Some Arithmetic Features of String Theory

Michael B. Green, University of Cambridge

ICTP: 50 Years of Science for the Future
6-9 October 2014

erc

UNIVERSITY OF CAMBRIDGE

Happy 50th birthday to ICTP

Happy 50th birthday to ICTP

Any visit to ICTP brings back happy memories of the times spent here – and specially fond memories of Abdus Salam, who was such a remarkable human being.

HAPPY 50TH BIRTHDAY TO ICTP

Any visit to ICTP brings back happy memories of the times spent here – and specially fond memories of Abdus Salam, who was such a remarkable human being.

Thank you Fernando, for organising this wonderful celebration of ICTP's half century, which has emphasised how it is flourishing under your guidance,

HAPPY 50TH BIRTHDAY TO ICTP

Any visit to ICTP brings back happy memories of the times spent here – and specially fond memories of Abdus Salam, who was such a remarkable human being.

Thank you Fernando, for organising this wonderful celebration of ICTP's half century, which has emphasised how it is flourishing under your guidance,

BUT

HAPPY 50TH BIRTHDAY TO ICTP

Any visit to ICTP brings back happy memories of the times spent here – and specially fond memories of Abdus Salam, who was such a remarkable human being.

Thank you Fernando, for organising this wonderful celebration of ICTP's half century, which has emphasised how it is flourishing under your guidance,

BUT

WE MISS YOU IN CAMBRIDGE – AND WE WANT YOU BACK!!

String theory is proposed as a framework for explaining elementary particles and their forces, as well as large-scale cosmological phenomena.

String theory is proposed as a framework for explaining elementary particles and their forces, as well as large-scale cosmological phenomena.

However, the development of string theory has revealed profound connections with a broader range of problems in theoretical physics that are presently the subject of intense research.

String theory is proposed as a framework for explaining elementary particles and their forces, as well as large-scale cosmological phenomena.

However, the development of string theory has revealed profound connections with a broader range of problems in theoretical physics that are presently the subject of intense research.

Most impressively, many of the developments in string theory have involved a profound symbiosis with various areas of pure mathematics.

String theory is proposed as a framework for explaining elementary particles and their forces, as well as large-scale cosmological phenomena.

However, the development of string theory has revealed profound connections with a broader range of problems in theoretical physics that are presently the subject of intense research.

Most impressively, many of the developments in string theory have involved a profound symbiosis with various areas of pure mathematics.

Given the diverse backgrounds of the audience, this talk will be a sketchy attempt to illustrate a particular aspect of this connection with mathematics in a narrow area of string theory.

THEME:

STRING THEORY CORRECTIONS TO EINSTEIN GRAVITY AT LOW ENERGY

Theme:

String theory corrections to Einstein gravity at low energy

- Exact features of closed string theory connecting weak and strong coupling - dualities

THEME:

STRING THEORY CORRECTIONS TO EINSTEIN GRAVITY AT LOW ENERGY

- EXACT FEATURES OF CLOSED STRING THEORY CONNECTING WEAK AND STRONG COUPLING - DUALITIES

- CONNECTIONS WITH BEAUTIFUL MATHEMATICS:

  Modular Forms; Automorphic forms for higher-rank groups; Multi-Zeta Values; …….

THEME:

STRING THEORY CORRECTIONS TO EINSTEIN GRAVITY AT LOW ENERGY

- EXACT FEATURES OF CLOSED STRING THEORY CONNECTING WEAK AND STRONG COUPLING - DUALITIES

- CONNECTIONS WITH BEAUTIFUL MATHEMATICS:

  Modular Forms; Automorphic forms for higher-rank groups; Multi-Zeta Values; …….

MBG, Stephen Miller, Pierre Vanhove arXiv:1404.2192
MBG, Eric D'Hoker, arXiv:1308.4597
MBG, Eric D'Hoker, Boris Pioline, Rudolfo Russo; arXiv:1405.6226

REMINDER:

THE STRINGS OF STRING THEORY

REMINDER:

THE STRINGS OF STRING THEORY

OPEN (SUPER) strings - ground state VECTOR BOSON.

REMINDER:

THE STRINGS OF STRING THEORY

OPEN (SUPER) strings - ground state VECTOR BOSON.

At low energy reproduces (super) Yang-Mills theory.

REMINDER:

THE STRINGS OF STRING THEORY

OPEN (SUPER) strings - ground state VECTOR BOSON.

At low energy reproduces (super) Yang-Mills theory.

CLOSED (SUPER) strings – ground state GRAVITON.

REMINDER:

THE STRINGS OF STRING THEORY

OPEN (SUPER) strings - ground state VECTOR BOSON.

At low energy reproduces (super) Yang-Mills theory.

CLOSED (SUPER) strings – ground state GRAVITON.

At low energy reproduces EINSTEIN (super) gravity.

REMINDER:

THE STRINGS OF STRING THEORY

OPEN (SUPER) strings - ground state VECTOR BOSON.

At low energy reproduces (super) Yang-Mills theory.

CLOSED (SUPER) strings – ground state GRAVITON.

At low energy reproduces EINSTEIN (super) gravity.

Since open string endpoints can join to form closed strings, string theory exhibits profound interconnections between Yang-Mills theory and gravity.

REMINDER:

THE STRINGS OF STRING THEORY

OPEN (SUPER) strings - ground state VECTOR BOSON.

At low energy reproduces (super) Yang-Mills theory.

CLOSED (SUPER) strings – ground state GRAVITON.

At low energy reproduces EINSTEIN (super) gravity.

Since open string endpoints can join to form closed strings, string theory exhibits profound interconnections between Yang-Mills theory and gravity.

TWO KINDS OF APPROXIMATION

(Closed) String Perturbation Theory

(CLOSED) STRING PERTURBATION THEORY

Expansion in powers of the string coupling, $g_s = e^{\phi} \ll 1$.

i.e, around a boundary of moduli space (the space of scalar fields).

(CLOSED) STRING PERTURBATION THEORY

Expansion in powers of the string coupling, $g_s = e^{\phi} \ll 1$.

i.e, around a boundary of moduli space (the space of scalar fields).

STRINGY FEYNMAN DIAGRAMS for scattering amplitudes:

(CLOSED) STRING PERTURBATION THEORY

Expansion in powers of the string coupling, $g_s = e^{\phi} \ll 1$.

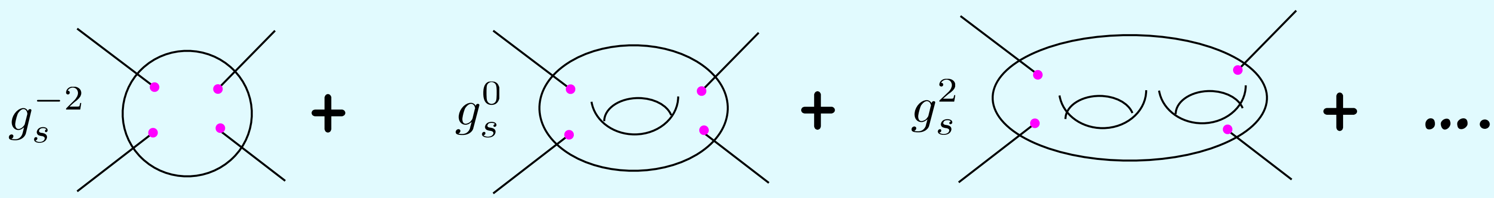
i.e, around a boundary of moduli space (the space of scalar fields).

STRINGY FEYNMAN DIAGRAMS for scattering amplitudes:

(world-sheets)

Sum of functional integrals over Riemann surfaces of arbitrary genus :

(c.f. FEYNMAN DIAGRAMS of quantum field theory)



(CLOSED) STRING PERTURBATION THEORY

Expansion in powers of the string coupling, $g_s = e^{\phi} \ll 1$.

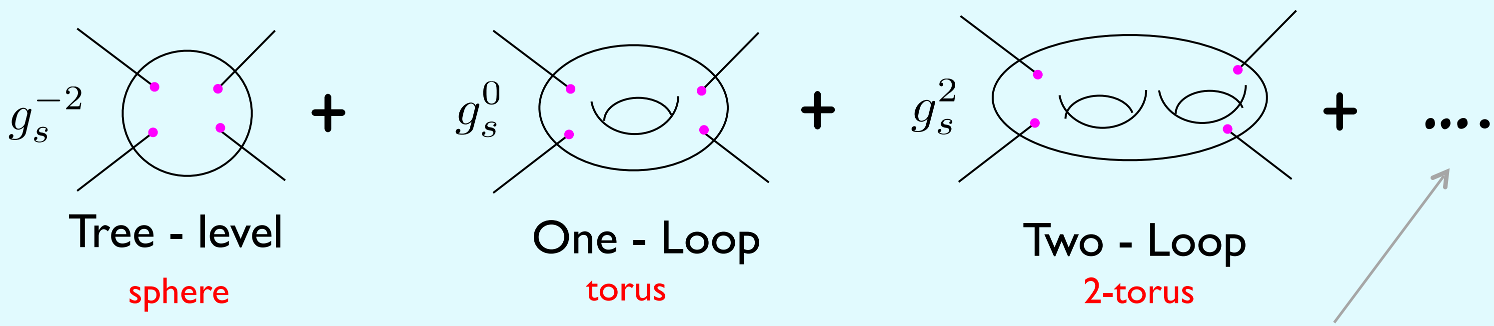
i.e, around a boundary of moduli space (the space of scalar fields).

STRINGY FEYNMAN DIAGRAMS for scattering amplitudes:

(world-sheets)

Sum of functional integrals over Riemann surfaces of arbitrary genus :

(c.f. FEYNMAN DIAGRAMS of quantum field theory)



$g_s^{2h-2} \times$ (genus-h Riemann surface) + ... includes nonperturbative (e.g. instanton) effects.

(CLOSED) STRING PERTURBATION THEORY

Expansion in powers of the string coupling, $g_s = e^{\phi} \ll 1$.

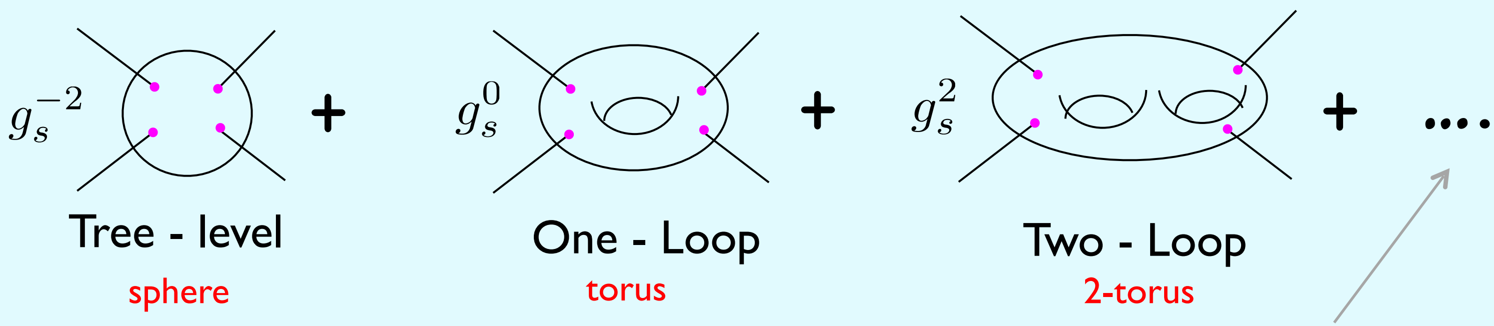
i.e, around a boundary of moduli space (the space of scalar fields).

STRINGY FEYNMAN DIAGRAMS for scattering amplitudes:

(world-sheets)

Sum of functional integrals over Riemann surfaces of arbitrary genus :

(c.f. FEYNMAN DIAGRAMS of quantum field theory)



g_s^{2h-2} $\times$ (genus-h Riemann surface) + ... includes nonperturbative (e.g. instanton) effects.

NO ULTRAVIOLET DIVERGENCES!!

(CLOSED) STRING PERTURBATION THEORY

Expansion in powers of the string coupling, $g_s = e^{\phi} \ll 1$.

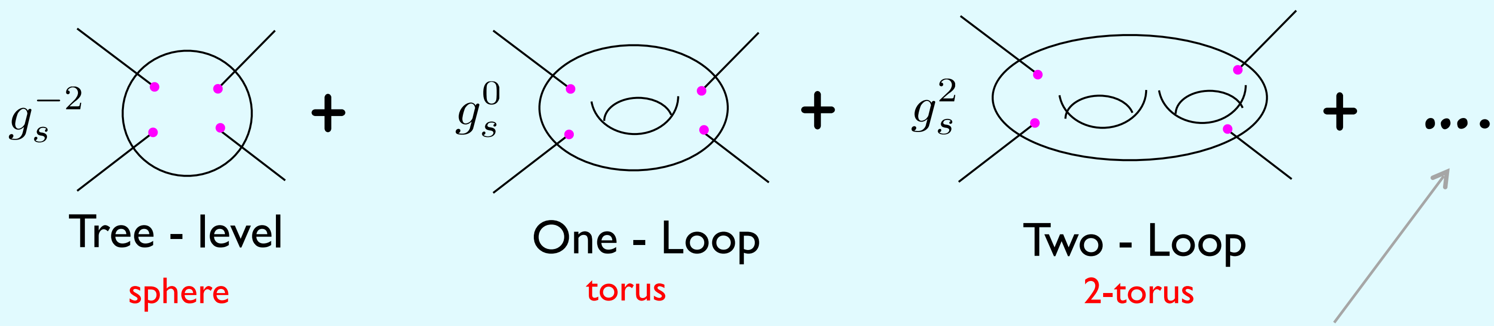
i.e, around a boundary of moduli space (the space of scalar fields).

STRINGY FEYNMAN DIAGRAMS for scattering amplitudes:

(world-sheets)

Sum of functional integrals over Riemann surfaces of arbitrary genus :

(c.f. FEYNMAN DIAGRAMS of quantum field theory)



$g_s^{2h-2} \times$ (genus-h Riemann surface) + ... includes nonperturbative (e.g. instanton) effects.

NO ULTRAVIOLET DIVERGENCES!!

c.f.

SUPERGRAVITY: CLASSICALLY low energy field theory limit of closed string theory.

(field theory) QUANTUM theory probably inconsistent – Ultraviolet divergences.

THE LOW ENERGY EXPANSION OF STRING THEORY

- LOWEST ORDER TERM reproduces the results of classical Einstein supergravity

ℓ_s is STRING LENGTH SCALE

$$\frac{1}{\ell_s^8}\int d^{10}x\sqrt{-\det G}\, e^{-2\phi}\, R + \ldots$$

curvature

several other supergravity fields

EINSTEIN-HILBERT ACTION

THE LOW ENERGY EXPANSION OF STRING THEORY

- LOWEST ORDER TERM reproduces the results of classical Einstein supergravity

ℓ_s is STRING LENGTH SCALE

$$\frac{1}{\ell_s^8}\int d^{10}x\sqrt{-\det G}\,e^{-2\phi}\,R + \ldots$$

curvature

several other supergravity fields

METRIC – $G_{\mu\nu}$

SCALAR FIELD - DILATON

$$e^{-\phi} = \frac{1}{g_s}$$

STRING COUPLING CONSTANT

EINSTEIN-HILBERT ACTION

The Low Energy Expansion of String Theory

- Lowest order term reproduces the results of classical Einstein supergravity

ℓ_s is STRING LENGTH SCALE

$$\frac{1}{\ell_s^8}\int d^{10}x\sqrt{-\det G}\,e^{-2\phi}\,R + \dots$$

curvature

several other supergravity fields

METRIC – $G_{\mu\nu}$

SCALAR FIELD - DILATON

$$e^{-\phi} = \frac{1}{g_s}$$

STRING COUPLING CONSTANT

EINSTEIN-HILBERT ACTION

Expanding the curvature in small fluctuations of the metric around D=10 Minkowski space gives contributions to "classical" MULTI-GRAVITON scattering amplitudes.

THE LOW ENERGY EXPANSION OF STRING THEORY

- LOWEST ORDER TERM reproduces the results of classical Einstein supergravity

ℓ_s is STRING LENGTH SCALE

$$\frac{1}{\ell_s^8}\int d^{10}x\sqrt{-\det G}\, e^{-2\phi}\, R + \ldots$$

curvature

several other supergravity fields

METRIC – $G_{\mu\nu}$

SCALAR FIELD - DILATON

$e^{-\phi} = \dfrac{1}{g_s}$ STRING COUPLING CONSTANT

EINSTEIN-HILBERT ACTION

Expanding the curvature in small fluctuations of the metric around D=10 Minkowski space gives contributions to "classical" MULTI-GRAVITON scattering amplitudes.

- HIGHER ORDER TERMS:

$$\frac{1}{\ell_s^2}\int d^{10}x\,\sqrt{-\det G}\,\mathcal{F}(\phi,\ldots)\,R^4 + \ldots$$

MODULI-DEPENDENT COEFFICIENT – function of scalar fields, or couplings. We want to understand these coefficients

THE LOW ENERGY EXPANSION OF STRING THEORY

- LOWEST ORDER TERM reproduces the results of classical Einstein supergravity

ℓ_s is STRING LENGTH SCALE

$$\frac{1}{\ell_s^8}\int d^{10}x\sqrt{-\det G}\, e^{-2\phi}\, R + \dots$$

curvature

several other supergravity fields

METRIC – $G_{\mu\nu}$

SCALAR FIELD - DILATON

$$e^{-\phi} = \frac{1}{g_s}$$

STRING COUPLING CONSTANT

EINSTEIN-HILBERT ACTION

Expanding the curvature in small fluctuations of the metric around D=10 Minkowski space gives contributions to "classical" MULTI-GRAVITON scattering amplitudes.

- HIGHER ORDER TERMS: $$\frac{1}{\ell_s^2}\int d^{10}x\,\sqrt{-\det G}\,\mathcal{F}(\phi,\dots)\,R^4 + \dots$$

- Expansion in powers of $\ell_s^2\, R\,,\ \ell_s^2\, D^2\,,\dots$

MODULI-DEPENDENT COEFFICIENT – function of scalar fields, or couplings. We want to understand these coefficients

THE LOW ENERGY EXPANSION OF STRING THEORY

- LOWEST ORDER TERM reproduces the results of classical Einstein supergravity

ℓ_s is STRING LENGTH SCALE

$$\frac{1}{\ell_s^8}\int d^{10}x\sqrt{-\det G}\, e^{-2\phi}\, R + \dots$$

curvature

several other supergravity fields

METRIC – $G_{\mu\nu}$

SCALAR FIELD - DILATON

$$e^{-\phi} = \frac{1}{g_s}$$

STRING COUPLING CONSTANT

EINSTEIN-HILBERT ACTION

Expanding the curvature in small fluctuations of the metric around D=10 Minkowski space gives contributions to "classical" MULTI-GRAVITON scattering amplitudes.

- HIGHER ORDER TERMS:

$$\frac{1}{\ell_s^2}\int d^{10}x\,\sqrt{-\det G}\,\mathcal{F}(\phi,\dots)\,R^4 + \dots$$

MODULI-DEPENDENT COEFFICIENT – function of scalar fields, or couplings. We want to understand these coefficients

- Expansion in powers of $\ell_s^2\, R\,,\ \ell_s^2\, D^2\,, \dots$

such as $D^4R^4 \quad D^6R^4 \quad D^8R^4 \quad \dots$ etc.

$R^5 \quad D^2R^4 \quad \dots$

Moduli and Duality groups in closed string theory

MODULI AND DUALITY GROUPS IN CLOSED STRING THEORY

Scalar fields – (geometric and non-geometric) MODULI parameterize symmetric space

Moduli and Duality groups in closed string theory

Scalar fields – (geometric and non-geometric) MODULI parameterize symmetric space

$$\mu_D \in G(\mathbb{R})/K(\mathbb{R})$$

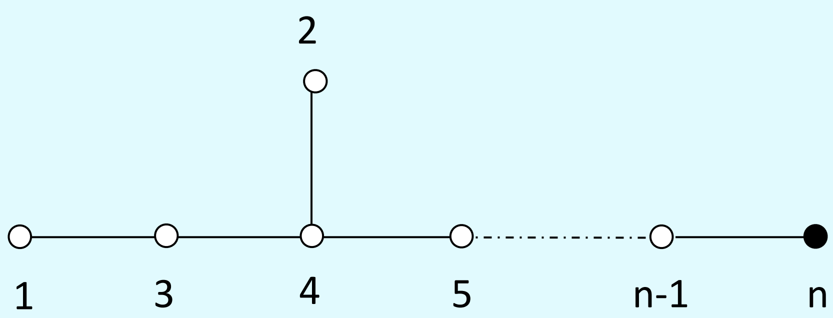
groups in E_n series
(real split forms)

(Cremmer, Julia)

Moduli and Duality groups in closed string theory

Scalar fields – (geometric and non-geometric) MODULI parameterize symmetric space

$\mu_D \in G(\mathbb{R})/K(\mathbb{R})$ groups in E_n series (real split forms) (Cremmer, Julia)



Dynkin diagram for E_n

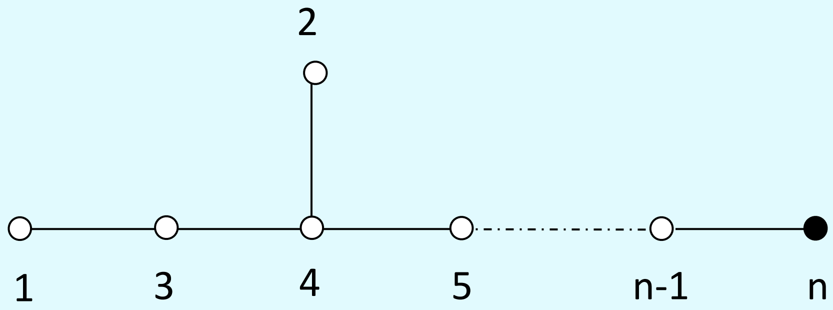
MODULI AND DUALITY GROUPS IN CLOSED STRING THEORY

Scalar fields – (geometric and non-geometric) MODULI parameterize symmetric space

$\mu_D \in G(\mathbb{R})/K(\mathbb{R})$ groups in E_n series (real split forms) (Cremmer, Julia)

Discrete identifications of scalar fields

$G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$



Dynkin diagram for E_n

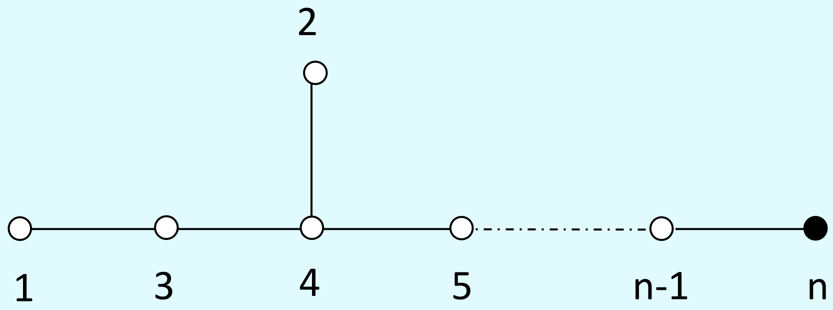
Moduli and Duality groups in closed string theory

Scalar fields – (geometric and non-geometric) MODULI parameterize symmetric space

$\mu_D \in G(\mathbb{R})/K(\mathbb{R})$ groups in E_n series (real split forms) (Cremmer, Julia)

Discrete identifications of scalar fields $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$

DUALITY GROUP $G(\mathbb{Z})$ Arithmetic subgroup of $G(\mathbb{R})$ is symmetry of string theory.



Dynkin diagram for E_n

MODULI AND DUALITY GROUPS IN CLOSED STRING THEORY

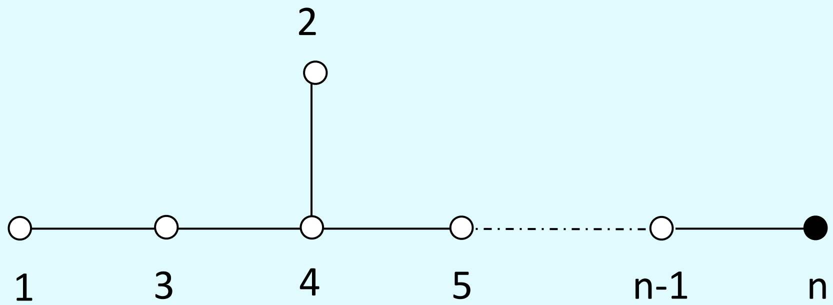
Scalar fields – (geometric and non-geometric) MODULI parameterize symmetric space

$\mu_D \in G(\mathbb{R})/K(\mathbb{R})$ groups in E_n series (real split forms) (Cremmer, Julia)

Discrete identifications of scalar fields $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$

DUALITY GROUP $G(\mathbb{Z})$

Arithmetic subgroup of $G(\mathbb{R})$ is symmetry of string theory.



Dynkin diagram for E_n

DISCRETE DUALITY GROUPS

Discrete Duality Groups	Space-time dimensions
$SL(2,\mathbb{Z})$	10B
$SL(2,\mathbb{Z})$	9
$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	8
$SL(5,\mathbb{Z})$	7
$SO(5,5,\mathbb{Z})$	6
$E_{6(6)}(\mathbb{Z})$	5
$E_{7(7)}(\mathbb{Z})$	4
$E_{8(8)}(\mathbb{Z})$	3

Compactify on $\mathcal{T}^{n-1}$

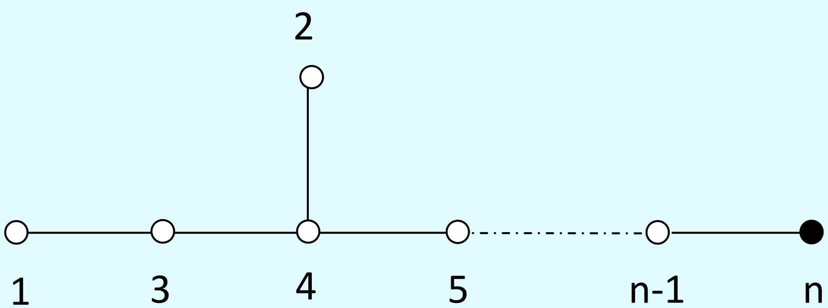
Moduli and Duality groups in closed string theory

Scalar fields – (geometric and non-geometric) MODULI parameterize symmetric space

$\mu_D \in G(\mathbb{R})/K(\mathbb{R})$ groups in E_n series (real split forms) (Cremmer, Julia)

Discrete identifications of scalar fields $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$

DUALITY GROUP $G(\mathbb{Z})$ Arithmetic subgroup of $G(\mathbb{R})$ is symmetry of string theory.



Dynkin diagram for E_n

DISCRETE DUALITY GROUPS

	Space-time dimensions
$SL(2,\mathbb{Z})$	10B
$SL(2,\mathbb{Z})$	9
$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	8
$SL(5,\mathbb{Z})$	7
$SO(5,5,\mathbb{Z})$	6
$E_{6(6)}(\mathbb{Z})$	5
$E_{7(7)}(\mathbb{Z})$	4
$E_{8(8)}(\mathbb{Z})$	3

Compactify on $\mathcal{T}^{n-1}$

DUALITIES relate theories in different regions of MODULI SPACE.

Moduli and Duality groups in closed string theory

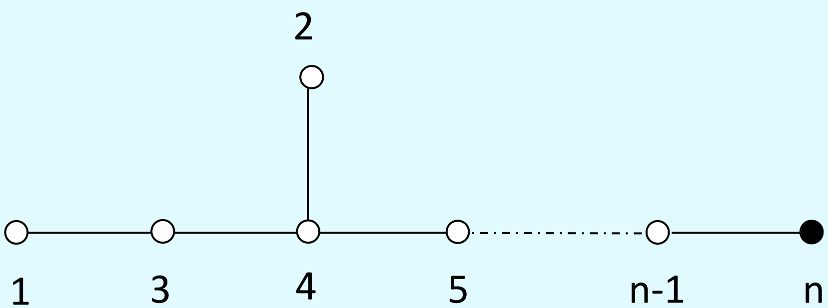
Scalar fields – (geometric and non-geometric) MODULI parameterize symmetric space

$\mu_D \in G(\mathbb{R})/K(\mathbb{R})$ ← groups in E_n series (real split forms) (Cremmer, Julia)

Discrete identifications of scalar fields $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$

DUALITY GROUP $G(\mathbb{Z})$

Arithmetic subgroup of $G(\mathbb{R})$ is symmetry of string theory.



Dynkin diagram for E_n

DISCRETE DUALITY GROUPS

$SL(2,\mathbb{Z})$	10B	Space-time dimensions
$SL(2,\mathbb{Z})$	9	
$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	8	
$SL(5,\mathbb{Z})$	7	Compactify on $\mathcal{T}^{n-1}$
$SO(5,5,\mathbb{Z})$	6	
$E_{6(6)}(\mathbb{Z})$	5	
$E_{7(7)}(\mathbb{Z})$	4	
$E_{8(8)}(\mathbb{Z})$	3	

DUALITIES relate theories in different regions of MODULI SPACE.

RICH DEPENDENCE ON MODULI

The power of supersymmetry and duality

THE POWER OF SUPERSYMMETRY AND DUALITY

DUALITIES typically relate weak and strong coupling –
connect perturbative and non-perturbative features in a highly nontrivial manner.

The power of supersymmetry and duality

Dualities typically relate weak and strong coupling –
connect perturbative and non-perturbative features in a highly nontrivial manner.

Dependence on moduli (or coupling constants) expressed in terms of automorphic forms for relevant duality groups.

THE POWER OF SUPERSYMMETRY AND DUALITY

DUALITIES typically relate weak and strong coupling –
connect perturbative and non-perturbative features in a highly nontrivial manner.

Dependence on moduli (or coupling constants) expressed in terms of automorphic forms for relevant duality groups.

e.g., $SL(2,\mathbb{Z})$ duality of type 2b theory - associated with modular group of large diffeomorphisms of a hidden (M-theory) torus. Acts on the single complex modulus (coupling constant) τ:

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \tau = x + iy \qquad y = e^{-\phi} = \frac{1}{g_s}$$

$$a, b, c, d \in \mathbb{Z} \qquad ad - bc = 1$$

THE POWER OF SUPERSYMMETRY AND DUALITY

DUALITIES typically relate weak and strong coupling –
connect perturbative and non-perturbative features in a highly nontrivial manner.

Dependence on moduli (or coupling constants) expressed in terms of automorphic forms for relevant duality groups.

e.g., $SL(2,\mathbb{Z})$ duality of type 2b theory - associated with modular group of large diffeomorphisms of a hidden (M-theory) torus. Acts on the single complex modulus (coupling constant) τ:

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \tau = x + iy \quad y = e^{-\phi} = \frac{1}{g_s} \qquad a, b, c, d \in \mathbb{Z} \quad ad - bc = 1$$

SUPERSYMMETRY leads to first order differential equations on moduli space relating automorphic coefficients in low energy expansion.
Determines moduli dependent coefficients, at least for low dimension interactions.

THE POWER OF SUPERSYMMETRY AND DUALITY

DUALITIES typically relate weak and strong coupling –
connect perturbative and non-perturbative features in a highly nontrivial manner.

Dependence on moduli (or coupling constants) expressed in terms of automorphic forms for relevant duality groups.

e.g., $SL(2,\mathbb{Z})$ duality of type 2b theory - associated with modular group of large diffeomorphisms of a hidden (M-theory) torus. Acts on the single complex modulus (coupling constant) τ:

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \tau = x + iy \quad y = e^{-\phi} = \frac{1}{g_s} \qquad a,b,c,d \in \mathbb{Z} \quad ad - bc = 1$$

SUPERSYMMETRY leads to first order differential equations on moduli space relating automorphic coefficients in low energy expansion.
Determines moduli dependent coefficients, at least for low dimension interactions.

e.g. R^4 coefficient satisfies Laplace eigenvalue equation

$$\Delta_\tau \mathcal{F}_{\frac{1}{2}}(\tau) = \tfrac{3}{4}\, \mathcal{F}_{\frac{1}{2}}(\tau) \qquad \Delta_\tau = y^2(\partial_x^2 + \partial_y^2)$$

THE POWER OF SUPERSYMMETRY AND DUALITY

DUALITIES typically relate weak and strong coupling –
connect perturbative and non-perturbative features in a highly nontrivial manner.

Dependence on moduli (or coupling constants) expressed in terms of automorphic forms for relevant duality groups.

e.g., $SL(2,\mathbb{Z})$ duality of type 2b theory - associated with modular group of large diffeomorphisms of a hidden (M-theory) torus. Acts on the single complex modulus (coupling constant) τ:

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \tau = x + iy \quad y = e^{-\phi} = \frac{1}{g_s} \qquad a,b,c,d \in \mathbb{Z} \quad ad - bc = 1$$

SUPERSYMMETRY leads to first order differential equations on moduli space relating automorphic coefficients in low energy expansion.
Determines moduli dependent coefficients, at least for low dimension interactions.

e.g. R^4 coefficient satisfies Laplace eigenvalue equation

$$\Delta_\tau \mathcal{F}_{\frac{1}{2}}(\tau) = \tfrac{3}{4}\, \mathcal{F}_{\frac{1}{2}}(\tau) \qquad \Delta_\tau = y^2(\partial_x^2 + \partial_y^2)$$

UNIQUE SOLUTION: given the moderate growth boundary condition as $y \to \infty$

NON-HOLOMORPHIC EISENSTEIN SERIES for $SL(2,\mathbb{Z})$,

Some properties of non-holomorphic $SL(2,\mathbb{Z})$ Eisenstein series

$$\tau = x + iy$$

$$y = e^{-\phi} = \frac{1}{g_s}$$

$$E_s(\tau) = \sum_{\gcd(p,q)=1} \frac{y^s}{|p+q\tau|^{2s}} = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} (\mathrm{Im}\gamma\tau)^s$$

Some properties of non-holomorphic $SL(2,\mathbb{Z})$ Eisenstein series

$\tau = x + iy$

$y = e^{-\phi} = \frac{1}{g_s}$

$$E_s(\tau) = \sum_{\gcd(p,q)=1} \frac{y^s}{|p+q\tau|^{2s}} = \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} (\mathrm{Im}\gamma\tau)^s$$

Parabolic subgroup

Poincare series – manifest $SL(2,\mathbb{Z})$ invariance – generalises to higher rank duality groups

SOME PROPERTIES OF NON-HOLOMORPHIC $SL(2,\mathbb{Z})$ EISENSTEIN SERIES

$\tau = x + iy$

$y = e^{-\phi} = \frac{1}{g_s}$

$$E_s(\tau) = \sum_{\gcd(p,q)=1} \frac{y^s}{|p+q\tau|^{2s}} = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} (\mathrm{Im}\gamma\tau)^s$$

Parabolic subgroup

Poincare series – manifest $SL(2,\mathbb{Z})$ invariance – generalises to higher rank duality groups

- Solution of LAPLACE EIGENVALUE EQN. (consequence of maximal supersymmetry)

$$\Delta_\tau E_s(\tau) = s(s-1)\, E_s(\tau)$$

$\Delta_\tau = y^2(\partial_x^2 + \partial_y^2)$

SOME PROPERTIES OF NON-HOLOMORPHIC $SL(2,\mathbb{Z})$ EISENSTEIN SERIES

$\tau = x + iy$

$y = e^{-\phi} = \frac{1}{g_s}$

$$E_s(\tau) = \sum_{\gcd(p,q)=1} \frac{y^s}{|p+q\tau|^{2s}} = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} (\mathrm{Im}\gamma\tau)^s$$

Parabolic subgroup

Poincare series – manifest $SL(2,\mathbb{Z})$ invariance – generalises to higher rank duality groups

- Solution of LAPLACE EIGENVALUE EQN. (consequence of maximal supersymmetry)

$$\Delta_\tau E_s(\tau) = s(s-1)\, E_s(\tau)$$

$\Delta_\tau = y^2(\partial_x^2 + \partial_y^2)$

- FOURIER SERIES

$$E_s(\tau) = 2\sum_{k=0}^{\infty} \mathcal{F}_k(y)\cos(2\pi ikx)$$

SOME PROPERTIES OF NON-HOLOMORPHIC $SL(2,\mathbb{Z})$ EISENSTEIN SERIES

$\tau = x + iy$

$y = e^{-\phi} = \frac{1}{g_s}$

$$E_s(\tau) = \sum_{\gcd(p,q)=1} \frac{y^s}{|p+q\tau|^{2s}} = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} (\mathrm{Im}\gamma\tau)^s$$

Parabolic subgroup

Poincare series – manifest $SL(2,\mathbb{Z})$ invariance – generalises to higher rank duality groups

- Solution of LAPLACE EIGENVALUE EQN. (consequence of maximal supersymmetry)

$$\Delta_\tau E_s(\tau) = s(s-1)\, E_s(\tau) \qquad \Delta_\tau = y^2(\partial_x^2 + \partial_y^2)$$

- FOURIER SERIES

$$E_s(\tau) = 2\sum_{k=0}^{\infty} \mathcal{F}_k(y)\cos(2\pi ikx)$$

- ZERO MODE $k=0$: TWO POWER-BEHAVED TERMS ("perturbative") :

$$\mathcal{F}_0 = y^s + \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\zeta(2s)\Gamma(s)}\, y^{s-1}$$

SOME PROPERTIES OF NON-HOLOMORPHIC $SL(2,\mathbb{Z})$ EISENSTEIN SERIES

$\tau = x + iy$

$y = e^{-\phi} = \frac{1}{g_s}$

$$E_s(\tau) = \sum_{\gcd(p,q)=1} \frac{y^s}{|p+q\tau|^{2s}} = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} (\mathrm{Im}\gamma\tau)^s$$

Parabolic subgroup

Poincare series – manifest $SL(2,\mathbb{Z})$ invariance – generalises to higher rank duality groups

- Solution of LAPLACE EIGENVALUE EQN. (consequence of maximal supersymmetry)

$$\Delta_\tau E_s(\tau) = s(s-1)\, E_s(\tau) \qquad \Delta_\tau = y^2(\partial_x^2 + \partial_y^2)$$

- FOURIER SERIES $\quad E_s(\tau) = 2\sum_{k=0}^{\infty} \mathcal{F}_k(y)\cos(2\pi i k x)$

- ZERO MODE $k=0$: TWO POWER-BEHAVED TERMS ("perturbative") :

$$\mathcal{F}_0 = y^s + \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\zeta(2s)\Gamma(s)}\, y^{s-1}$$

- NON-ZERO MODES $k>0$: D-INSTANTON SUM

K-Bessel function

$$\mathcal{F}_k = \frac{2\pi^s}{\zeta(2s)\Gamma(s)} |k|^{s-\frac{1}{2}}\,\sigma_{2s-1}(k)\, y^{\frac{1}{2}}\, K_{s-\frac{1}{2}}(2\pi|k|y)$$

$$\underset{y\to\infty}{\sim} \frac{\pi^{s-\frac{1}{2}}}{\zeta(2s)\Gamma(s)} |k|^{s-1}\sigma_{2s-1}(k)\, e^{-2\pi|k|y}(1+O(y^{-1}))$$

measure

$$\sigma_n(k) = \sum_{p|k} p^n$$

½-BPS INTERACTION

$$\mathcal{F}_{\frac{1}{2}}(\tau)\, R^4 = 2\zeta(3)\, E_{\frac{3}{2}}(\tau)\, R^4$$

Einstein frame

½-BPS INTERACTION

$$\mathcal{F}_{\frac{1}{2}}(\tau)\, R^4 = 2\zeta(3)\, E_{\frac{3}{2}}(\tau)\, R^4$$

Einstein frame

where $\Delta_\tau \mathcal{F}_{\frac{1}{2}}(\tau) = \frac{3}{4}\mathcal{F}_{\frac{1}{2}}(\tau)$, so

$$e^{-c/g_s}$$

$$g_s^{-\frac{1}{2}}\, \mathcal{F}_{\frac{1}{2}} = 2\zeta(3)\, g_s^{-\frac{1}{2}}\, E_{\frac{3}{2}}(\tau) \sim 2\zeta(3) g_s^{-2} + 4\zeta(2)\, g_s^0 + \mathrm{D-instantons}$$

String frame

½-BPS INTERACTION $\mathcal{F}_{\frac{1}{2}}(\tau)\, R^4 = 2\zeta(3)\, E_{\frac{3}{2}}(\tau)\, R^4$ Einstein frame

where $\Delta_\tau \mathcal{F}_{\frac{1}{2}}(\tau) = \frac{3}{4}\mathcal{F}_{\frac{1}{2}}(\tau)$, so

e^{-c/g_s}

$$g_s^{-\frac{1}{2}}\, \mathcal{F}_{\frac{1}{2}} = 2\zeta(3)\, g_s^{-\frac{1}{2}}\, E_{\frac{3}{2}}(\tau) \sim 2\zeta(3) g_s^{-2} + 4\zeta(2)\, g_s^0 + \mathrm{D-instantons}$$ String frame

Two perturbative terms: tree-level genus-one exponential

NON-RENORMALISATION BEYOND 1 LOOP FOR R^4

½-BPS INTERACTION

$$\mathcal{F}_{\frac{1}{2}}(\tau)\, R^4 = 2\zeta(3)\, E_{\frac{3}{2}}(\tau)\, R^4$$

Einstein frame

where $\Delta_\tau \mathcal{F}_{\frac{1}{2}}(\tau) = \frac{3}{4}\mathcal{F}_{\frac{1}{2}}(\tau)$, so

$$e^{-c/g_s}$$

$$g_s^{-\frac{1}{2}}\, \mathcal{F}_{\frac{1}{2}} = 2\zeta(3)\, g_s^{-\frac{1}{2}}\, E_{\frac{3}{2}}(\tau) \sim 2\zeta(3) g_s^{-2} + 4\zeta(2)\, g_s^0 + \mathrm{D-instantons}$$

String frame

Two perturbative terms: tree-level genus-one exponential

NON-RENORMALISATION BEYOND 1 LOOP FOR R^4

- Coefficients, which are rational multiples of zeta values, agree with explicit calculations in string perturbation theory

½-BPS INTERACTION $\mathcal{F}_{\frac{1}{2}}(\tau)\, R^4 = 2\zeta(3)\, E_{\frac{3}{2}}(\tau)\, R^4$ Einstein frame

where $\Delta_\tau \mathcal{F}_{\frac{1}{2}}(\tau) = \frac{3}{4}\mathcal{F}_{\frac{1}{2}}(\tau)$, so

$$e^{-c/g_s}$$

$$g_s^{-\frac{1}{2}}\,\mathcal{F}_{\frac{1}{2}} = 2\zeta(3)\, g_s^{-\frac{1}{2}}\, E_{\frac{3}{2}}(\tau) \sim 2\zeta(3) g_s^{-2} + 4\zeta(2)\, g_s^0 + \mathrm{D-instantons}$$ String frame

Two perturbative terms: tree-level genus-one exponential

NON-RENORMALISATION BEYOND 1 LOOP FOR R^4

- Coefficients, which are rational multiples of zeta values, agree with explicit calculations in string perturbation theory
- Very detailed instanton measure – related via AdS/CFT to multi-instanton measure in maximally supersymmetric $SU(N)$ Yang-Mills in $N \to \infty$ limit.

½-BPS INTERACTION $\mathcal{F}_{\frac{1}{2}}(\tau)\, R^4 = 2\zeta(3)\, E_{\frac{3}{2}}(\tau)\, R^4$ Einstein frame

where $\Delta_\tau \mathcal{F}_{\frac{1}{2}}(\tau) = \frac{3}{4}\mathcal{F}_{\frac{1}{2}}(\tau)$, so

$$g_s^{-\frac{1}{2}}\,\mathcal{F}_{\frac{1}{2}} = 2\zeta(3)\, g_s^{-\frac{1}{2}}\, E_{\frac{3}{2}}(\tau) \sim 2\zeta(3) g_s^{-2} + 4\zeta(2)\, g_s^0 + \mathrm{D-instantons}$$

String frame; D-instantons $\sim e^{-c/g_s}$

Two perturbative terms: tree-level genus-one exponential

NON-RENORMALISATION BEYOND 1 LOOP FOR R^4

- Coefficients, which are rational multiples of zeta values, agree with explicit calculations in string perturbation theory
- Very detailed instanton measure – related via AdS/CFT to multi-instanton measure in maximally supersymmetric $SU(N)$ Yang-Mills in $N \to \infty$ limit.

¼-BPS INTERACTION $\mathcal{F}_{\frac{1}{4}}(\tau)\, D^4 R^4 = 2\zeta(3)\, E_{\frac{3}{2}}(\tau)\, D^4 R^4$

½-BPS INTERACTION $\mathcal{F}_{\frac{1}{2}}(\tau)\,R^4 = 2\zeta(3)\,E_{\frac{3}{2}}(\tau)\,R^4$ Einstein frame

where $\Delta_\tau \mathcal{F}_{\frac{1}{2}}(\tau) = \frac{3}{4}\mathcal{F}_{\frac{1}{2}}(\tau)$, so

$$e^{-c/g_s}$$

$$g_s^{-\frac{1}{2}}\,\mathcal{F}_{\frac{1}{2}} = 2\zeta(3)\,g_s^{-\frac{1}{2}}\,E_{\frac{3}{2}}(\tau) \sim 2\zeta(3)g_s^{-2} + 4\zeta(2)\,g_s^0 + \mathrm{D}-\mathrm{instantons}$$ String frame

Two perturbative terms: tree-level genus-one exponential

NON-RENORMALISATION BEYOND 1 LOOP FOR R^4

- Coefficients, which are rational multiples of zeta values, agree with explicit calculations in string perturbation theory
- Very detailed instanton measure – related via AdS/CFT to multi-instanton measure in maximally supersymmetric $SU(N)$ Yang-Mills in $N \to \infty$ limit.

¼-BPS INTERACTION $\mathcal{F}_{\frac{1}{4}}(\tau)\,D^4R^4 = 2\zeta(3)\,E_{\frac{3}{2}}(\tau)\,D^4R^4$

$$g_s^{\frac{1}{2}}\,\mathcal{F}_{\frac{1}{4}}(\tau) = \zeta(5)\,g_s^{\frac{1}{2}}\,E_{\frac{5}{2}}(\tau) \sim \zeta(5)g_s^{-2} + \frac{4}{3}\zeta(4)\,g_s^2 + \mathrm{D}-\mathrm{instantons}$$

½-BPS INTERACTION $\mathcal{F}_{\frac{1}{2}}(\tau)\, R^4 = 2\zeta(3)\, E_{\frac{3}{2}}(\tau)\, R^4$ Einstein frame

where $\Delta_\tau \mathcal{F}_{\frac{1}{2}}(\tau) = \frac{3}{4}\mathcal{F}_{\frac{1}{2}}(\tau)$, so

e^{-c/g_s}

$$g_s^{-\frac{1}{2}}\, \mathcal{F}_{\frac{1}{2}} = 2\zeta(3)\, g_s^{-\frac{1}{2}}\, E_{\frac{3}{2}}(\tau) \sim 2\zeta(3) g_s^{-2} + 4\zeta(2)\, g_s^0 + \mathrm{D} - \mathrm{instantons}$$ String frame

Two perturbative terms: tree-level genus-one exponential

NON-RENORMALISATION BEYOND 1 LOOP FOR R^4

- Coefficients, which are rational multiples of zeta values, agree with explicit calculations in string perturbation theory
- Very detailed instanton measure – related via AdS/CFT to multi-instanton measure in maximally supersymmetric $SU(N)$ Yang-Mills in $N \to \infty$ limit.

¼-BPS INTERACTION $\mathcal{F}_{\frac{1}{4}}(\tau)\, D^4 R^4 = 2\zeta(3)\, E_{\frac{3}{2}}(\tau)\, D^4 R^4$

$$g_s^{\frac{1}{2}}\, \mathcal{F}_{\frac{1}{4}}(\tau) = \zeta(5)\, g_s^{\frac{1}{2}}\, E_{\frac{5}{2}}(\tau) \sim \zeta(5) g_s^{-2} + \frac{4}{3}\zeta(4)\, g_s^2 + \mathrm{D} - \mathrm{instantons}$$

Perturbative terms: tree-level genus-two

(no genus-one term)

½-BPS INTERACTION $\mathcal{F}_{\frac{1}{2}}(\tau)\, R^4 = 2\zeta(3)\, E_{\frac{3}{2}}(\tau)\, R^4$ Einstein frame

where $\Delta_\tau \mathcal{F}_{\frac{1}{2}}(\tau) = \frac{3}{4}\mathcal{F}_{\frac{1}{2}}(\tau)$, so

$$e^{-c/g_s}$$

$$g_s^{-\frac{1}{2}}\, \mathcal{F}_{\frac{1}{2}} = 2\zeta(3)\, g_s^{-\frac{1}{2}}\, E_{\frac{3}{2}}(\tau) \sim 2\zeta(3) g_s^{-2} + 4\zeta(2)\, g_s^0 + \mathrm{D}-\mathrm{instantons}$$ String frame

Two perturbative terms: tree-level genus-one exponential

NON-RENORMALISATION BEYOND 1 LOOP FOR R^4

- Coefficients, which are rational multiples of zeta values, agree with explicit calculations in string perturbation theory
- Very detailed instanton measure – related via AdS/CFT to multi-instanton measure in maximally supersymmetric $SU(N)$ Yang-Mills in $N \to \infty$ limit.

¼-BPS INTERACTION $\mathcal{F}_{\frac{1}{4}}(\tau)\, D^4 R^4 = 2\zeta(3)\, E_{\frac{3}{2}}(\tau)\, D^4 R^4$

$$g_s^{\frac{1}{2}}\, \mathcal{F}_{\frac{1}{4}}(\tau) = \zeta(5)\, g_s^{\frac{1}{2}}\, E_{\frac{5}{2}}(\tau) \sim \zeta(5) g_s^{-2} + \frac{4}{3}\zeta(4)\, g_s^2 + \mathrm{D}-\mathrm{instantons}$$

Perturbative terms: tree-level genus-two
(no genus-one term)

NON-RENORMALISATION BEYOND 2 LOOPS FOR $D^4 R^4$

½-BPS INTERACTION $\mathcal{F}_{\frac{1}{2}}(\tau)\, R^4 = 2\zeta(3)\, E_{\frac{3}{2}}(\tau)\, R^4$ Einstein frame

where $\Delta_\tau \mathcal{F}_{\frac{1}{2}}(\tau) = \frac{3}{4}\mathcal{F}_{\frac{1}{2}}(\tau)$, so

$$e^{-c/g_s}$$

$$g_s^{-\frac{1}{2}}\,\mathcal{F}_{\frac{1}{2}} = 2\zeta(3)\, g_s^{-\frac{1}{2}}\, E_{\frac{3}{2}}(\tau) \sim 2\zeta(3) g_s^{-2} + 4\zeta(2)\, g_s^0 + \mathrm{D} - \mathrm{instantons}$$ String frame

Two perturbative terms: tree-level genus-one exponential

NON-RENORMALISATION BEYOND 1 LOOP FOR R^4

- Coefficients, which are rational multiples of zeta values, agree with explicit calculations in string perturbation theory
- Very detailed instanton measure – related via AdS/CFT to multi-instanton measure in maximally supersymmetric $SU(N)$ Yang-Mills in $N \to \infty$ limit.

¼-BPS INTERACTION $\mathcal{F}_{\frac{1}{4}}(\tau)\, D^4 R^4 = 2\zeta(3)\, E_{\frac{3}{2}}(\tau)\, D^4 R^4$

$$g_s^{\frac{1}{2}}\,\mathcal{F}_{\frac{1}{4}}(\tau) = \zeta(5)\, g_s^{\frac{1}{2}}\, E_{\frac{5}{2}}(\tau) \sim \zeta(5) g_s^{-2} + \frac{4}{3}\zeta(4)\, g_s^2 + \mathrm{D} - \mathrm{instantons}$$

Perturbative terms: tree-level genus-two
(no genus-one term)

NON-RENORMALISATION BEYOND 2 LOOPS FOR $D^4 R^4$

- Coefficients again agree with explicit calculations in string perturbation theory

1/8-BPS Next order $g_s\,\mathcal{F}_{\frac{1}{8}}(\tau)\;\; D^6 R^4$

1/8-BPS Next order $g_s \, \mathcal{F}_{\frac{1}{8}}(\tau) \;\; D^6 R^4$

INHOMOGENEOUS LAPLACE EQUATION:

$$(\Delta_\tau - 12)\, \mathcal{F}_{\frac{1}{8}}(\tau) = -\left(2\zeta(3)\, E_{\frac{3}{2}}(\tau)\right)^2$$

→ The square of the coefficient of R^4

1/8-BPS Next order $g_s\,\mathcal{F}_{\frac{1}{8}}(\tau)\;\; D^6 R^4$

INHOMOGENEOUS LAPLACE EQUATION:

$$(\Delta_\tau - 12)\,\mathcal{F}_{\frac{1}{8}}(\tau) = -\Big(2\zeta(3)\,E_{\frac{3}{2}}(\tau)\Big)^2$$

→ The square of the coefficient of R^4

Detailed structure not yet derived in detail from supersymmetry but is based on duality of string theory with M-theory on a torus. [Derive from supersymmetry?]

1/8-BPS Next order $g_s\,\mathcal{F}_{\frac{1}{8}}(\tau)\;\;D^6 R^4$

INHOMOGENEOUS LAPLACE EQUATION:

$$(\Delta_\tau - 12)\,\mathcal{F}_{\frac{1}{8}}(\tau) = -\Big(2\zeta(3)\,E_{\frac{3}{2}}(\tau)\Big)^2$$

→ The square of the coefficient of R^4

Detailed structure not yet derived in detail from supersymmetry but is based on duality of string theory with M-theory on a torus. [Derive from supersymmetry?]

An interesting equation with a rather nontrivial solution.

1/8-BPS   Next order $g_s\,\mathcal{F}_{\frac{1}{8}}(\tau)\;\; D^6 R^4$

INHOMOGENEOUS LAPLACE EQUATION:

$$(\Delta_\tau - 12)\,\mathcal{F}_{\frac{1}{8}}(\tau) = -\Big(2\zeta(3)\,E_{\frac{3}{2}}(\tau)\Big)^2$$

→ The square of the coefficient of R^4

Detailed structure not yet derived in detail from supersymmetry but is based on duality of string theory with M-theory on a torus. [Derive from supersymmetry?]

An interesting equation with a rather nontrivial solution.

Solution can be expressed as a Poincare series:

(MBG, Miller, Vanhove)
+Zagier

$$\mathcal{F}_{\frac{1}{8}}(\tau) = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \Phi(\gamma\tau)$$

1/8-BPS Next order $g_s \, \mathcal{F}_{\frac{1}{8}}(\tau) \;\; D^6 R^4$

INHOMOGENEOUS LAPLACE EQUATION:

$$(\Delta_\tau - 12)\, \mathcal{F}_{\frac{1}{8}}(\tau) = -\Big(2\zeta(3)\, E_{\frac{3}{2}}(\tau)\Big)^2$$

→ The square of the coefficient of R^4

Detailed structure not yet derived in detail from supersymmetry but is based on duality of string theory with M-theory on a torus. [Derive from supersymmetry?]

An interesting equation with a rather nontrivial solution.

Solution can be expressed as a Poincare series: (MBG, Miller, Vanhove) +Zagier

$$\mathcal{F}_{\frac{1}{8}}(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \Phi(\gamma\tau)$$

where

$$\Phi(\tau) = a_0(y) + \sum_{n \neq 0} a_n(y) e^{2\pi i n x}$$

$a_n(y)$ linear in $K_0\,,\;\; K_1$ Bessel fns.

Fourier modes

$$\mathcal{F}_{\frac{1}{8}}(\tau) = \sum_k \hat{f}_k(y)\, e^{2\pi i k x}$$

FOURIER MODES $\mathcal{F}_{\frac{1}{8}}(\tau) = \sum_k \hat{f}_k(y)\, e^{2\pi i k x}$

ZERO MODE

$$\hat{f}_0(y) = \frac{2\zeta(3)^2}{3}\, y^3 + \frac{4\,\zeta(2)\,\zeta(3)}{3}\, y + 4\,\zeta(4) y^{-1} + \frac{4\,\zeta(6)}{27} y^{-3} + \sum_{m\neq 0} \hat{f}_0^m(y)$$

FOURIER MODES $\mathcal{F}_{\frac{1}{8}}(\tau) = \sum_k \hat{f}_k(y)\, e^{2\pi i k x}$

ZERO MODE

$$\hat{f}_0(y) = \frac{2\zeta(3)^2}{3}\, y^3 + \frac{4\,\zeta(2)\,\zeta(3)}{3}\, y + 4\,\zeta(4) y^{-1} + \frac{4\,\zeta(6)}{27} y^{-3} + \sum_{m\neq 0} \hat{f}_0^m(y)$$

GENUS	0	1	2	3	Non-Perturbative

Four power-behaved terms

FOURIER MODES $\mathcal{F}_{\frac{1}{8}}(\tau) = \sum_k \hat{f}_k(y)\, e^{2\pi i k x}$

ZERO MODE

$$\hat{f}_0(y) = \frac{2\zeta(3)^2}{3}\, y^3 + \frac{4\,\zeta(2)\,\zeta(3)}{3}\, y + 4\,\zeta(4) y^{-1} + \frac{4\,\zeta(6)}{27} y^{-3} + \sum_{m \neq 0} \hat{f}_0^m(y)$$

GENUS	0	1	2	3	Non-Perturbative

Four power-behaved terms

- PERTURBATIVE CONTRIBUTIONS AGREE WITH EXPLICIT STRING THEORY CALCULATIONS

(BUT GENUS 3 string calculation needs RE-CHECKING)

FOURIER MODES $\mathcal{F}_{\frac{1}{8}}(\tau) = \sum_k \hat{f}_k(y)\, e^{2\pi i k x}$

ZERO MODE

$$\hat{f}_0(y) = \frac{2\zeta(3)^2}{3}\, y^3 + \frac{4\,\zeta(2)\,\zeta(3)}{3}\, y + 4\,\zeta(4) y^{-1} + \frac{4\,\zeta(6)}{27} y^{-3} + \sum_{m\neq 0} \hat{f}_0^m(y)$$

GENUS 0 1 2 3 Non-Perturbative

Four power-behaved terms

- PERTURBATIVE CONTRIBUTIONS AGREE WITH EXPLICIT STRING THEORY CALCULATIONS

  (BUT GENUS 3 string calculation needs RE-CHECKING)

- NON-PERTURBATIVE terms $\hat{f}_0(y) \underset{y\to\infty}{\sim} e^{-4\pi|m|y} \left(\frac{\sigma_2(|m|)^2}{|m|^5 y^2} + O(y^-3) \right)$

FOURIER MODES $\mathcal{F}_{\frac{1}{8}}(\tau) = \sum_k \hat{f}_k(y)\, e^{2\pi i k x}$

ZERO MODE

$$\hat{f}_0(y) = \frac{2\zeta(3)^2}{3}\, y^3 + \frac{4\,\zeta(2)\,\zeta(3)}{3}\, y + 4\,\zeta(4) y^{-1} + \frac{4\,\zeta(6)}{27} y^{-3} + \sum_{m\neq 0} \hat{f}_0^m(y)$$

GENUS	0	1	2	3	Non-Perturbative

Four power-behaved terms

- PERTURBATIVE CONTRIBUTIONS AGREE WITH EXPLICIT STRING THEORY CALCULATIONS

  (BUT GENUS 3 string calculation needs RE-CHECKING)

- NON-PERTURBATIVE terms $\hat{f}_0(y) \underset{y\to\infty}{\sim} e^{-4\pi|m|y} \left(\frac{\sigma_2(|m|)^2}{|m|^5 y^2} + O(y^-3) \right)$

Suggestive of charge-zero INSTANTON / ANTI-INSTANTON pairs.

FOURIER MODES $\mathcal{F}_{\frac{1}{8}}(\tau) = \sum_k \hat{f}_k(y)\, e^{2\pi i k x}$

ZERO MODE

$$\hat{f}_0(y) = \frac{2\zeta(3)^2}{3}\, y^3 + \frac{4\,\zeta(2)\,\zeta(3)}{3}\, y + 4\,\zeta(4) y^{-1} + \frac{4\,\zeta(6)}{27} y^{-3} + \sum_{m\neq 0} \hat{f}_0^m(y)$$

GENUS	0	1	2	3	Non-Perturbative

Four power-behaved terms

- PERTURBATIVE CONTRIBUTIONS AGREE WITH EXPLICIT STRING THEORY CALCULATIONS

(BUT GENUS 3 string calculation needs RE-CHECKING)

- NON-PERTURBATIVE terms $\hat{f}_0(y) \underset{y\to\infty}{\sim} e^{-4\pi|m|y} \left(\frac{\sigma_2(|m|)^2}{|m|^5 y^2} + O(y^-3) \right)$

Suggestive of charge-zero INSTANTON / ANTI-INSTANTON pairs.

NON-ZERO MODES

$$\hat{f}_k(y) \underset{y\to\infty}{\sim} \sum_{\substack{k_1+k_2=k \\ (k_1,k_2)\neq(0,0)}} (\dots) e^{-2\pi(|k_1|+|k_2|)y}$$

FOURIER MODES $\mathcal{F}_{\frac{1}{8}}(\tau) = \sum_k \hat{f}_k(y)\, e^{2\pi i k x}$

ZERO MODE

$$\hat{f}_0(y) = \frac{2\zeta(3)^2}{3}\, y^3 + \frac{4\,\zeta(2)\,\zeta(3)}{3}\, y + 4\,\zeta(4) y^{-1} + \frac{4\,\zeta(6)}{27} y^{-3} + \sum_{m\neq 0} \hat{f}_0^m(y)$$

GENUS	0	1	2	3	Non-Perturbative

Four power-behaved terms

- PERTURBATIVE CONTRIBUTIONS AGREE WITH EXPLICIT STRING THEORY CALCULATIONS

(BUT GENUS 3 string calculation needs RE-CHECKING)

- NON-PERTURBATIVE terms $\hat{f}_0(y) \underset{y \to \infty}{\sim} e^{-4\pi|m|y} \left(\frac{\sigma_2(|m|)^2}{|m|^5 y^2} + O(y^-3) \right)$

Suggestive of charge-zero INSTANTON / ANTI-INSTANTON pairs.

NON-ZERO MODES

$$\hat{f}_k(y) \underset{y \to \infty}{\sim} \sum_{\substack{k_1+k_2=k \\ (k_1,k_2)\neq(0,0)}} (\dots) e^{-2\pi(|k_1|+|k_2|)y}$$

INSTANTONS AND ANTI-INSTANTONS

HIGHER-RANK DUALITY GROUPS

Compactify M-theory on a n-torus to D=10-n dimensions

MBG, Miller, Russo, Vanhove

Pioline

HIGHER-RANK DUALITY GROUPS

Compactify M-theory on a n-torus to D=10-n dimensions

MBG, Miller, Russo, Vanhove

Pioline

Duality Group $G(\mathbb{Z})$	space-time dimension
1	10A
$SL(2,\mathbb{Z})$	10B
$SL(2,\mathbb{Z})$	9
$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	8
$SL(5,\mathbb{Z})$	7
$SO(5,5,\mathbb{Z})$	6
$E_{6(6)}(\mathbb{Z})$	5
$E_{7(7)}(\mathbb{Z})$	4
$E_{8(8)}(\mathbb{Z})$	3

HIGHER-RANK DUALITY GROUPS

Compactify M-theory on a n-torus to D=10-n dimensions

MBG, Miller, Russo, Vanhove
Pioline

Duality Group $G(\mathbb{Z})$	space-time dimension
1	10A
$SL(2,\mathbb{Z})$	10B
$SL(2,\mathbb{Z})$	9
$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	8
$SL(5,\mathbb{Z})$	7
$SO(5,5,\mathbb{Z})$	6
$E_{6(6)}(\mathbb{Z})$	5
$E_{7(7)}(\mathbb{Z})$	4
$E_{8(8)}(\mathbb{Z})$	3

Automorphic functions for higher-rank groups ;

Langlands Eisenstein series' associated with maximal parabolic subgroups of G.

HIGHER-RANK DUALITY GROUPS

Compactify M-theory on a n-torus to D=10-n dimensions

MBG, Miller, Russo, Vanhove
Pioline

Duality Group $G(\mathbb{Z})$	space-time dimension
$\mathbf{1}$	10A
$SL(2,\mathbb{Z})$	10B
$SL(2,\mathbb{Z})$	9
$SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$	8
$SL(5,\mathbb{Z})$	7
$SO(5,5,\mathbb{Z})$	6
$E_{6(6)}(\mathbb{Z})$	5
$E_{7(7)}(\mathbb{Z})$	4
$E_{8(8)}(\mathbb{Z})$	3

Automorphic functions for higher-rank groups ;

Langlands Eisenstein series' associated with maximal parabolic subgroups of G.

$$E^G_{s_1,\ldots,s_r}$$

rank r $\quad s_1, \ldots, s_r \in \mathbb{C}$

labels associated with nodes of Dynkin diagram

HIGHER-RANK DUALITY GROUPS

Compactify M-theory on a n-torus to D=10-n dimensions

MBG, Miller, Russo, Vanhove

Pioline

Duality Group $G(\mathbb{Z})$	space-time dimension
1	10A
$SL(2,\mathbb{Z})$	10B
$SL(2,\mathbb{Z})$	9
$SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$	8
$SL(5,\mathbb{Z})$	7
$SO(5,5,\mathbb{Z})$	6
$E_{6(6)}(\mathbb{Z})$	5
$E_{7(7)}(\mathbb{Z})$	4
$E_{8(8)}(\mathbb{Z})$	3

Automorphic functions for higher-rank groups ;

Langlands Eisenstein series' associated with maximal parabolic subgroups of G.

$$E^G_{s_1,\dots,s_r}$$

rank r $s_1,\ .\ .\ .\ ,\ s_r\in\mathbb{C}$

labels associated with nodes of Dynkin diagram

For D=3, 4, 5 $E^G_{\frac{3}{2},0,\dots,0}(\{\mu_D\})\,R^4$ $E^G_{\frac{5}{2},0,\dots,0}(\{\mu_D\})\,D^4R^4$

HIGHER-RANK DUALITY GROUPS

Compactify M-theory on a n-torus to D=10-n dimensions

MBG, Miller, Russo, Vanhove

Pioline

Duality Group $G(\mathbb{Z})$	space-time dimension
$\mathbf{1}$	10A
$SL(2,\mathbb{Z})$	10B
$SL(2,\mathbb{Z})$	9
$SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$	8
$SL(5,\mathbb{Z})$	7
$SO(5,5,\mathbb{Z})$	6
$E_{6(6)}(\mathbb{Z})$	5
$E_{7(7)}(\mathbb{Z})$	4
$E_{8(8)}(\mathbb{Z})$	3

Automorphic functions for higher-rank groups ;

Langlands Eisenstein series' associated with maximal parabolic subgroups of G.

$$E^G_{s_1,\dots,s_r}$$

rank r $\quad s_1,\,.\,.\,.\,,\,s_r\in\mathbb{C}$

labels associated with nodes of Dynkin diagram

For D=3, 4, 5 $\quad E^G_{\frac{3}{2},0,\dots,0}(\{\mu_D\})\,R^4 \qquad E^G_{\frac{5}{2},0,\dots,0}(\{\mu_D\})\,D^4R^4$

$$\mathcal{F}^G_{\frac{1}{8}}(\{\mu_D\})\,D^6R^4$$

Satisfies inhomogeneous Laplace equation for G

HIGHER-RANK DUALITY GROUPS

Compactify M-theory on a n-torus to D=10-n dimensions

MBG, Miller, Russo, Vanhove

Pioline

Duality Group $G(\mathbb{Z})$	space-time dimension
1	10A
$SL(2,\mathbb{Z})$	10B
$SL(2,\mathbb{Z})$	9
$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	8
$SL(5,\mathbb{Z})$	7
$SO(5,5,\mathbb{Z})$	6
$E_{6(6)}(\mathbb{Z})$	5
$E_{7(7)}(\mathbb{Z})$	4
$E_{8(8)}(\mathbb{Z})$	3

Automorphic functions for higher-rank groups ;

Langlands Eisenstein series' associated with maximal parabolic subgroups of G.

$$E^G_{s_1,\dots,s_r}$$

rank r $\quad s_1, \dots, s_r \in \mathbb{C}$

labels associated with nodes of Dynkin diagram

For D=3, 4, 5 $\quad E^G_{\frac{3}{2},0,\dots,0}(\{\mu_D\})\,R^4 \quad E^G_{\frac{5}{2},0,\dots,0}(\{\mu_D\})\,D^4R^4$

$\mathcal{F}^G_{\frac{1}{8}}(\{\mu_D\})\,D^6R^4$ Satisfies inhomogeneous Laplace equation for G

- Encodes perturbative string results in compactified theories.

HIGHER-RANK DUALITY GROUPS

Compactify M-theory on a n-torus to D=10-n dimensions

MBG, Miller, Russo, Vanhove

Pioline

Duality Group $G(\mathbb{Z})$	space-time dimension
1	10A
$SL(2,\mathbb{Z})$	10B
$SL(2,\mathbb{Z})$	9
$SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$	8
$SL(5,\mathbb{Z})$	7
$SO(5,5,\mathbb{Z})$	6
$E_{6(6)}(\mathbb{Z})$	5
$E_{7(7)}(\mathbb{Z})$	4
$E_{8(8)}(\mathbb{Z})$	3

Automorphic functions for higher-rank groups ;

Langlands Eisenstein series' associated with maximal parabolic subgroups of G.

$$E^G_{s_1,\dots,s_r}$$

rank r $\quad s_1,\ . \ . \ . \ ,\ s_r \in \mathbb{C}$

labels associated with nodes of Dynkin diagram

For D=3, 4, 5 $\quad E^G_{\frac{3}{2},0,\dots,0}(\{\mu_D\})\,R^4 \quad E^G_{\frac{5}{2},0,\dots,0}(\{\mu_D\})\,D^4R^4$

$$\mathcal{F}^G_{\frac{1}{8}}(\{\mu_D\})\,D^6R^4$$

Satisfies inhomogeneous Laplace equation for G

- Encodes perturbative string results in compactified theories.
- D-INSTANTONS fill out expected fractional BPS orbits – minimal, next-to-minimal, ...

COMMENTS:

COMMENTS:

- Intriguing interrelations between properties of perturbative and non-perturbative string theory and other areas of mathematics (multipolylogarithms , elliptic multipolylogarithms, motivic multi-zeta values, ……)

COMMENTS:

- Intriguing interrelations between properties of perturbative and non-perturbative string theory and other areas of mathematics (multipolylogarithms , elliptic multipolylogarithms, motivic multi-zeta values, ……)

- Can we go to higher orders in the low energy expansion? To what extent do supersymmetry, duality and unitarity determiine the low energy expansion.

COMMENTS:

- Intriguing interrelations between properties of perturbative and non-perturbative string theory and other areas of mathematics (multipolylogarithms , elliptic multipolylogarithms, motivic multi-zeta values, ……)

- Can we go to higher orders in the low energy expansion? To what extent do supersymmetry, duality and unitarity determiine the low energy expansion.

- String theory:
  (i) is free of ultraviolet divergences at all orders;
  (ii) all supergravity field theory Feynman diagrams are packaged into a single world-sheet diagram.

COMMENTS:

- Intriguing interrelations between properties of perturbative and non-perturbative string theory and other areas of mathematics (multipolylogarithms , elliptic multipolylogarithms, motivic multi-zeta values, ……)

- Can we go to higher orders in the low energy expansion? To what extent do supersymmetry, duality and unitarity determiine the low energy expansion.

- String theory:
  (i) is free of ultraviolet divergences at all orders;
  (ii) all supergravity field theory Feynman diagrams are packaged into a single world-sheet diagram.

FANTASY: COULD ONE DETERMINE PROPERTIES OF SUPERGRAVITY FEYNMAN DIAGRAMS BY SUITABLE LIMIT OF STRING THEORY DIAGRAMS