Topics in theoretical physics I

self-organization on scale hierarchy

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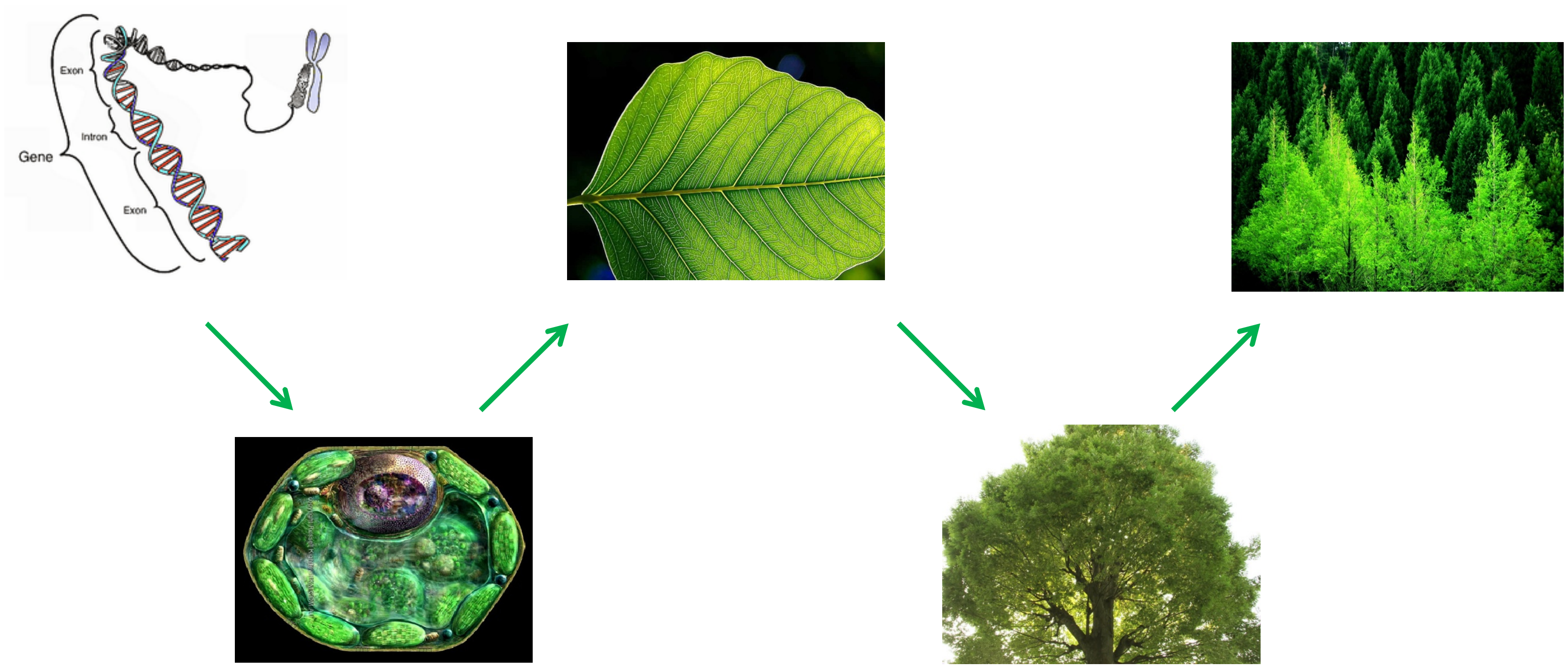
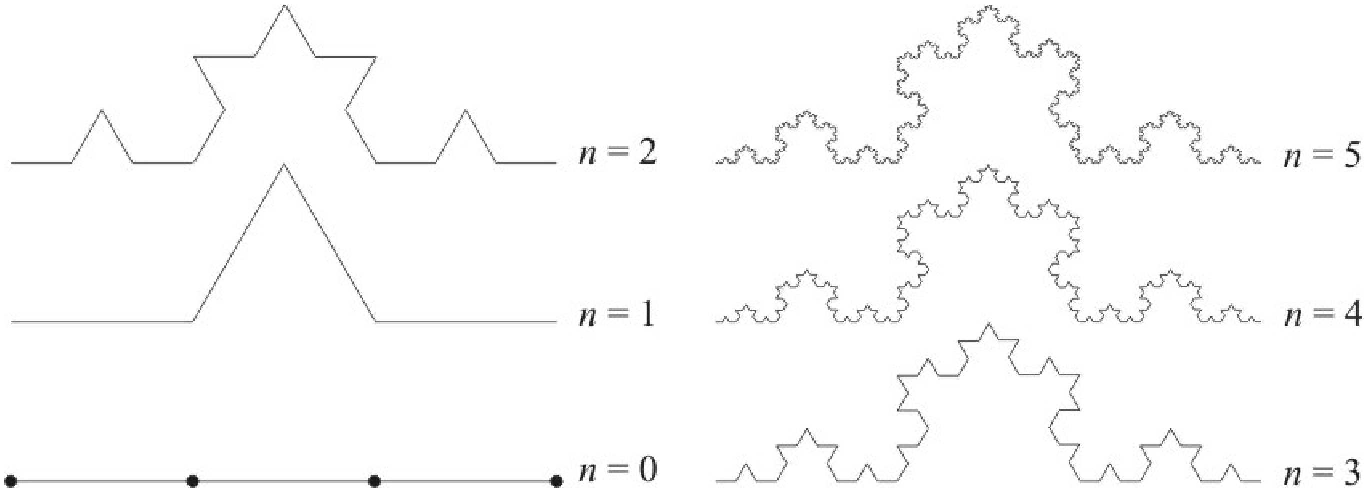
Scale hierarchy



- Surprisingly diverse structures and mechanisms on different scales
- How “macro” can be different from “micro” ?
- Physics: What is the mechanism that works without blueprints ?

subject defines a scale

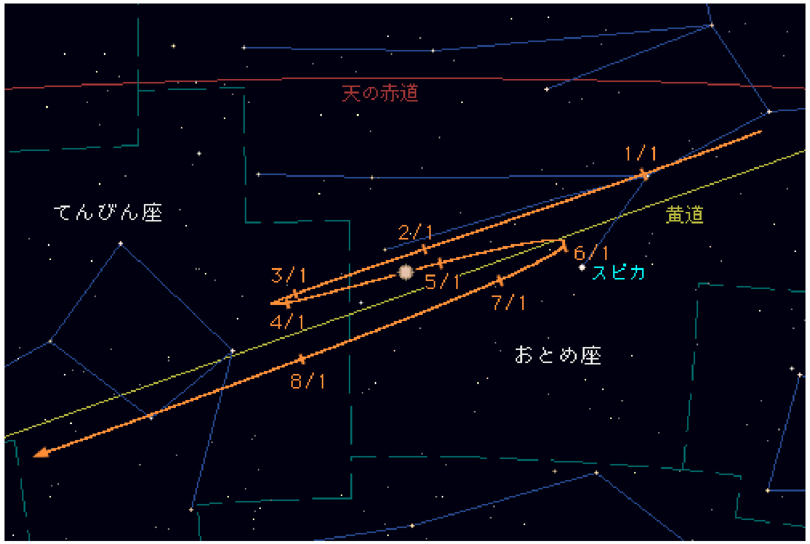


The definition (representation) of matter depends on the subject=scale.

Space-time: *a priori* of theory (?)

- *Kopernikanische Wendung*

  a priori of recognition



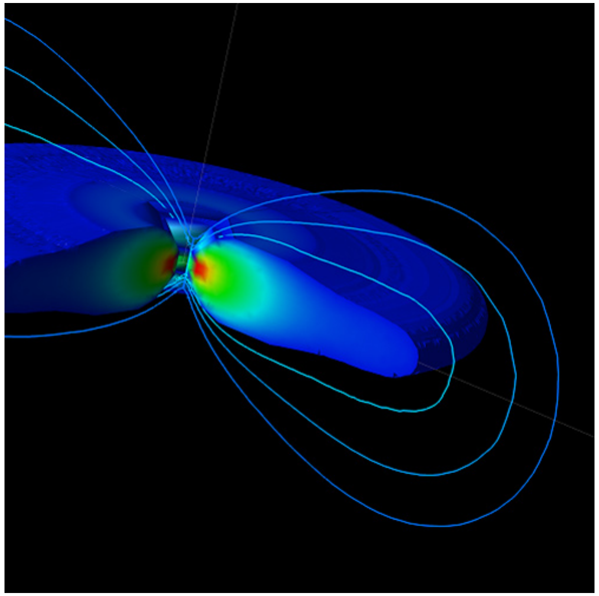
Immanuel Kant
(1724-1804)

phenomena (event) → framework of recognition (space-time)

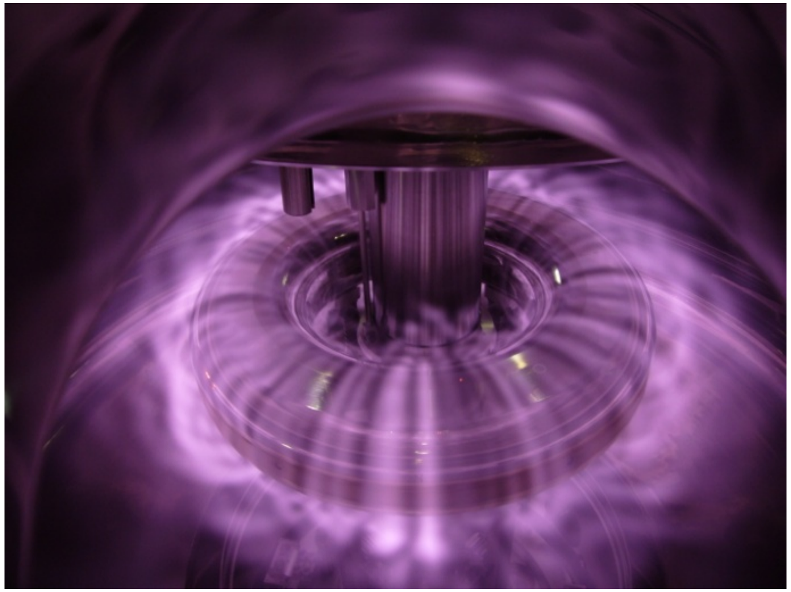
Vortex = space-time distortion

kurasse.jp/member/little-kinoko778/note/93577

vortex → *self-organized confinement*

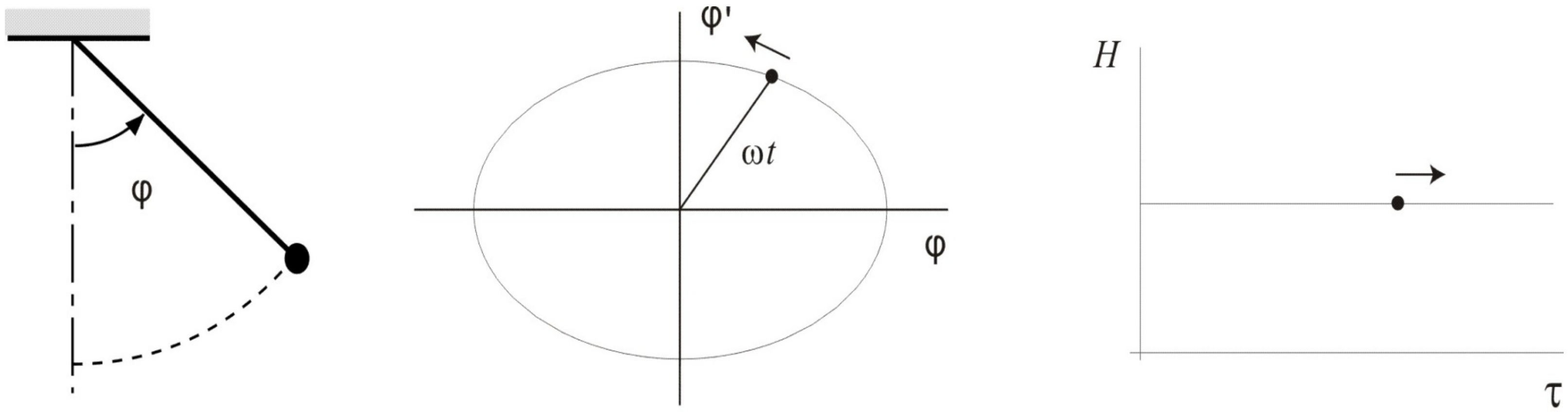


Jovian magnetosphere



RT-1 magnetospheric plasma

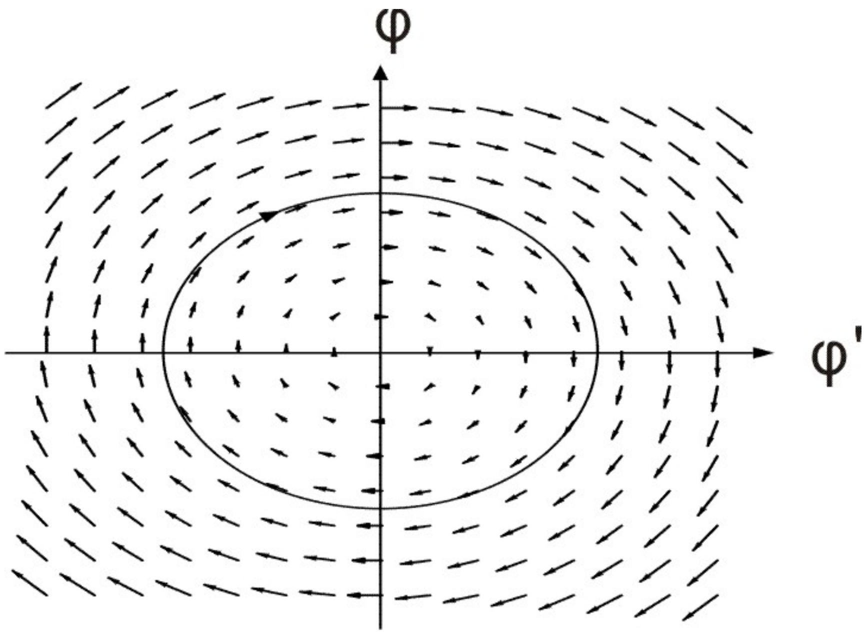
“canonical vortex” = micro scale



(a) (b) (c)

repetition circulation continuation

Hamiltonian formalism



$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Symplectic group

→ canonical vortex

matter = energy = Hamiltonian *H*

space-time = symplectic geometry *J*

$$\frac{d}{dt}F = \{H, F\}$$

$$\{H, F\} = \langle \partial_z H, J\partial_z F \rangle$$

What is *macroscopic* space-time ?

- How can magnetic field "confine" a plasma, despite the fact that $f = \exp(-\beta E)$?

→ This is possible because of "scale separation".

- What is scale hierarchy?
- What is "state" and what is "space"?

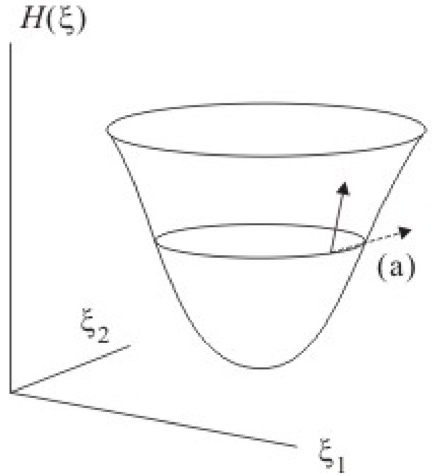
.

micro = canonical / macro = non-canonical

General form of Hamiltonian systems

- Hamiltonian mechanics is dictated by
  J (Poisson operator) and H (Hamiltonian)

$$\boxed{\frac{d}{dt} z = J \partial_z H(z)}$$



Poisson bracket: $\{G, F\} = \langle \partial_z G(z), J \partial_z F(z) \rangle$

$$\frac{d}{dt} F(z) = \{H, F\}$$

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$$

Examples of Hamiltonian systems

classical mechanics:

$$\frac{d}{dt}\boldsymbol{z} = \frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix} = J\partial_z H$$

quantum mechanics:

$$\partial_t \psi = -i\partial_\psi \left\langle \mathcal{H}\ \psi, \psi \right\rangle / 2 = J\partial_\psi H$$

These examples are "canonical" because *J* are regular operators.

Non-canonical Hamiltonian mechanics

- Non-canonicality : Ker(J)

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \boxed{\begin{matrix} 0 & I \\ -I & 0 \end{matrix}} \end{pmatrix}$$

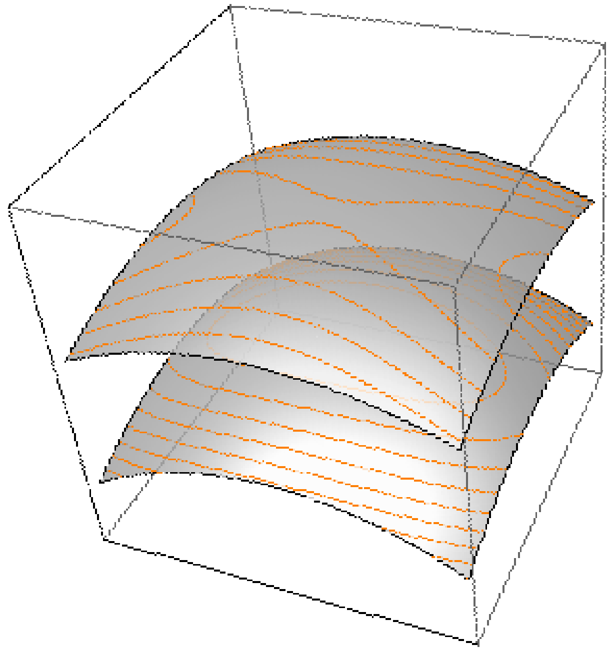
Effective phase space

- Ker(J) = Coker(J) → "topological defect"

non-canonical Hamiltonian mechanics

- Ker (J) = Coker(J) → "topological constraint"
- Foliation of Ker (J) → Casimir elements

$$\exists C \ s.t. \left\{G, C\right\} = 0 \ (\forall G)$$
$$i.e. \quad \partial_z C \in \mathrm{Ker}(J)$$



Scale hierarchy of magnetized particles

$$H = \frac{m}{2}\left(V_c^2 + V_\parallel^2 + V_\perp^2\right) + q\phi$$

$$H_c = \mu\omega_c + \frac{m}{2}\left(V_\parallel^2 + V_\perp^2\right) + q\phi$$

$$= \mu\omega_c + \frac{\left(P_\theta - q\psi\right)^2}{2mr^2} + \frac{p_\parallel^2}{2m} + q\phi$$

$$\boldsymbol{z} = (\vartheta_c, \mu; \zeta, p_\parallel; \theta, P_\theta)$$

Coarse graining → macro-hierarchy

$$\boldsymbol{z} = (\vartheta_c, p_c; \zeta, p_\parallel; \theta, P_\theta) \rightarrow (\cancel{\vartheta_c}, \mu; \zeta, p_\parallel; \theta, P_\theta)$$

Coarse-graining → non-canonicalizetion

$$J = \begin{pmatrix} J_c & 0 & 0 \\ 0 & J_c & 0 \\ 0 & 0 & J_c \end{pmatrix} \rightarrow J_{nc} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & J_c & 0 \\ 0 & 0 & J_c \end{pmatrix}$$

Boltzmann distribution on Casimir leaf

$$\boxed{\delta\left(S-\alpha N-\beta E-\gamma M\right)=0}$$

$$S=-\int f\log f\,d^6z$$

$$N=\int f\,d^6z$$

$$E=\int H_c f\,d^6z$$

$$M=\int \mu f\,d^6z$$

$$\boxed{f=c\exp\left(-\beta H_c-\gamma\mu\right)}$$

Chemical potential Quasi-particle number

Embedding into the lab-frame space

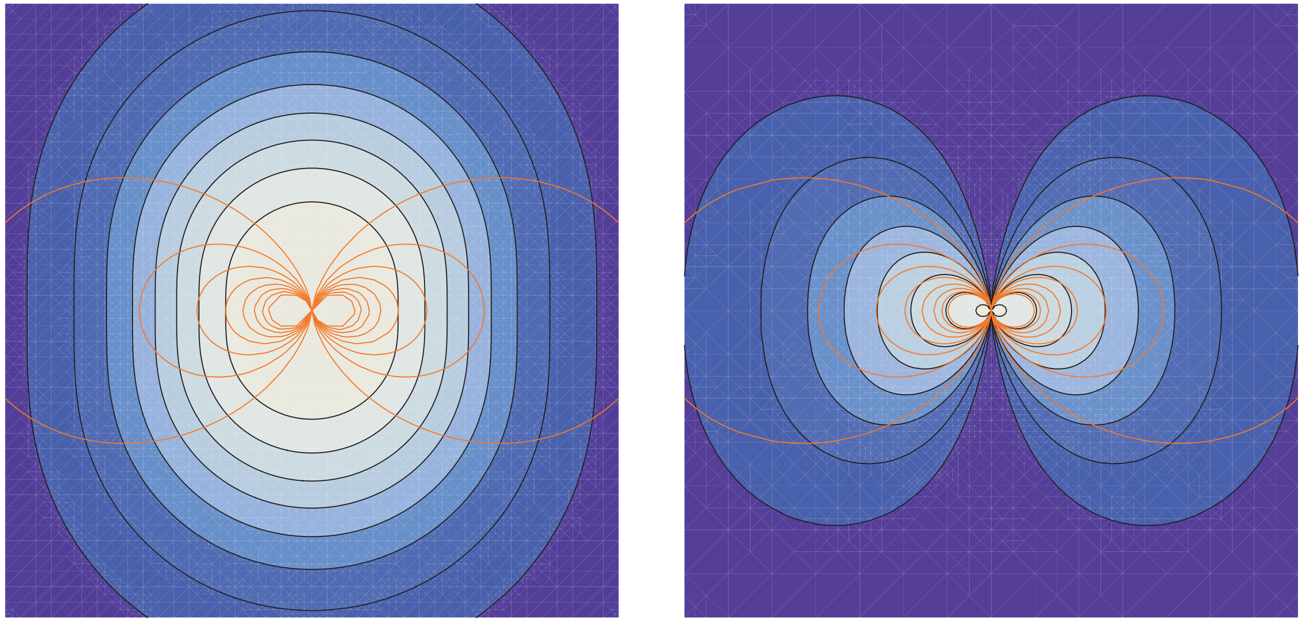
$$\rho(x) = \int f\, d^3v = \int f\,(2\pi\omega_c / m)\, d\mu\, dv_{\|}\, dv_\theta$$

$$= c\int \exp\left(-\beta H_c - \gamma\mu\right)\frac{2\pi\omega_c\, d\mu}{m}\, dv_{\|}\, dv_\theta$$

$$= \frac{\omega_c(x)}{\omega_c(x) + \gamma}$$

Density clump in lab-frame space

Adiabatic invariants = *number* of quasi-particles
→ thermodynamic distribution on a Casimir leaf



(A) (B)

Topics in theoretical physics II
self-organization in foliated phasespace

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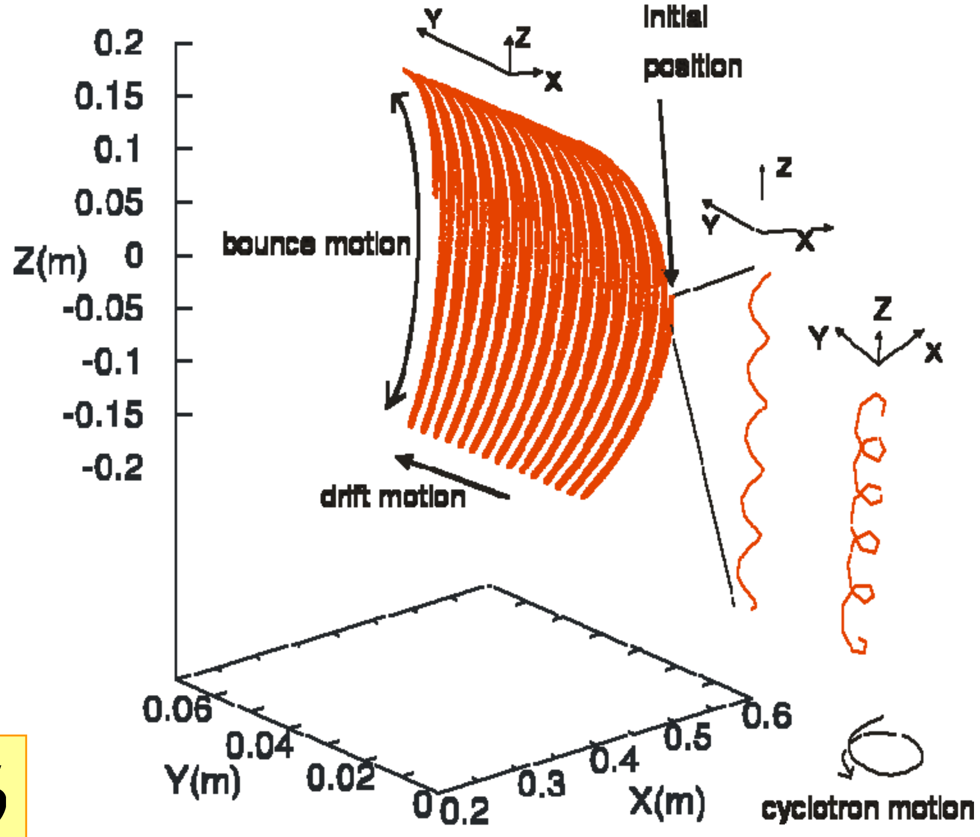
Scale hierarchy of magnetized particles

$$H = \frac{m}{2}\left(V_\perp^2 + V_\parallel^2\right) + q\phi$$

Guiding center = quasi-particle

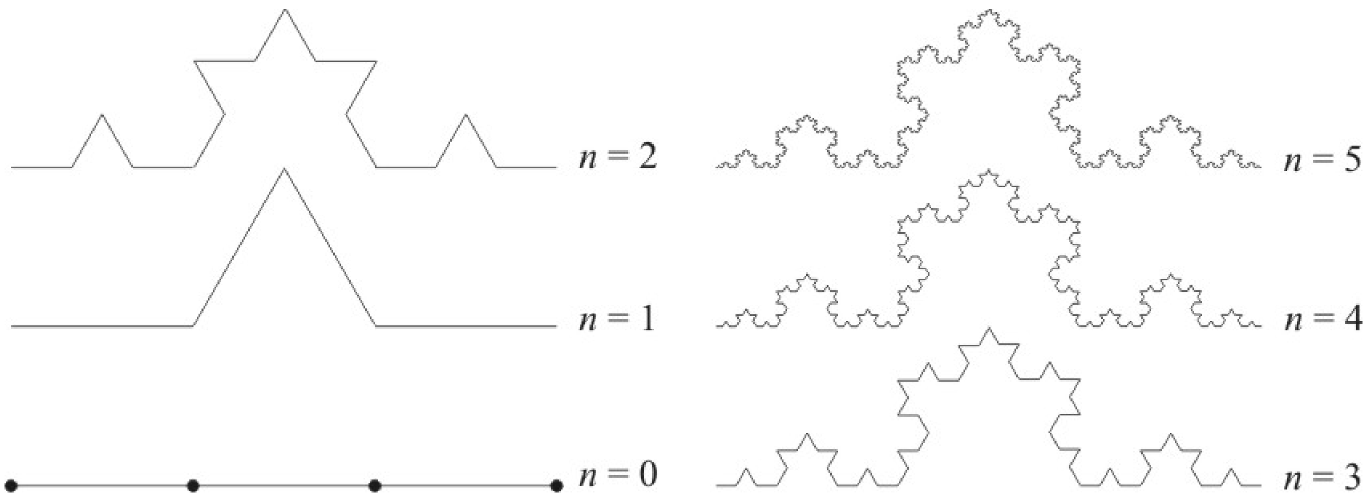
$$H_{gc} = \omega_c \mu + \omega_b J_\parallel + q\phi$$



subject defines a scale

$$\boldsymbol{z} = (\vartheta_c, p_c; \zeta, p_{\|}; \theta, P_\theta) \rightarrow (\cancel{\vartheta_c}, \mu; \zeta, p_{\|}; \theta, P_\theta)$$

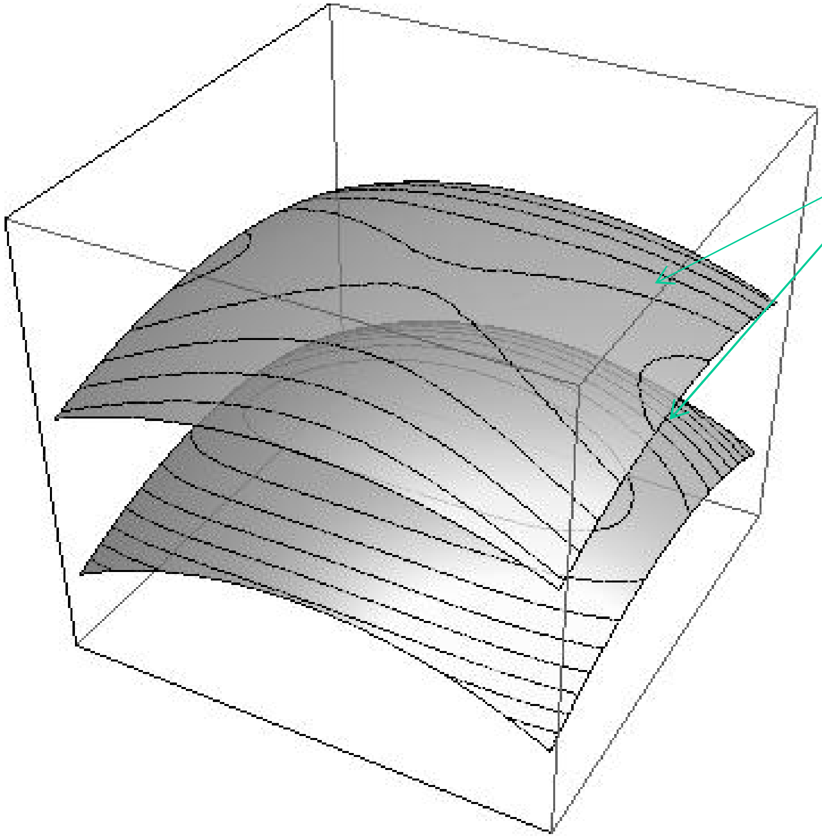
The definition (representation) of matter depends on the subject=scale.



Foliated phase space of *non-canonical* Hamiltonian system

- Ker(J) = { grad C (z) }

Casimir leaf



The “effective” phase space of constrained dynamics

•Equilibrium points?
•Stability?
•Thermal equilibrium?

Grand-canonical distribution on Casimir leaf

$$\boxed{\delta\left(S-\alpha N-\beta E-\gamma M\right)=0}$$

$$S=-\int f\log f\,d^6z$$

$$N=\int f\,d^6z$$

$$E=\int H_c f\,d^6z$$

$$M=\int \mu f\,d^6z$$

$$\boxed{f=c\exp\left[-\beta\left(H_c+\gamma'\mu\right)\right]}$$

Chemical potential

Quasi-particle number

Embedding into the lab-frame space

$$\rho(x) = \int f\, d^3 v = \int f \ \frac{2\pi\omega_c d\mu}{m} dv_{\parallel}$$

$$= c\int \exp\left(-\beta H_c - \gamma\mu\right)\frac{2\pi\omega_c d\mu}{m} dv_{\parallel}$$

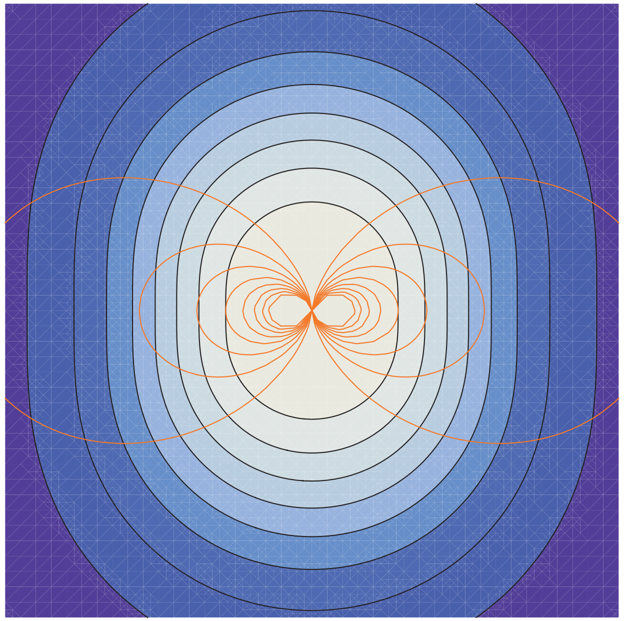
$$= \frac{\omega_c(x)}{\omega_c(x) + \gamma}$$

Density clump in lab-frame space

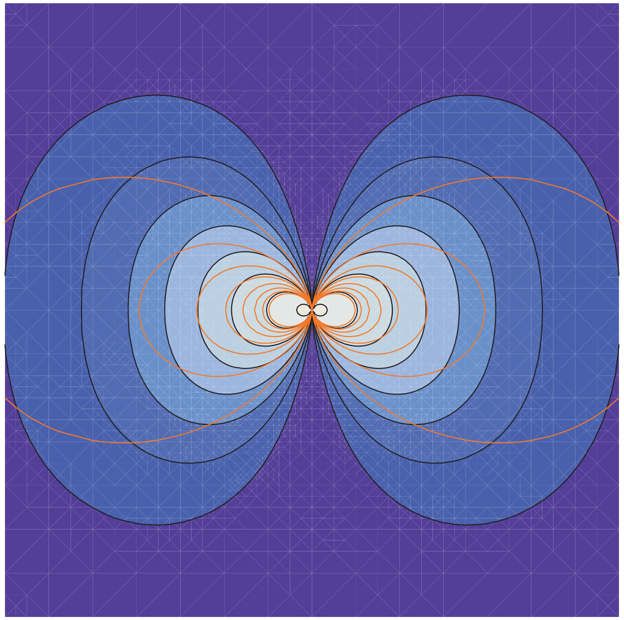
Adiabatic invariants

→ *phase space* of "quasi-particles"

→ thermodynamic distribution on a Casimir leaf



(A)



(B)

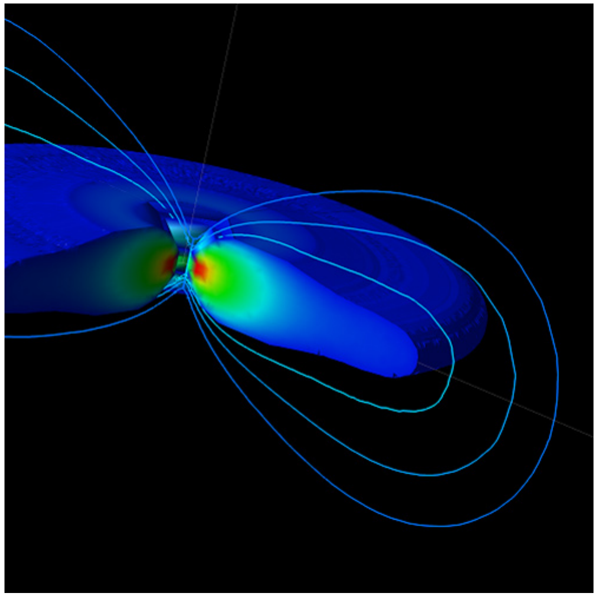
Conclusion (I)

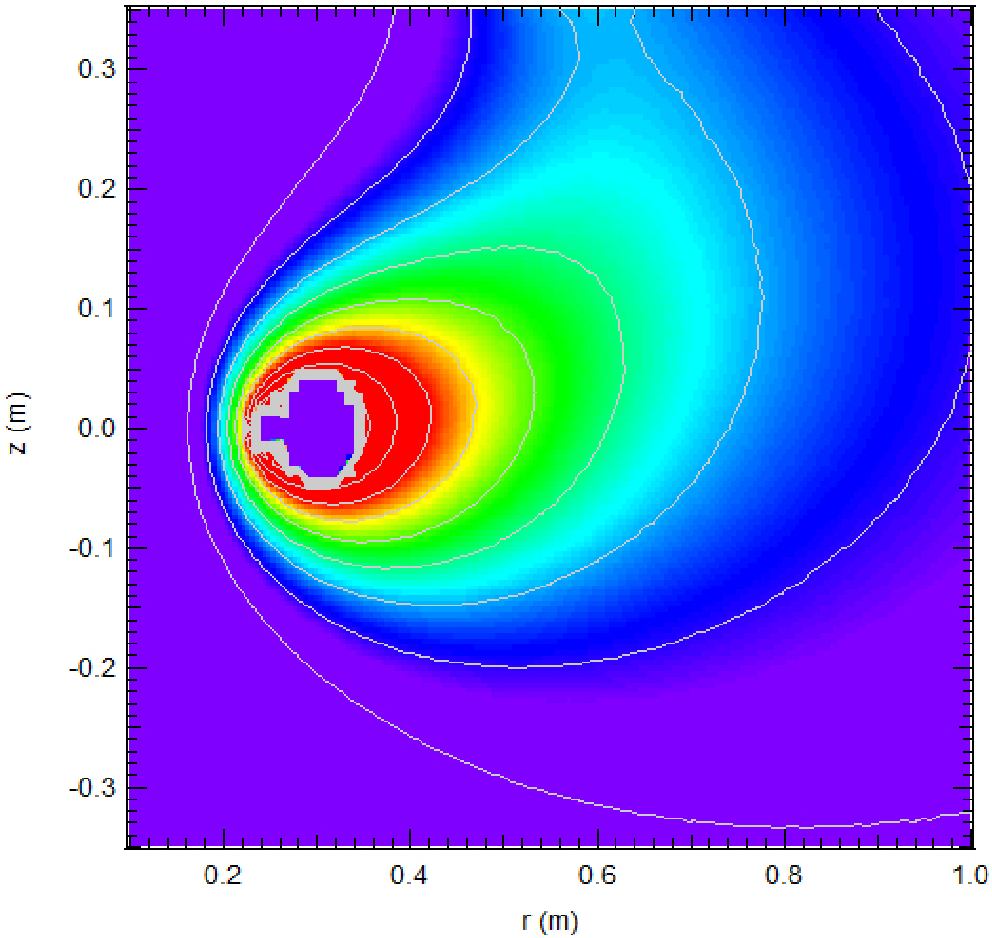
- Scale hierarchy = foliated phase space
- Adiabatic invariant = *Casimir element*
  → foliation
- Distorted metric on a leaf → structure

Z.Yoshida & S.M. Mahajan, *Self-organization in foliated phase space: construction of a scale hierarchy by adiabatic invariants of magnetized particles*, Prog. Theor. Exp. Phys. **2014** (2014), 073J01

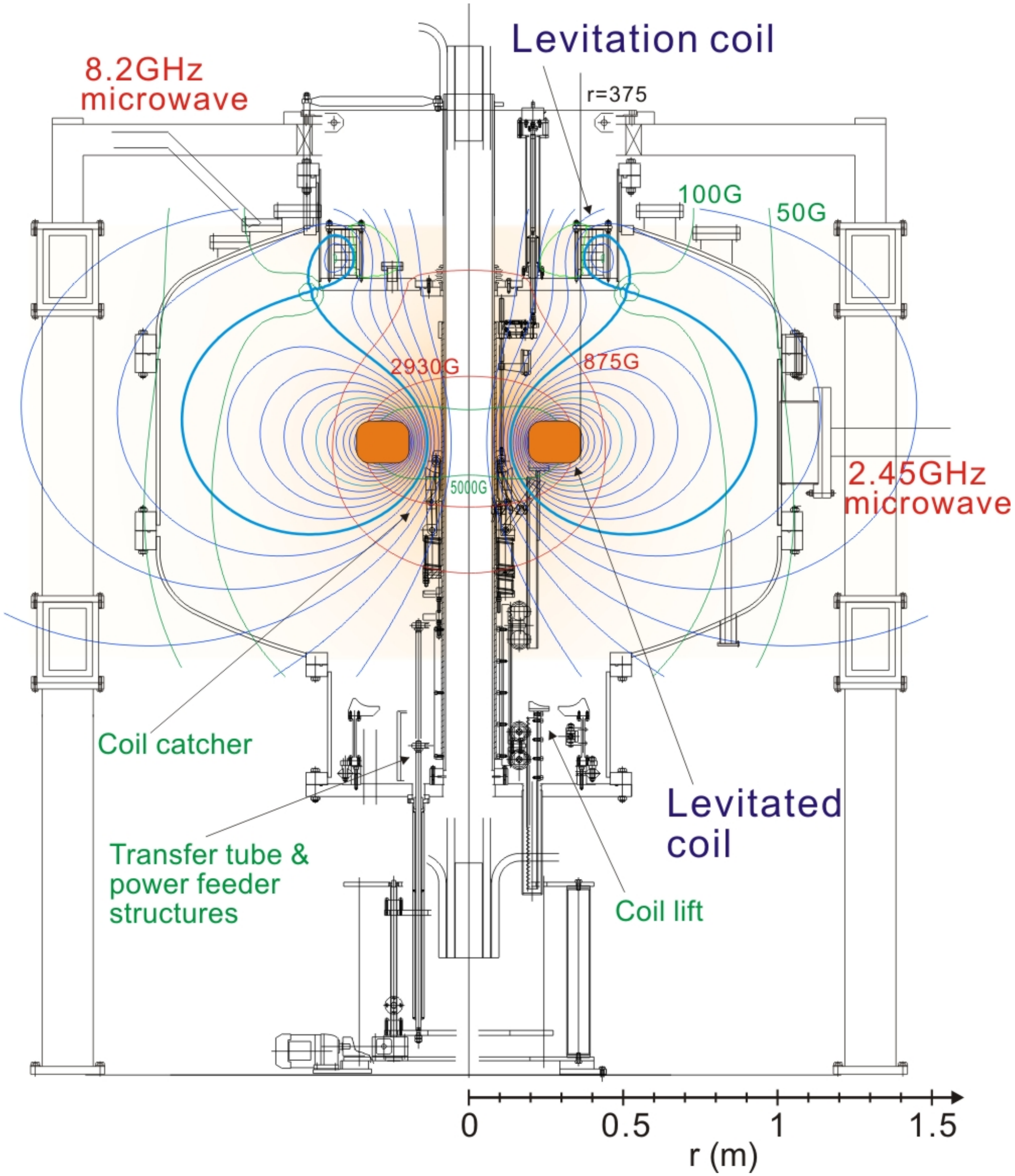
A magnetosphere on the Earth

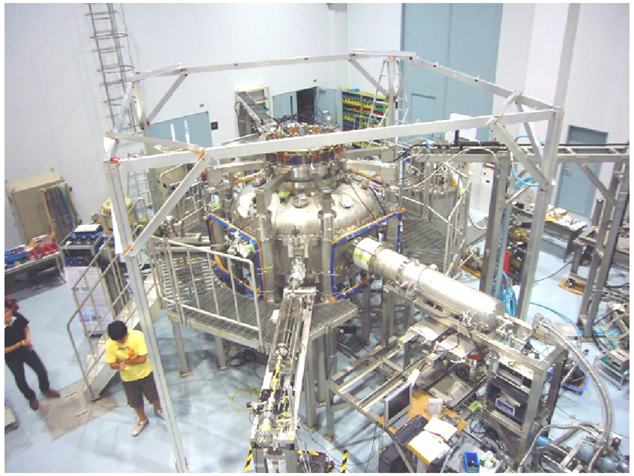
RT-1 project

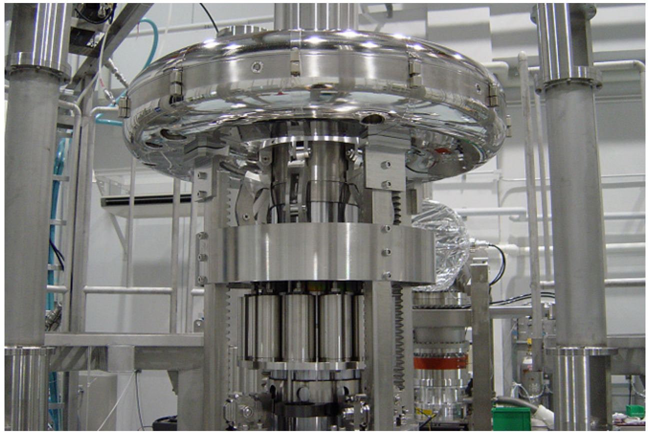




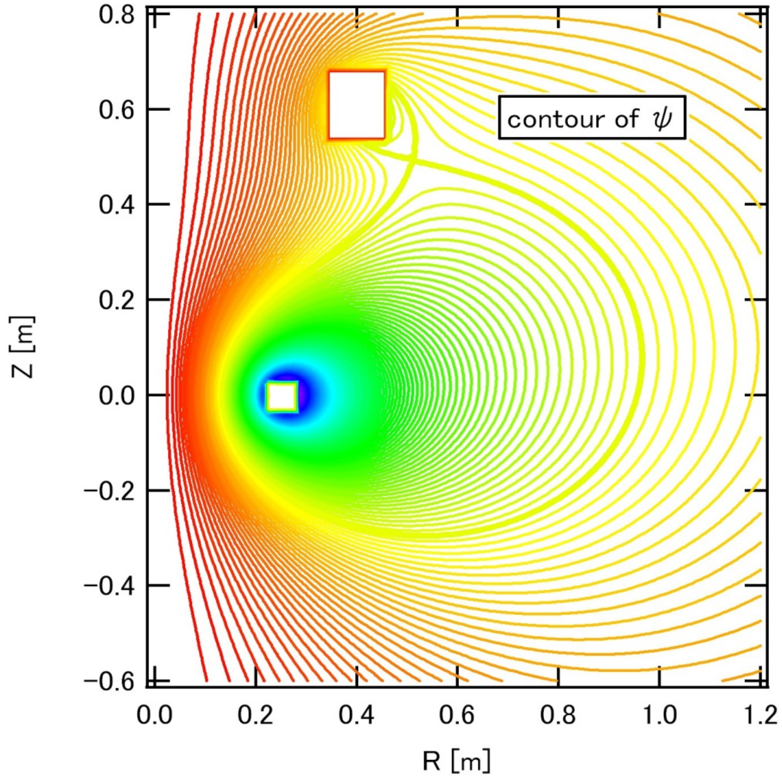
Levitating HTC superconducting magnet system

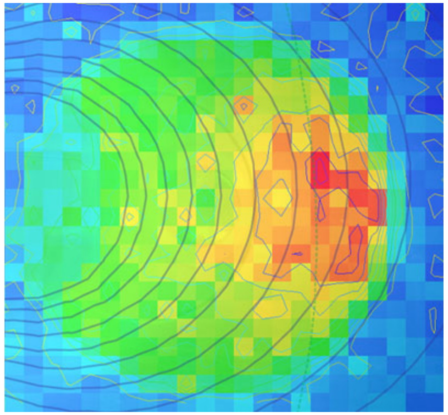






High-beta plasma confinement

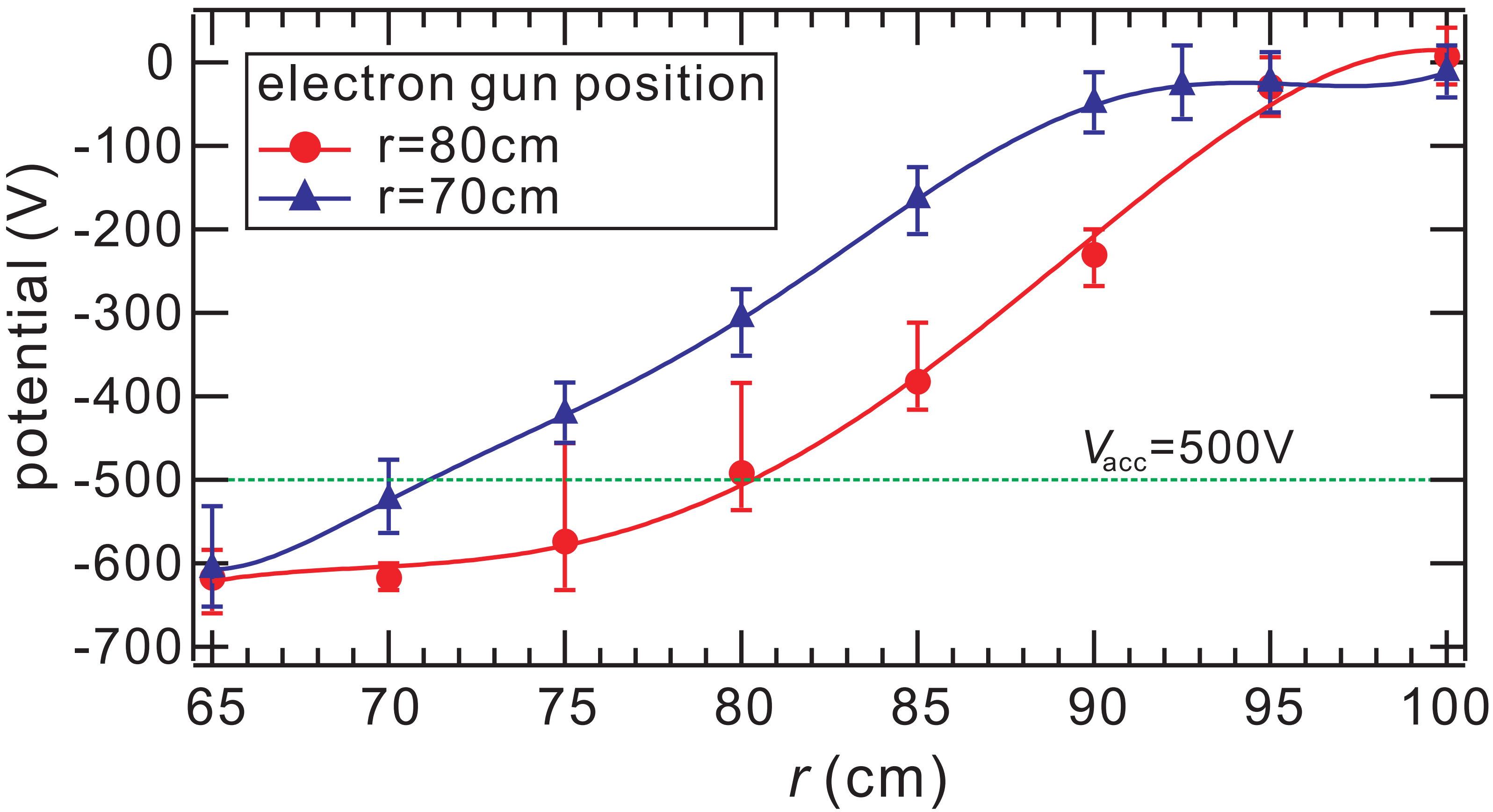




$$T_e \approx 10 \sim 30\text{keV}, \quad n_e = 10^{16} - 10^{17}\ \text{m}^{-3}$$

$$\beta \approx \beta_e \approx 0.7, \quad \tau_E \approx 0.5\,\text{sec}$$

Inward (up-hill) diffusion



ZY *et al.*, Phys. Rev. Lett. **104** (2010), 235004 .

space for “confinement”

kurasse.jp/member/little-kinoko778/note/93577

Conclusion (II)

- Experimental proof of *self-organized confinement*.

- Experimental evidence of *inward diffusion*.

MHD (incompressible)

- Naïve form:

$$\boxed{\begin{aligned}\partial_t \boldsymbol{V} &= -P_\sigma(\nabla\times\boldsymbol{V})\times\boldsymbol{V} + P_\sigma(\nabla\times\boldsymbol{B})\times\boldsymbol{B},\\ \partial_t \boldsymbol{B} &= \nabla\times(\boldsymbol{V}\times\boldsymbol{B}).\end{aligned}}$$

- State vector:

$$\boxed{u = (\boldsymbol{V},\boldsymbol{B}) \in L^2_\sigma(\Omega)\times L^2_\sigma(\Omega)}$$

$$L^2_\sigma(\Omega) := \left\{\boldsymbol{v}\in L^2_\sigma(\Omega); \nabla\cdot\boldsymbol{v}=0, \boldsymbol{n}\cdot\boldsymbol{v}=0\right\}$$

$$P_\sigma : L^2(\Omega)\to L^2_\sigma(\Omega)$$

Hamiltonian form of MHD

- Hamiltonian

$$H(u) = \frac{1}{2}\int_{\Omega}\left(V^2 + B^2\right)dx = \frac{1}{2}\|u\|^2$$

- Poisson operator

$$J(u) = \begin{pmatrix} -P_\sigma(\nabla \times \boldsymbol{V}) \times \circ & P_\sigma(\nabla \times \circ) \times \boldsymbol{B} \\ \nabla \times (\circ \times \boldsymbol{B}) & 0 \end{pmatrix}$$

- *Casimir elements:*

$$C_1(u) = \frac{1}{2}\langle \boldsymbol{A}, \boldsymbol{B} \rangle, \quad C_2(u) = \langle \boldsymbol{V}, \boldsymbol{B} \rangle$$

Beltrami equilibria on helicity leaves

- Beltrami equilibrium:

$$\boxed{\partial_u H_\mu(u) = 0 \quad (H_\mu = H - \mu C_1).}$$

$$\rightarrow \quad \boxed{\nabla \times \boldsymbol{B} = \mu \boldsymbol{B}, \quad \boldsymbol{B} \in L^2_\sigma(\Omega).}$$

- Two classes of Beltrami eigenvalues:

(1) Self-adjoint curl operator S

$\rightarrow$ discrete real eigenvalues $\mu \in \{\lambda_1, \lambda_2, \cdots\}$

(2) Non-self-adjoint extension T

(if Ω is multiply connected) $\rightarrow$ $\forall \mu \in \mathbf{C}$

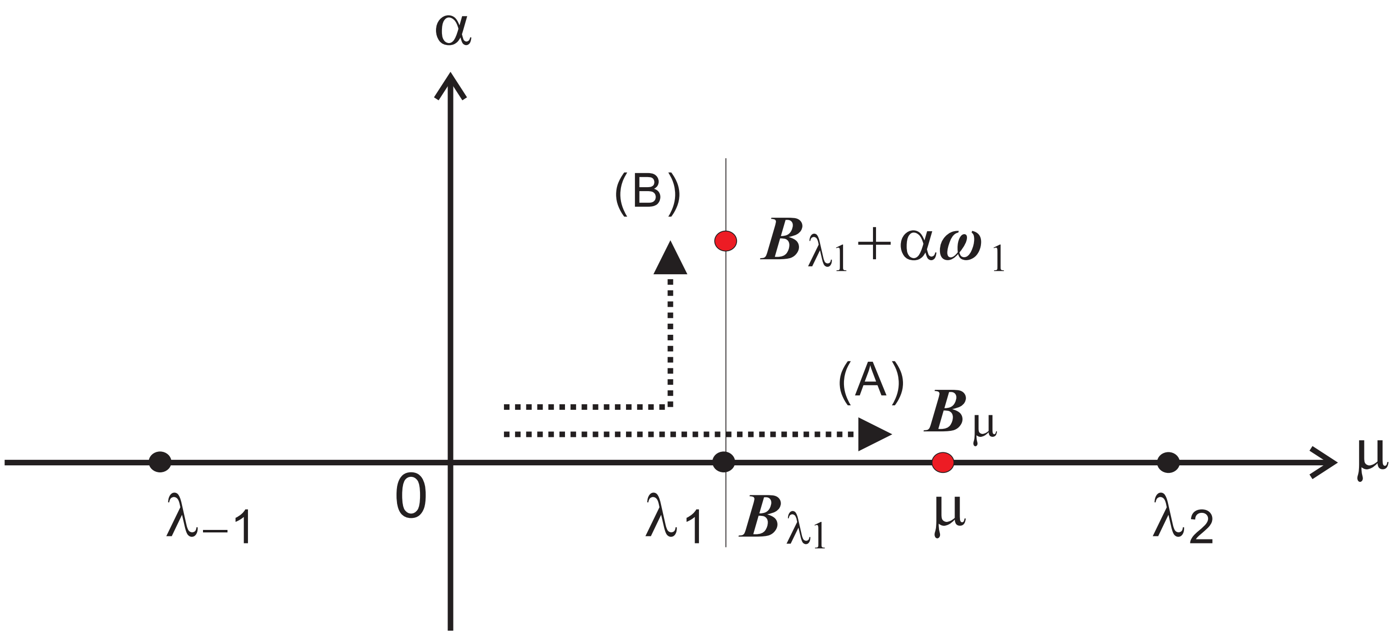
ZY & Y. Giga, Math. Z. **204** (1990) 235.

Bifurcation theorem

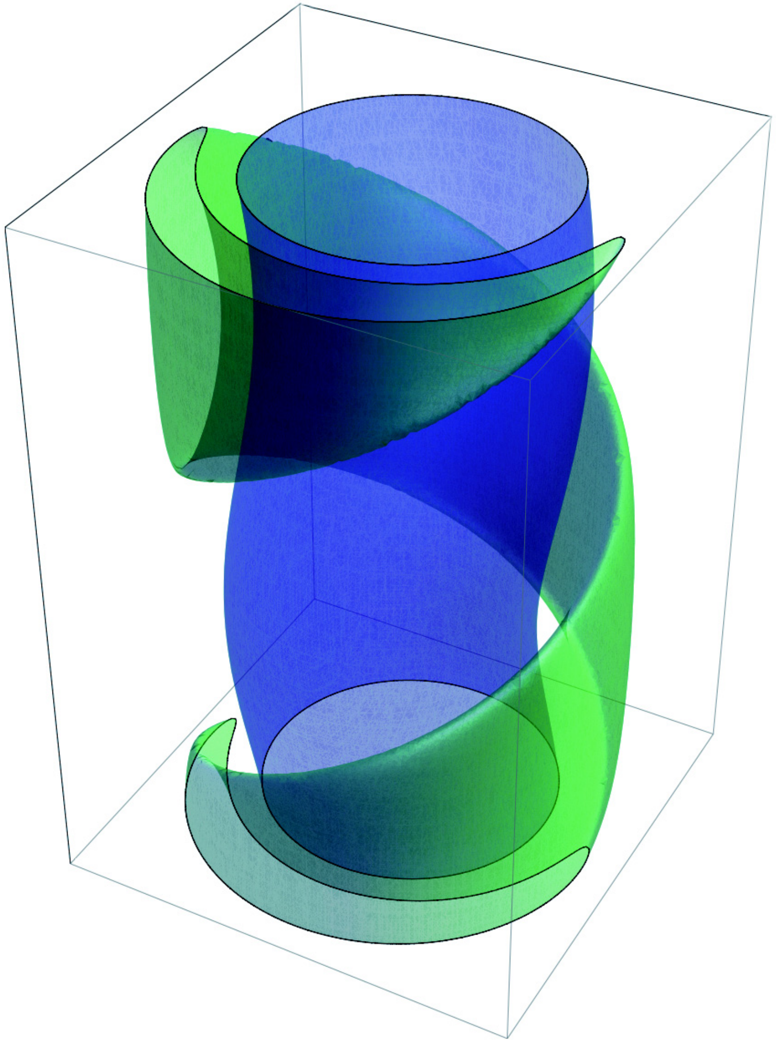
Theorem 1

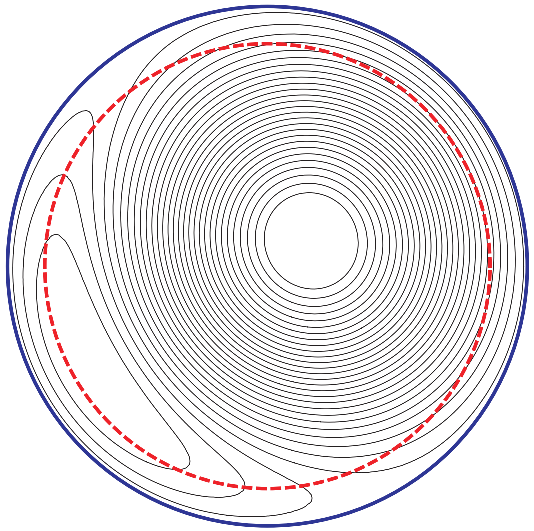
Let $\boldsymbol{A}_{\mathrm{H}}$ *be the vector potential of* $\boldsymbol{B}_{\mathrm{H}}$ (*cohomology*).

If $\langle \boldsymbol{A}_{\mathrm{H}}, \boldsymbol{\omega}_j \rangle = 0$, *then branch-(B) bifurcates at* $\mu = \lambda_j \in \sigma_p(S)$.



Bifurcated Beltrami equilibrium



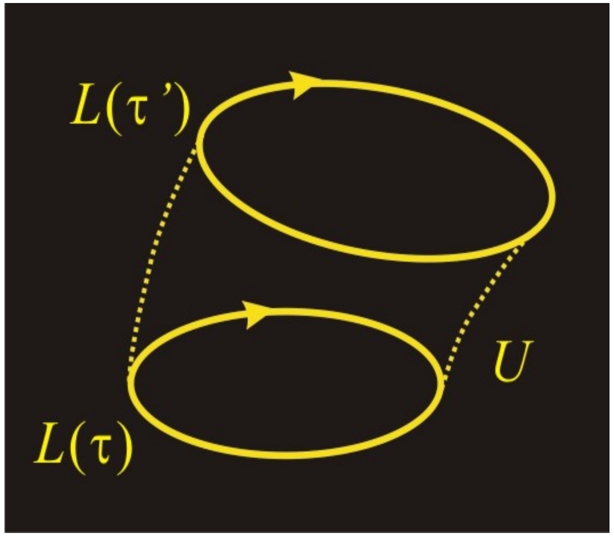


Conclusion (III)

- The *helicity* is a Casimir element that foliates the phase space.
- We find bifurcated equilibrium points on helicity leaves, because helicity leaves are distorted with respect to the energy norm.
- A tearing mode is an equilibrium on a leaf of singular (resonant) Casimir element

Z. Yoshida & R. L. Dewar; J. Phys. A **45** (2012), 365502 1-36.

"Loops" in a fluid / plasma



$$\frac{d}{dt}\left(\oint_{L(t)} \boldsymbol{P}\cdot d\boldsymbol{x}\right) = \oint_{L(t)} \left[\partial_t \boldsymbol{P} + (\nabla\times\boldsymbol{P})\times\boldsymbol{V}\right]\cdot d\boldsymbol{x} = 0$$

if canonical momentum $\boldsymbol{P} = m\boldsymbol{V} + (q/c)\boldsymbol{A}$ obeys

$$\partial_t \boldsymbol{P} + (\nabla\times\boldsymbol{P})\times\boldsymbol{V} = -\nabla\varphi \quad (\varphi = H + h).$$

Vortex and Entropy

$$-\oint \delta W$$

$$\frac{d}{dt}\left(\oint_{L(t)} \boldsymbol{P}\cdot d\boldsymbol{x}\right) = \oint_{L(t)} \left[\partial_t \boldsymbol{P} + (\nabla \times \boldsymbol{P}) \times \boldsymbol{V}\right]\cdot d\boldsymbol{x}$$

$$= \oint_{L(t)} -\nabla(H+h)\cdot d\boldsymbol{x} + \boxed{\oint_{L(t)} T\nabla S \cdot d\boldsymbol{x}}$$

0

$$-\oint dE$$

$$\oint \delta Q$$

Relativistic circulation theorem

$$\frac{d}{ds}\left(\oint_{L(s)} P^{\mu} \cdot dx_{\mu}\right) = \oint_{L(s)} \left(\partial^{\mu} P^{\nu} - \partial^{\nu} P^{\mu}\right) U_{\nu} dx_{\mu}$$

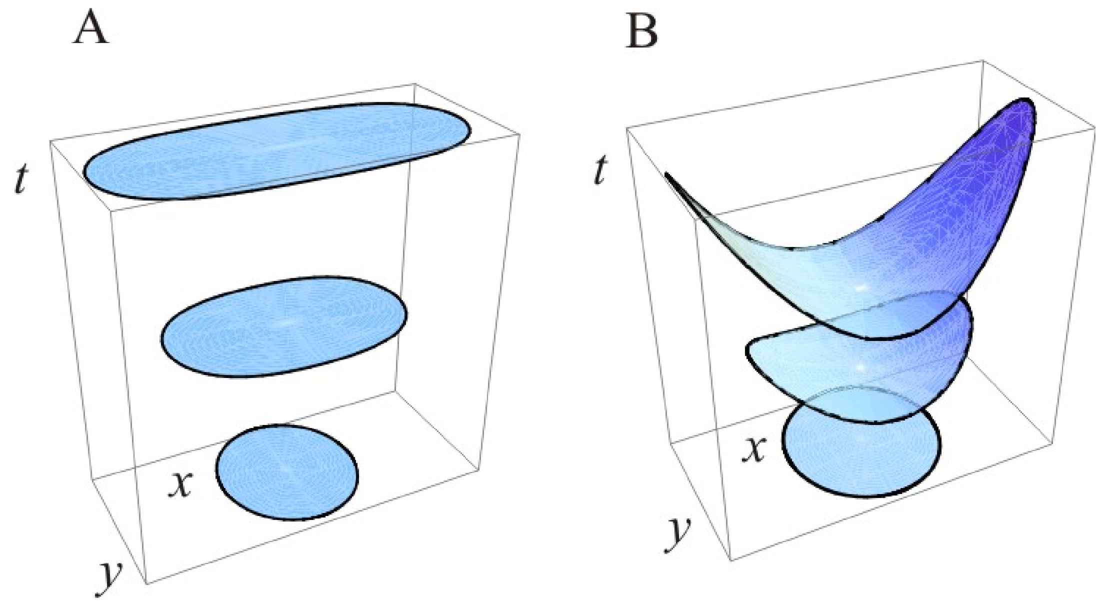
s : proper time, $U^{\mu} = (\gamma, \gamma V^{j} / c)$

relativistic equation of motion

$$\left(\partial^{\mu} P^{\nu} - \partial^{\nu} P^{\mu}\right) U_{\nu} = T \partial^{\mu} S$$

Relativistic circulation theorem applies on space-time: relativistic distortion of space-time

Relativistic distortion of space-time



$$\boxed{\frac{d}{dt}x_\mu = V_\mu}$$

$$\boxed{\frac{d}{ds}x_\mu = U_\mu}$$

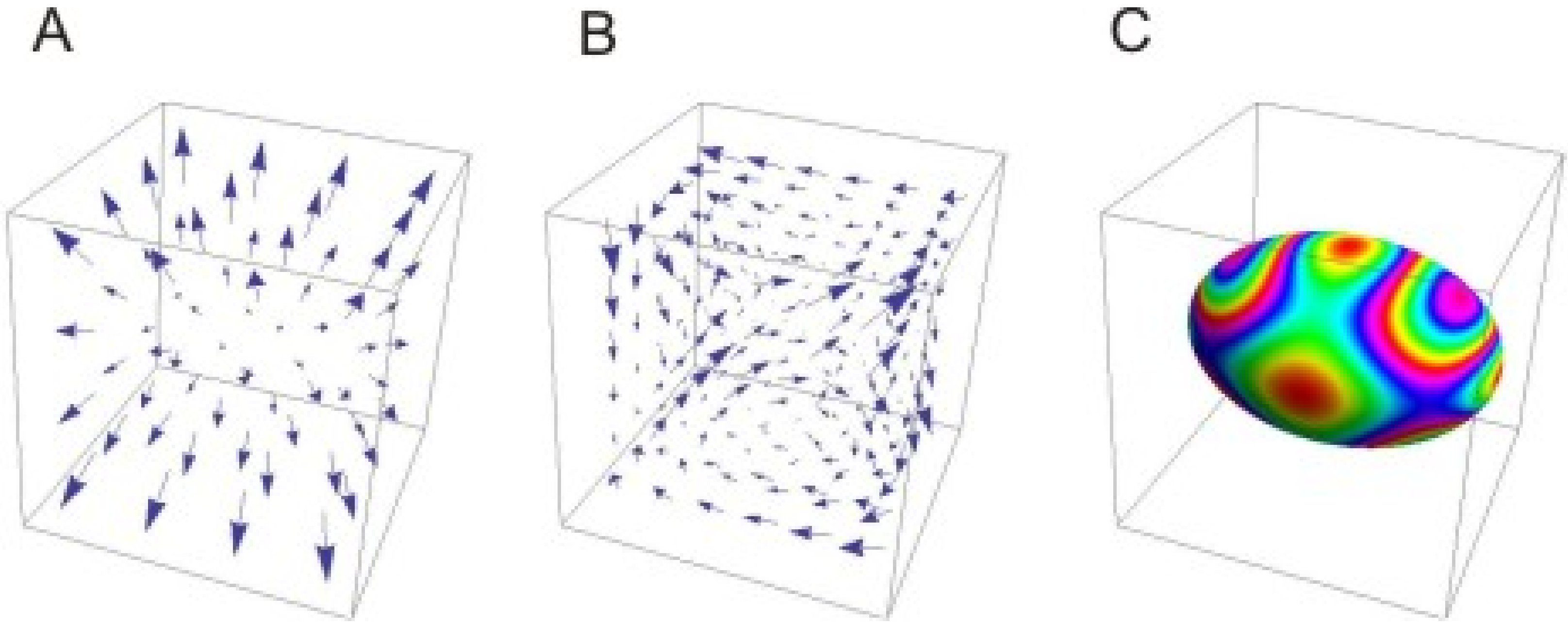
Relativity destroys the "synchronity" of a cycle.

→ "twists" space-time

→ generate circulation

$$\boxed{\oint_{L(t)} \gamma^{-1} T dS}$$

scalar → axial vector (vortex/magnetic field)
cosmological origin of magnetic field



$$\boldsymbol{V} \propto \nabla \varphi \quad \Rightarrow \quad \boldsymbol{\Omega} \propto \nabla \gamma^{-1} \times \nabla \varphi$$

S. M. Mahajan & ZY, Phys. Rev. Lett. **105** (2010), 095005.

Conclusion (III)

- Unified (canonical) Vortex
  $\rightarrow$ *flow-field (EM) coupling* .

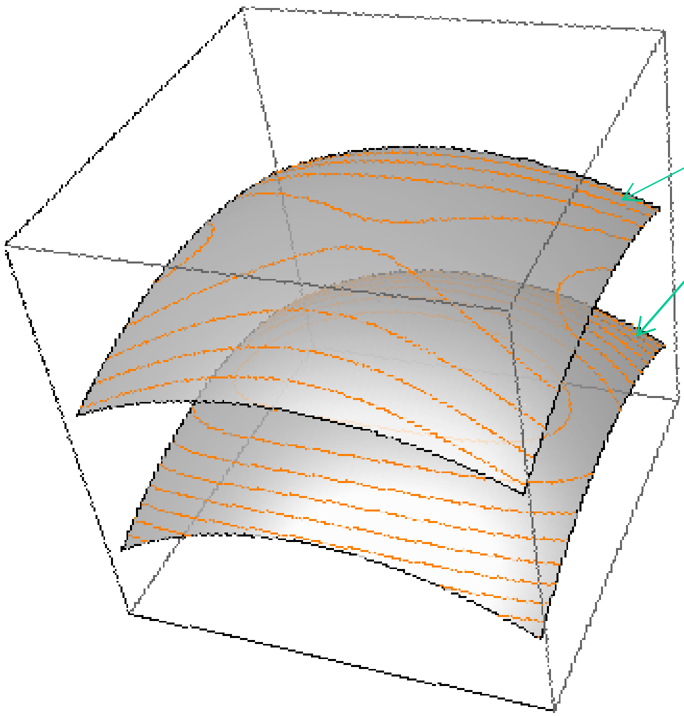
- Circulation conserves as a quantum
  (*helicity* is a Casimir)

- Relativity → space-time distortion
  → *creation* of vorticity (magnetic field)

Foliated phase space of *non-canonical* Hamiltonian system

- $\mathrm{Ker}(J) = \mathrm{Coker}(J) = \{\ \mathrm{grad}\, C\,(\boldsymbol{u})\ \}$

Casimir element



The "effective" phase space of constrained dynamics

•Equilibrium points?
•Stability?
•Thermal equilibrium?

Summary

- Macro = non-canonical
- Macro hierarchy = Casimir leaf
- Foliation → distorted space-time

Newton-Kant "space" → relativity, foliation
and beyond

The Euler eq. and vorticity eq.

- Euler equation of ideal fluid:

$$\partial_t u + (u \cdot \nabla) u = -\nabla p, \quad \nabla \cdot u = 0,$$
$$n \cdot u = 0$$

- Vorticity formulation:

$$\partial_t \omega = \nabla \times (u \times \omega) \quad (\omega := \nabla \times u)$$
$$= \{\omega, \phi\} \qquad (\mathrm{2D} : u = \mathrm{curl}\,\phi, \omega = -\Delta\phi)$$

- *Vorticity eq. in $H^{-1}(\Omega)$ is equivalent to Euler eq. in $L^2{}_\sigma(\Omega)$.*

Hamiltonian formalism of the Euler eq.

- Hamiltonian formalism

$$\boxed{\partial_t \omega = J(\omega)\partial_\omega H(\omega) \quad [\text{in } H^{-1}(\Omega)]}$$

- Hamiltonian

$$H(\omega) = \frac{1}{2}\int_\Omega (K\omega)\cdot\omega\, dx \quad [K := -\Delta^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)]$$

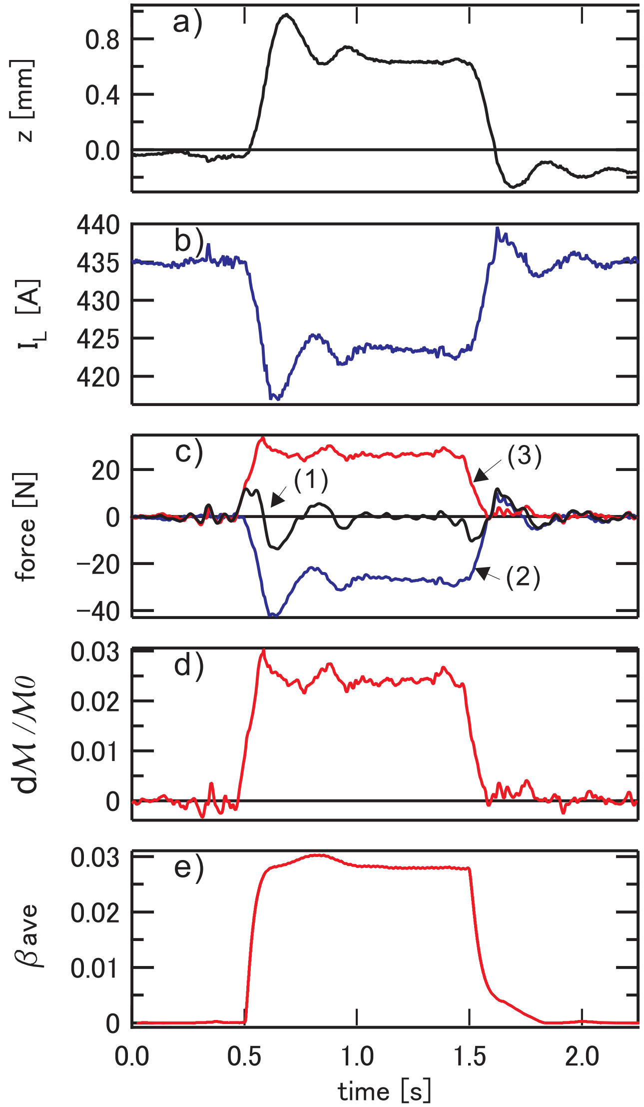
- Poisson operator

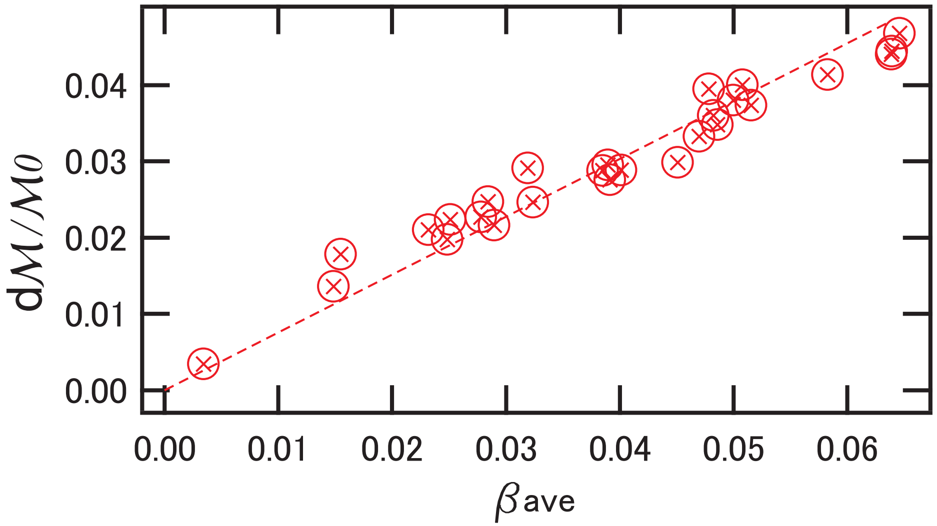
$$J(\omega) = \{\omega, \cdot\} : \; H_0^1(\Omega) \to H^{-1}(\Omega)$$

$$[\psi, \varphi] = \langle J(\omega)\psi, \varphi\rangle := \boxed{\langle \omega, \{\psi, \varphi\}\rangle} \quad [\omega \in C(\Omega)]$$

- *Known to be classically solvable for a Hölder continuous initial value: T. Kato (1967)*

Creation of magnetic moment by heating





$$\delta Q_{\mathrm{ECH}} = \overline{B}\,\mathrm{d}M = \overline{B}\sum_j \mathrm{d}\mu_j$$