plasma Dynamics

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# General - 1

## Conservation Laws - Conserved Quantities

Certain combinations of dynamical variables that remain "unchanged" during the physical evolution of a system.

The existence of such invariants restricts the dynamics

This restriction is of great essence

Very often it is the existence of an invariant (invariants) that facilitates (or guarantees) the solution of a dynamical problem.

You all know about the relationship of 'integrability' with invariants

In this lecture we will explore the subject, primarily, in the background of plasma dynamics

# General - 2

There are two main sources of Conservation laws

(1) Symmetries of the underlying Lagrangian
- space-time symmetries
- gauge invariance

The existence of such symmetries, for instance, leads to what are know as "Noether Currents".
These currents are 'conserved'.

(2) Then there are topological invariants
These invariants do not follow from an explicit ~~invaria~~ symmetry of the Lagrangian:
The nature and function of this second class of invariants is very different from the first:
They are often associated with the stability of solutions.
(2) also has conserved currents but these are not "Noether Currents".

# Conserved Currents.

Remember the entire purpose of this exploration is to look for some physical quantity which does not change with time.

There is a simple recipe for achieving this

I will deliberately deal only with four vector currents (though we could have any tensorial currents)

Let $J^{\mu} = \{J^{0}, \underset{\sim}{J}\}$ be a four current

( $\mu = 0 - 3$ and latin indices $i = 1-3$ )

For it to lead to a conservation law, we must have

$$\partial_{\mu} J^{\mu} = 0$$

i.e, the current is divergence less (four)

Let us see how ?

# Conserved Currents - Invariants

$$\partial_t J^0 + \underset{\sim}{\nabla} \cdot \underset{\sim}{J} = 0$$

$$\int \partial_t J^0 d^3x \equiv \frac{d}{dt} Q + \int \underbrace{\nabla \cdot \underset{\sim}{J} \, d^3x}_{\text{surface term} = 0} = 0$$

$$\Rightarrow \qquad \frac{dQ}{dt} = 0 \,, \quad Q = \int J^0 d^3x \text{ is conserved}$$

Some examples :

(1) The Continuity equation is called a Conservation law - So what is conserved ? what is the current ? — The flux $\Gamma^\mu = (n, n\underset{\sim}{u})$

Clearly then, the conserved "charge", corresponding to $\Gamma^\mu$ is $= \int \Gamma^0 d^3x$,

$$N = \int n \, d^3x = \text{the number of particles}$$

(2) So in the equation of motion :

$$\partial_\mu T^{\mu\nu} = 0$$

$T^{\mu\nu}$ is the total energy momentum.

Conserved ~~Qua~~ "charge"

$\cdot \; P^\nu = \int T^{0\nu} d^3x$ — Energy - Momentum

# Conservation Laws - An aside

If we have an equation of the form

$$\partial_\mu \underline{J}^\mu = \nabla \cdot \underset{\sim}{S} \qquad (a)$$

Then again, ~~be~~ if $S_n = 0 = J_n$ at $\pm\infty$,

$$Q = \int \underline{J}^0 d^3x$$

is conserved.

There are situations when an Equation of ~~the~~ type (a) may be allowed by Lorentz invariance.

For instance if $S^0$ is time independent and the we could have an

$$S^\mu = (S^0, \underset{\sim}{S})$$

$$\partial_\mu S^\mu \equiv \nabla \cdot S$$

Theres is no non-trivial 'conserved' 'charge' with such an $S^\mu$.

Finding Topological Invariants in a Dynamics Constitutes fundamental progress

By forbidding certain classes of motion, the topological invariants may gaurantee the stability of solutions.!

# Topological Invariants - A simple Model

## The Sine-Gordon Kink

Scalar field in 1+1 dimensions:

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + a \sin b\phi = 0 \qquad (a)$$

The Lagrangian for S.G

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - V(\phi) \qquad (b)$$

The Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + V(\phi) \qquad (c)$$

$$V(\phi) = \frac{a}{b}\left(1 - \cos b\phi\right) \qquad (d)$$

It possesses stationary as well as moving soln.

$$\phi(x,t) = f(x - vt) = f(\xi) \qquad (e)$$

$$f(\xi) = \frac{4}{b} \arctan^{\pm r\xi} \qquad (f)$$

with $r = (1-v^2)^{-1/2}$ — A solitary wave

It also has an infinite no. of constant solutions

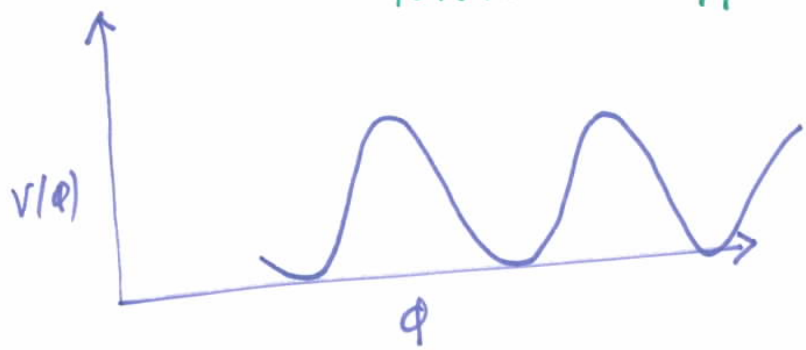
$$\phi = \frac{2\pi n}{b} \qquad n = 0 \text{ or an integer} \qquad (g)$$

For (g), $V = 0$, and $\therefore$ $\mathcal{H} = 0$

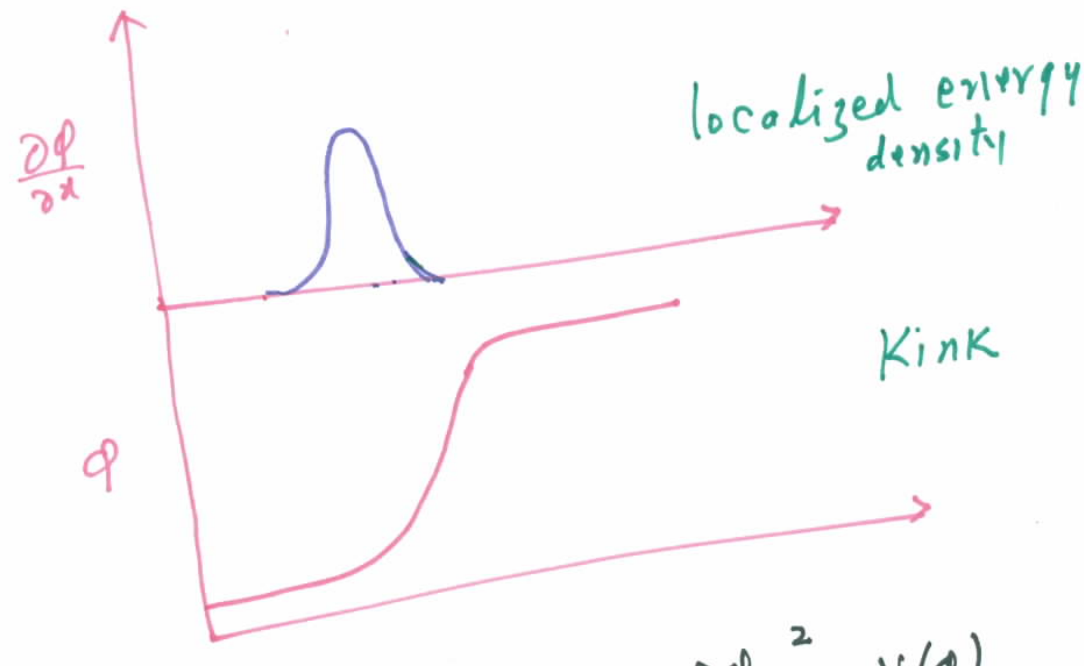
These are zero energy solutions $\Rightarrow$ Degenerate Vacua

# Sine-Gordon Kink

Potential energy vs. $\varphi$




For the stationary Kink ($\partial_t = 0$)




$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial V}{\partial \varphi} \Rightarrow \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 = V(\varphi)$$

Let us calculate the energy of this stationary kink.

# Sine-Gordon Kink

$$E = \int \mathcal{H}\, dx = \int \left[\frac{1}{2}\left(\frac{\partial q}{\partial x}\right)^2 + V(q)\right] dx$$

$$= \int 2V(q)\, dx = \int_0^{2\pi/b} \left[2V(q)\right]^{1/2} dq$$

where we have chosen to integrate between $n=0$ and $n=1$ of the degenerate vacua (we could have chosen any two)

$$E = 8\left(\frac{a}{b^3}\right)^{1/2} \equiv \text{finite}$$

This is a stable soliton solution with finite energy

Why is it 'mathematically' stable ?

The answer lies in the boundary conditions:

# Sine- Gordon Kink ↔ Stability

space, here, is an infinite line

• . . . . . . •

$n=0$ $n=1$

1- Kink

Any such system ($n=0$, $n=1$) cannot be continously deformed, for instance, to the ground state $n=0$, $n=0$

The stability, therefore, depends on the topological properties of the space : The boundary points of this space is a discrete set

Just like this Kink, the stability of soliton solutions in nonlinear field theories is a consequence of topology

But where is the Conservation Law - The Conserved Current . . . . .

It is obvious that, in this example, the conserved 'charge' $Q$ must be an integer $N$.

Let

$$J^{\mu} = \frac{b}{2\pi} \epsilon^{\mu\nu} \partial_\nu \varphi$$

Remember $\mu = (0,1)$. $\epsilon^{\mu\nu}$ is an antisymmetric unit tensor with $\epsilon^{01} = 1$

So we have

$$J^0 = \frac{b}{2\pi} \epsilon^{0\nu} \partial_\nu \varphi = \frac{b}{2\pi} \frac{\partial \varphi}{\partial x}$$

$$Q = \int_{-\infty}^{+\infty} J^0 dx = \frac{b}{2\pi} \int_{-\infty}^{+\infty} \frac{\partial \varphi}{\partial x} dx$$

$$= \frac{b}{2\pi} \left[ \varphi(\infty) - \varphi(-\infty) \right] = N$$

$N = n_1 - n_2$.

$N$ is a topological label for a Kink - Since it cannot be continously deformed to any other number, the Kink is stable.

# Plasmas

## Hot charged fluids.

Topological invariants of a hot charged fluid interacting with an electromagnetic field.

You have all heard of helicity in classical as well as well as in quantum mechanics.

Definition:

The helicity of a vector field $\underset{\sim}{B}$

$$h = \int \underset{\sim}{B} \cdot (\nabla \times^{-1} \underset{\sim}{B})\, d^3x \qquad (a)$$

And if $\underset{\sim}{B}$ is the standard magnetic field, then $\nabla \times^{-1} \underset{\sim}{B} = \underset{\sim}{A}$, the vector potential

$$h_m = \int \underset{\sim}{A} \cdot \underset{\sim}{B}\, d^3x \qquad (b)$$

the familiar expression.

Expression (b) has no reference to any fluid attributes!

# Helicity - Generalized Helicity

Question: One can show that in classical electromagnetism the magnetic helicity $h_m$ is conserved: Is there a corresponding Generalized helicity $G$ for a charged fluid - electromagnetic field system

Naturally for this $G$ to be interesting, it must be a constant of the motion.

In the rest of this lecture we will develope an extremely general method for constructing $G$ and then end it by discussing some of the consequences of $G$.

# A Hot Relativistic Charged Fluid

(S.M. Mahajan, Phys. Rev. Lett 90 (3), Jan 2003)

## A unified Magneto-Fluid Formalism

## Equation of Motion

$$T\partial^{\nu}\sigma = q\, U_{\mu} M^{\nu\mu} \qquad (a)$$

$q$ = charge, $T$ is the temperature, $\sigma$ is the entropy

$$U^{\mu} = \{\gamma, \gamma \underset{\sim}{v}\} \qquad (b)$$

is the four velocity and

$$M^{\nu\mu} = F^{\nu\mu} + \frac{m}{q} S^{\nu\mu} \qquad (c)$$

is the unified magnetofluid tensor,

$$S^{\mu\nu} = \partial^{\mu} f U^{\nu} - \partial^{\nu} f U^{\mu} \qquad (d)$$

is the fluid tensor (fully antisymmetric) constructed in analogy to the Faraday tensor

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \qquad (e)$$

$f U^{\mu}$ is the fluid equivalent of the e.m four potential $A^{\mu}$. $f = f(T)$ is related to the enthalpy density.

# Magneto - Fluid Equations



Just to get a better feel, let us examine the unified tensor $M^{\mu\nu}$:

We know that, ordinarily, $(F^{23}, F^{31}, F^{12})$ define the magnetic field $\underset{\sim}{B}$.

$(M^{23}, M^{31}, M^{12})$ define

$$\underset{\sim}{\hat{B}} = \underset{\sim}{B} + \frac{m}{q} \nabla \times (f r \underset{\sim}{v})$$

Generalized Magnetic field (GM)

or

$$\frac{q \underset{\sim}{\hat{B}}}{m} = \frac{q \underset{\sim}{B}}{m} + \nabla \times f r \underset{\sim}{v}$$

Generalized Vorticity (GV)

The role of the two terms on determining the dynamics is exactly the same [ The vorticity-term could cause Zeeman splitting just as $\underset{\sim}{B}$ does ]

Generalized Electric fields . . .

Remarkable thing is that the creation ⇒

# Magneto - Fluids

of a unified four potential for a hot fluid

$$\hat{A}^\mu = A^\mu + \frac{m}{q} f U^\mu \qquad (a)$$

follows exactly the prescription of the 'minimal coupling' in particle dynamics.

We will exploit this analogy to determine a conserved four vector for this magnetofluid.

Without any fluid, for the pure field

$$h_m = \int \underset{\sim}{A} \cdot \underset{\sim}{B} \, d^3x = \int P^0 d^3x \qquad (b)$$

the only $P^\mu$ made entirely from the field variables $A_\mu$ $F^{\mu\nu}$ (or its $\mathcal{F}^{\mu\nu}$). that has $\underset{\sim}{A} \cdot \underset{\sim}{B}$ as $P^0$ is

$$P^\nu = A_\mu \mathcal{F}^{\mu\nu} \qquad (c)$$

For the magnetic fluid, then, in analogy with (a), we must explore the four vector

$$K^\mu = \hat{A}_\nu \mathcal{M}^{\nu\mu} = \left(A_\nu + \frac{m}{q} f U_\nu\right) \mathcal{M}^{\nu\mu} \qquad (d)$$

Will this be Conserved ?

If $\partial_\mu K^\mu = 0$, $\partial_\mu K^\mu = \nabla \cdot S$ 

In either case

As long as $K_n = 0 = S_n$ at the surface of the domain

$$G = \int K^0 d^3x$$

$$= \int d^3x \left[ \hat{\underset{\sim}{A}} \cdot \hat{\underset{\sim}{B}} \right]$$

$$= \int d^3x \left[ \underset{\sim}{A} + \frac{m}{q} f \gamma \underset{\sim}{v} \right] \cdot \left[ \underset{\sim}{B} + \frac{m}{q} \nabla \times f \gamma \underset{\sim}{v} \right]$$

is conserved.

In the helicity density $K^0$, $f$ represents temperature and $\gamma$ the kinematic relativistic factor. It holds for 'arbitrary' speeds and temperatures.

# Conserved Four Vector - Conserved GH (GV)

First step: Calculate $\partial_\mu K^\mu$

* From SMM-03, one learns

$$\partial_\mu \mathcal{H}^{\nu\mu} = 0$$

This is because the Homogeneous Maxwell equation are contained in $\partial_\mu \mathcal{F}^{\mu\nu} = 0$, and $\mathcal{H}^{\nu\mu}$ was constructed totally in analogy.

*
$$\begin{aligned}\partial_\mu K^\mu &= (\partial_\mu A'_\nu)\, \mathcal{H}^{\nu\mu} \\ &= \frac{1}{2} M^{i}_{\mu\nu} \mathcal{H}^{\nu\mu} \\ &= 2\, \hat{\underset{\sim}{E}} \cdot \hat{\underset{\sim}{B}}\end{aligned}$$

Again in analogy with simple fluid-free case

$$F_{\mu\nu} \mathcal{F}^{\mu\nu} = -4\, \underset{\sim}{E} \cdot \underset{\sim}{B}$$

* What is $\hat{\underset{\sim}{E}} \cdot \hat{\underset{\sim}{B}}$ ?

We must go back to the equation of motion 13(a).

# Divergence of $K^\mu$

Remembering that $U^\mu = (\gamma, \gamma \underset{\sim}{u})$, the vector part of 13 (a) yields

$$T \nabla \sigma = q\gamma \left( \underset{\sim}{\hat{E}} + v \times \underset{\sim}{\hat{B}} \right) \qquad (a)$$

Well let us dot both sides with

$$T \; \underset{\sim}{\hat{B}} \cdot \nabla \sigma = q \gamma \underset{\sim}{\hat{E}} \cdot \underset{\sim}{\hat{B}}$$

$$\therefore \; \partial_\mu K^\mu = 2 \underset{\sim}{\hat{E}} \cdot \underset{\sim}{\hat{B}} = \frac{2T}{q\gamma} \underset{\sim}{\hat{B}} \cdot \nabla \sigma \qquad (b)$$

is the fundamental equation at the heart of Helicity conservation.

Notice that 17(b) is a remarkably general equation

it holds for a relativistic (kinematically as well as statistical) charged fluid.

# Helicities – In different approximations

## Non relativistic Limit :

In this Limit $\gamma \to 1$, $f \to 1$

$$\hat{\underset{\sim}{B}} = \underset{\sim}{B} + \frac{m}{q} \nabla \times \underset{\sim}{v}$$

$$\partial_\mu K^\mu = \frac{2T}{q} \hat{\underset{\sim}{B}} \cdot \nabla \sigma \qquad (a)$$

$$K^0 = \left(\underset{\sim}{A} + \frac{m}{q} \underset{\sim}{v}\right) \cdot \left(\underset{\sim}{B} + \frac{m}{q} \nabla \times \underset{\sim}{u}\right)$$

* If the fluid is homentropic (const. entropy)

$$\partial_\mu K^\mu = 0$$

G is, then conserved.

* If entropy is not uniform, but there exists an equation of state

$$\sigma = \sigma(T) \Rightarrow$$

$$T \nabla \sigma = \nabla \varsigma$$

$$\partial_\mu K^\mu = \nabla \cdot \underset{\sim}{S} \qquad (b)$$

with $\underset{\sim}{S} = \frac{2}{q} \hat{\underset{\sim}{B}} \, \varsigma$ ($\varsigma$ is the solution of $\frac{d\varsigma}{dT} = T \frac{d\sigma}{dT}$)

Then Again G is conserved

Only baroclinic fluids do not conserve helicity in the non relativistic limit

# Relativistic Helicity

As we go from the non-rel to the relativistic fluids, the situation changes dramatically

Even when we have an equation of state $\sigma = \sigma(T)$, Eq. 17(b) becomes

$$\partial_\mu K^\mu = \frac{1}{\gamma} \nabla \cdot S \qquad (a)$$

$$S = (2/q)\, \varsigma\, \hat{\underset{\sim}{B}}$$

For any dynamical relativistic fluid with nonuniform velocity field $\gamma$ cannot be absorbed within the divergence and

$$\frac{dG}{dt} = \frac{d}{dt}\int K^0 \, d^3x \neq 0$$

The fact that special relativity breaks the invariance of a general equilibrium ideal fluid, opens a channel for vorticity or Mag. field generation.

# Helicity Density - Evolving Dynamics

Let us see the evolution of helicity density as the dynamics becomes more and more general

$$K^0 = (\underset{\sim}{A} + \frac{m}{q} f \gamma \underset{\sim}{u}) \cdot (\underset{\sim}{B} + \frac{m}{q} \nabla \times f \gamma \underset{\sim}{u})$$

Non-relativistic

* Pure electromagnetic field $K^0 = \underset{\sim}{A} \cdot \underset{\sim}{B}$
* fluid with mass m $K^0 = (\underset{\sim}{A} + \frac{m}{q} \underset{\sim}{u}) \cdot (\underset{\sim}{B} + \frac{m}{q} \nabla \times \underset{\sim}{u})$

The fact that $K^0 = \underset{\sim}{A} \cdot \underset{\sim}{B}$ is taken to hold for a plasma is because it is the electron helicity with $m_e \to 0$.

*** Each fluid-species of a plasma contributes a helicity invariant (under app. circumstances)

* Electron-ion plasma $G_e = \langle (\underset{\sim}{A} - \frac{m_e}{e} \underset{\sim}{v_e}) \cdot (\underset{\sim}{B} - \frac{m_e}{e} \nabla \times \underset{\sim}{v_e}) \rangle$

$$G_i = \langle (\underset{\sim}{A} + \frac{M}{e} \underset{\sim}{u_i}) \cdot (\underset{\sim}{B} + \frac{M}{e} \nabla \times \underset{\sim}{u_i}) \rangle$$

## Relativistic :

For an electron-positron plasma

$$G_\pm = \langle (\underset{\sim}{A} \pm f_\pm \gamma_\pm \frac{m}{e} \underset{\sim}{v_\pm}) \cdot (\underset{\sim}{B} \pm \frac{m}{e} \nabla \times f_\pm \gamma_\pm \underset{\sim}{v_\pm}) \rangle .$$

Measuring the knottedness of the field (flow) lines, Helicity is a fundamental concept of Plasma / Fluid Dynamics and generally 'limits' the states accessible.