Workshop on Megathrust Earthquakes and Tsunami

Prof. Thorne Lay

Begin at the beginning; basic elasticity

Develop the basic equations for P and S waves.

CONCEPTS:

Continuum
Stress Tensor
Equation of Motion
Strain Tensor

Supplementary Reading (Optional, for more details/rigor)
Lay and Wallace, *Modern Global Seismology*, Ch. 2
Stein and Wysession, *An Introduction to Seismology, Earthquakes and Earth Structure*, Ch. 2

The overview talk was built on the idea that tectonic processes, like faulting, can produce observable signals (ground shaking from elastic waves) useful to study the tectonics.

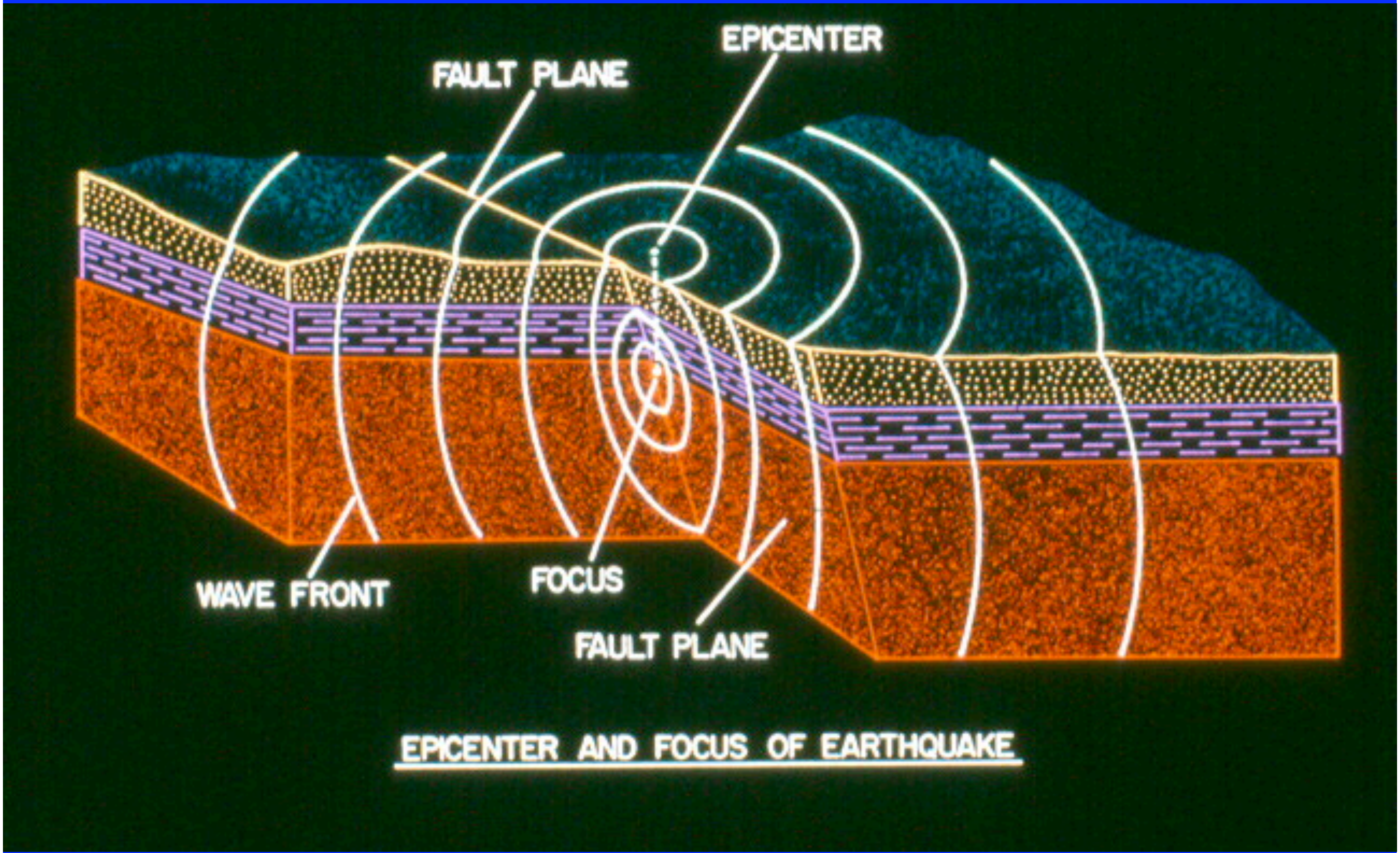
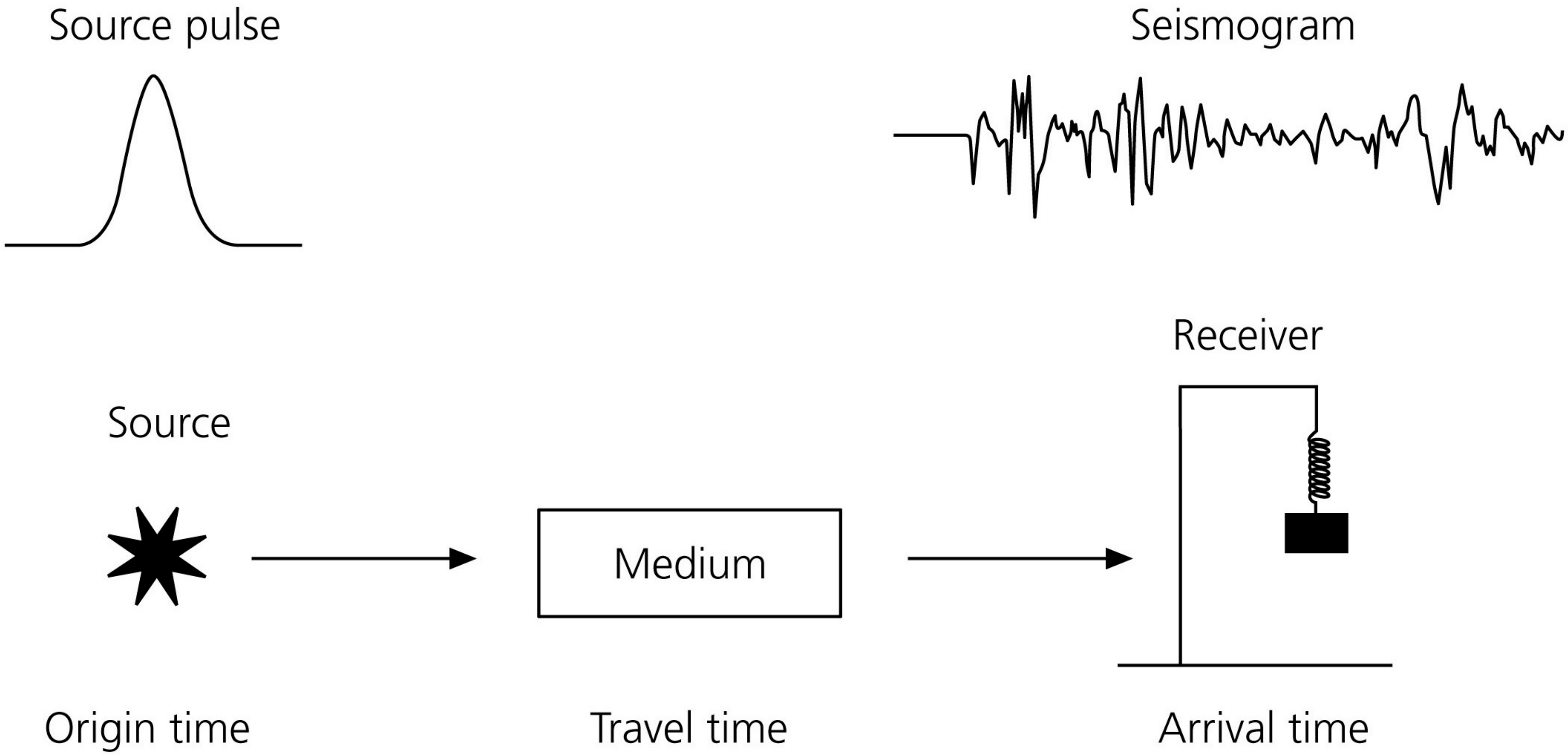


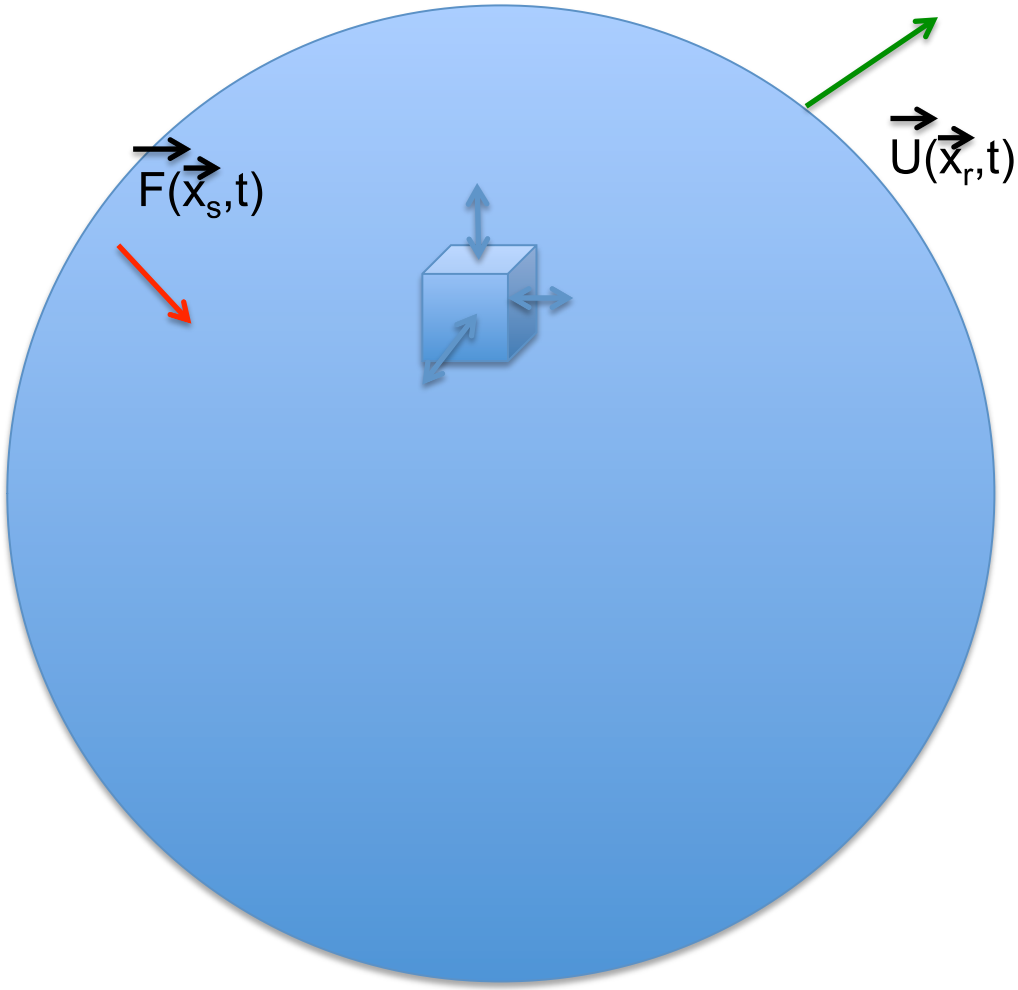
Figure 1.1-1: Schematic geometry of a seismic experiment.



There is a release of energy beginning at some time, lasting for some interval. We can think of this as pulse of energy input into the medium.

The energy release applies forces to the surrounding rock that cause accelerations (F=ma). P and S wave motions expand through the medium. It takes time to reach a location.

The P and S waves arrive at a location at times given by the distance and wave velocities, and the ground experiences accelerations. These can be recorded to give a seismogram; a record of ground motion as a function of time.



A force, associated with some form of energy release at position x_s, in or on a body will produce a motion at position x_r, somehow.

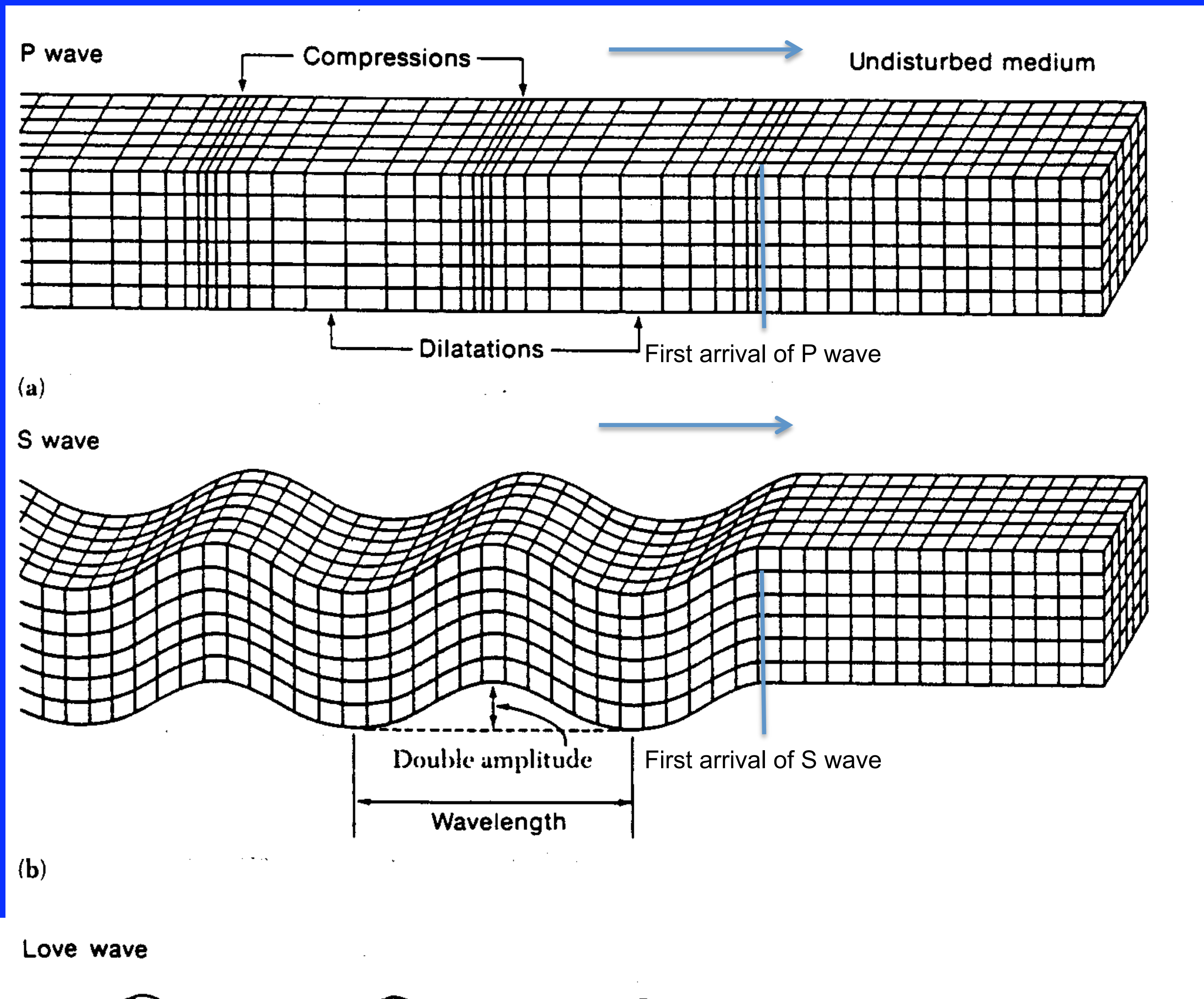
If the body is perfectly rigid, the motion will be an instantaneous translation, rotation or both. Use F=ma.

If the body is elastic, we still use F=ma, but the action of the force spreads through the body causing internal deformations that vary in space and time. Do describe these we need general representations of internal forces and deformations.

We use stress and strain.

$\vec{U}(\vec{x}_r,t)$ is vector ground displacement at position $\vec{x}_r$, defined in some coordinate system, as a function of time. This is what we can record.

An elastic medium will have only two types of waves, along with near-source permanent deformations if the forces at the source have permanent values.



We imagine the medium is made of continuous matter (ignore atomic nature). This is called a continuum. Mathematical representations of displacement, velocity, acceleration, stress, etc. are then continuous functions over three-dimensional space and time. We can compute their space and time derivatives and have continuous functions.

Continuum Mechanics:
Force --> per unit volume
Mass --> per unit volume

$$\mathbf{F} = m\mathbf{a}$$

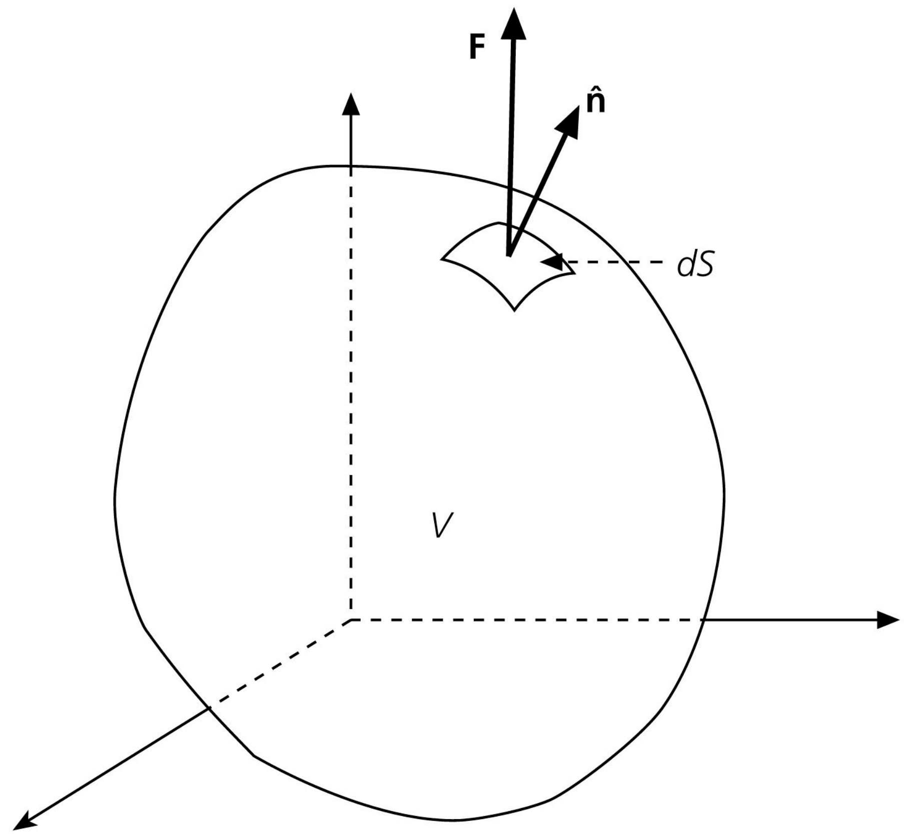
As vector:

$$\mathbf{f}(\mathbf{x}, t) = \rho \frac{\partial^2 \mathbf{u}(\mathbf{x}, t)}{\partial t^2}$$

As components:

$$f_i(\mathbf{x}, t) = \rho \frac{\partial^2 u_i(\mathbf{x}, t)}{\partial t^2} = \rho \ddot{u}_i$$

Figure 2.3-1: Surface force on a volume element.



Two classes of forces:

Body forces (e.g., gravity)

Surface forces (e.g., pressure underwater, stresses)

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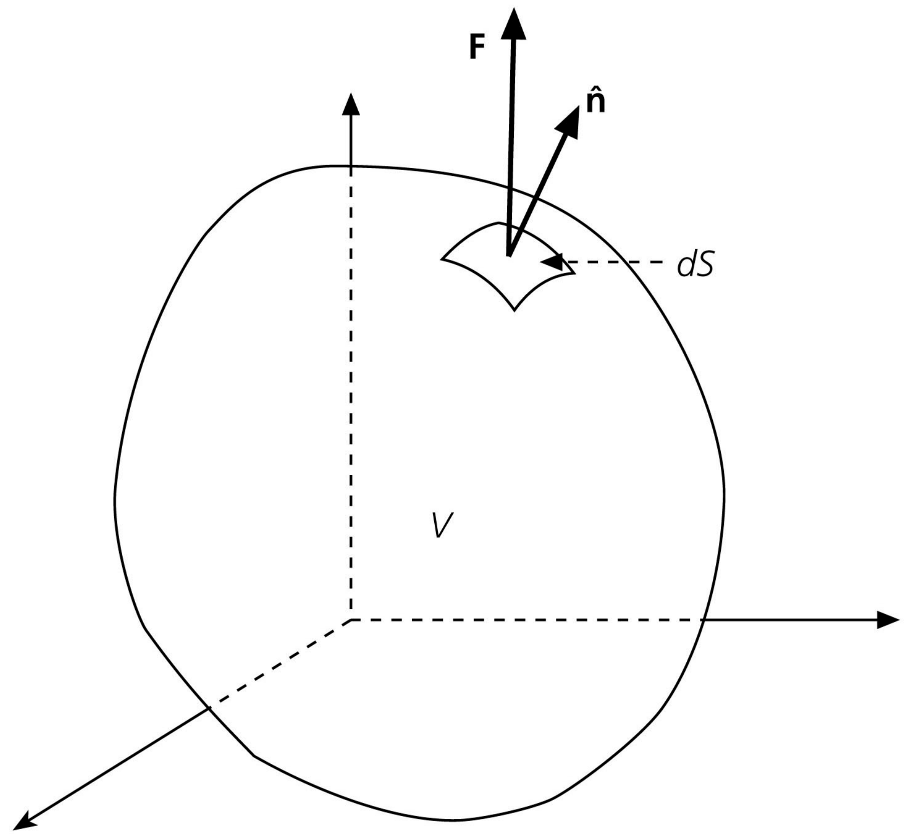

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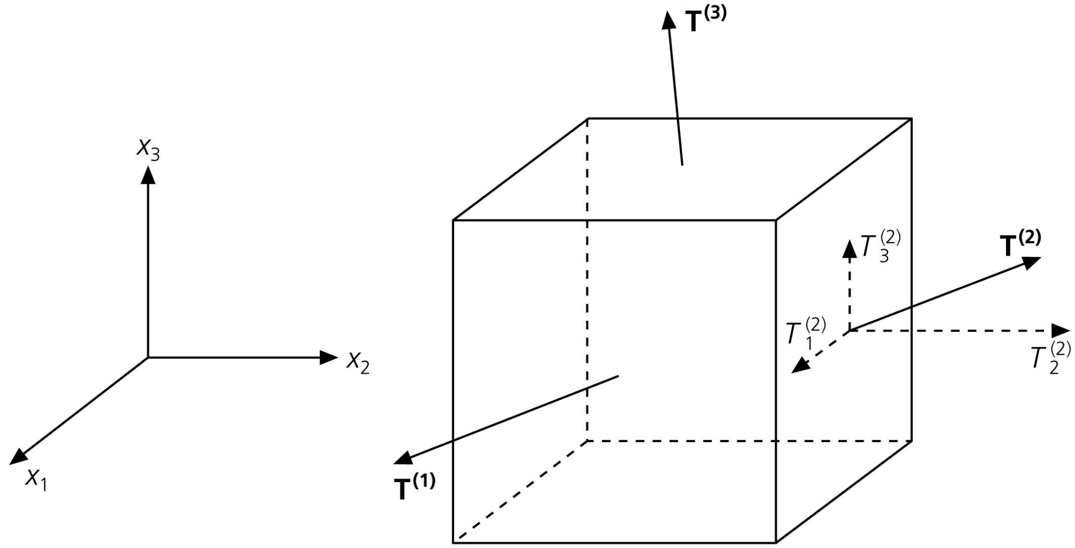
Body forces (e.g., gravity)

Surface forces (e.g., pressure underwater, stresses)

$$\mathbf{T}(\hat{\mathbf{n}}) = \lim_{dS \to 0} \frac{\mathbf{F}}{dS}$$

The traction vector has the same orientation as the force, and is a function of the unit normal vector $\hat{\mathbf{n}}$

Figure 2.3-2: Traction vectors on the faces of a volume element.



$\mathbf{T}^{(j)}$ is the traction vector acting on the surface whose outward normal is in the positive $\hat{\mathbf{e}}_j$ direction.

The components of the three traction vectors are $T_i^{(j)}$, where the upper index (j) indicates the surface and the lower (i) index indicates the component.

Example, $T_3^{(1)}$ is the x_3 component of the traction on the surface whose normal is $\hat{\mathbf{e}}_1$.

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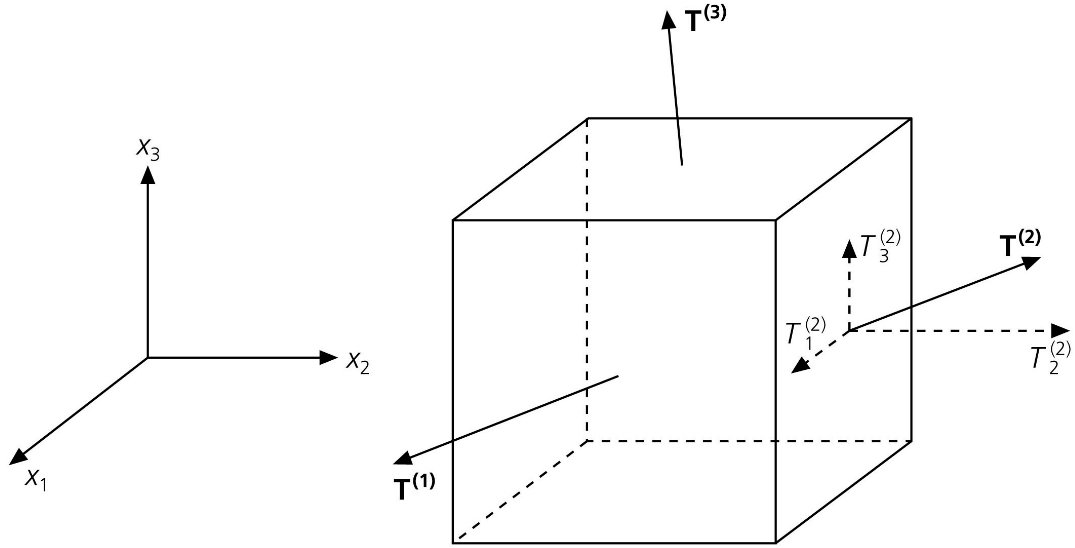

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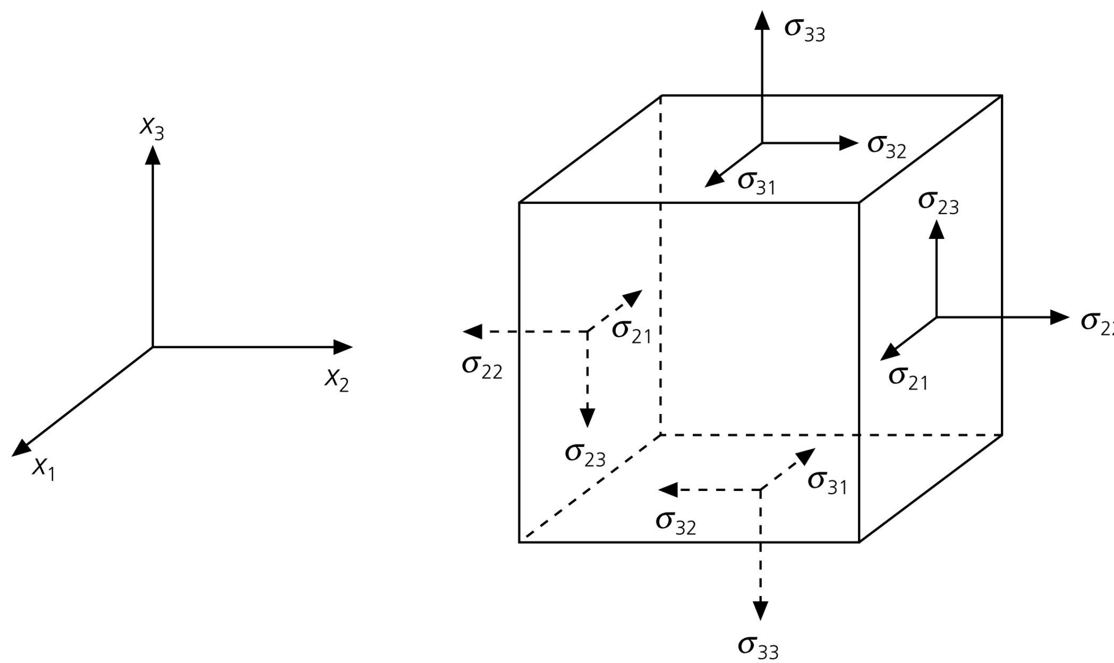
Stress tensor, σ_{ji}:

$$\sigma_{ji} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{T}^{(1)} \\ \mathbf{T}^{(2)} \\ \mathbf{T}^{(3)} \end{pmatrix} = \begin{pmatrix} T_1^{(1)} & T_2^{(1)} & T_3^{(1)} \\ T_1^{(2)} & T_2^{(2)} & T_3^{(2)} \\ T_1^{(3)} & T_2^{(3)} & T_3^{(3)} \end{pmatrix}$$

The tensor's rows are the three traction vectors.

The stress is the force per unit area that the material on the outside of the surface (the side to which $\hat{\mathbf{n}}$ points) exerts on the material inside.

Figure 2.3-4: Stress components on the faces of a volume element.



Normal stresses: σ_{11}, σ_{22}, σ_{33}

Shear stresses: σ_{12}, σ_{13}, σ_{21}, σ_{23}, σ_{31}, σ_{32}

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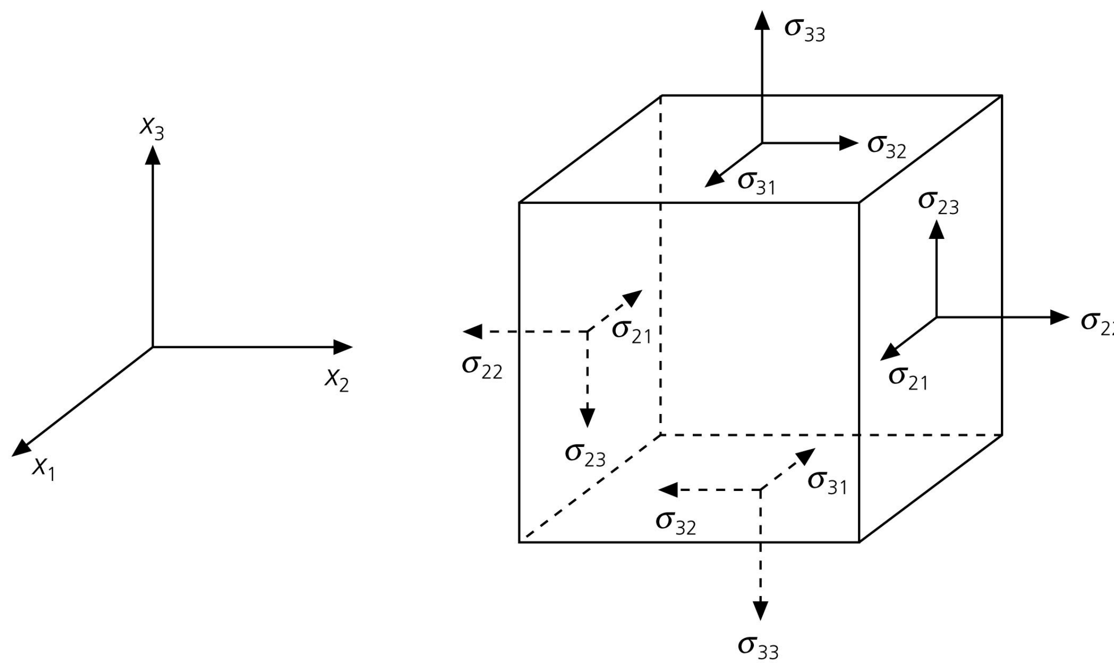


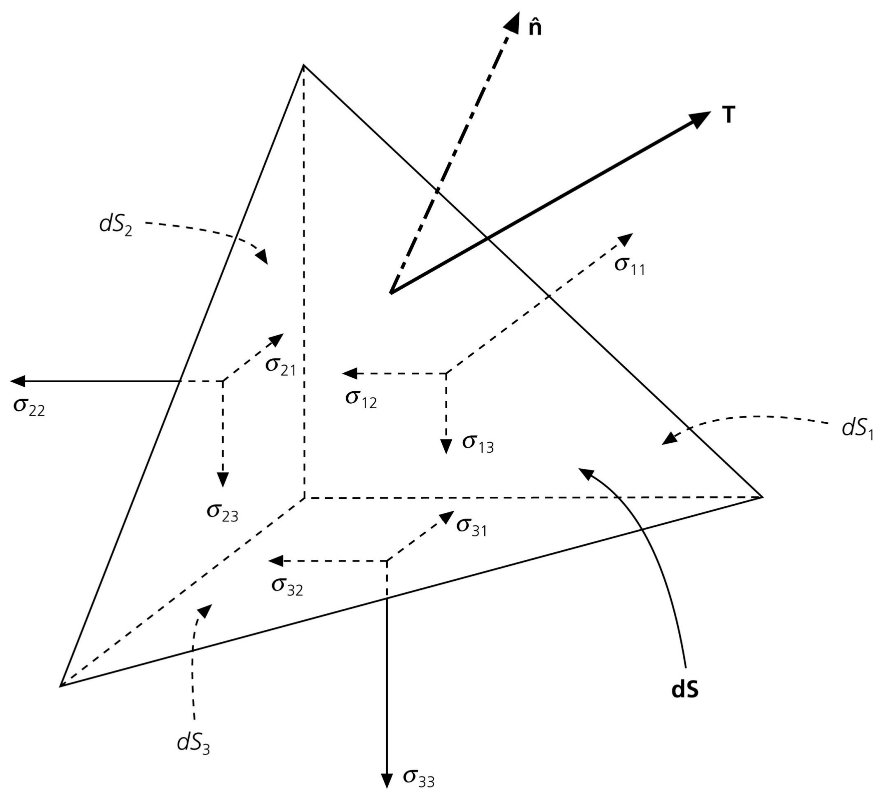
Normal stresses: σ_{11}, σ_{22}, σ_{33}

Shear stresses: σ_{12}, σ_{13}, σ_{21}, σ_{23}, σ_{31}, σ_{32}

Normal stresses are positive outward, and expand the volume.
---> *Tension*

Normal stresses are negative inward, and contract the volume.
---> *Compression*
(Holds true for Earth's interior)

Figure 2.3-3: Stress components on the faces of a tetrahedron.



The stress tensor gives the traction vector **T** acting on any surface within the medium.

Example/ The traction on an arbitrary element of surface dS, whose normal $\hat{\mathbf{n}}$ is not along a coordinate axis, is found by multiplying each component of the traction by the area of the face it acts on and summing over the faces.

$$T_i = \sigma_{1i}n_1 + \sigma_{2i}n_2 + \sigma_{3i}n_3 = \sum_{j=1}^{3} \sigma_{ji}n_j = \sigma_{ji}n_j$$

Index notation:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i$$

Einstein summation convention:

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i$$

$$|\mathbf{u}|^2 = u_i u_i .$$

The stress tensor is *symmetric:*

$\sigma_{ij} = \sigma_{ji}$

Otherwise: torques --> rotations!

Allows us to define tractions as

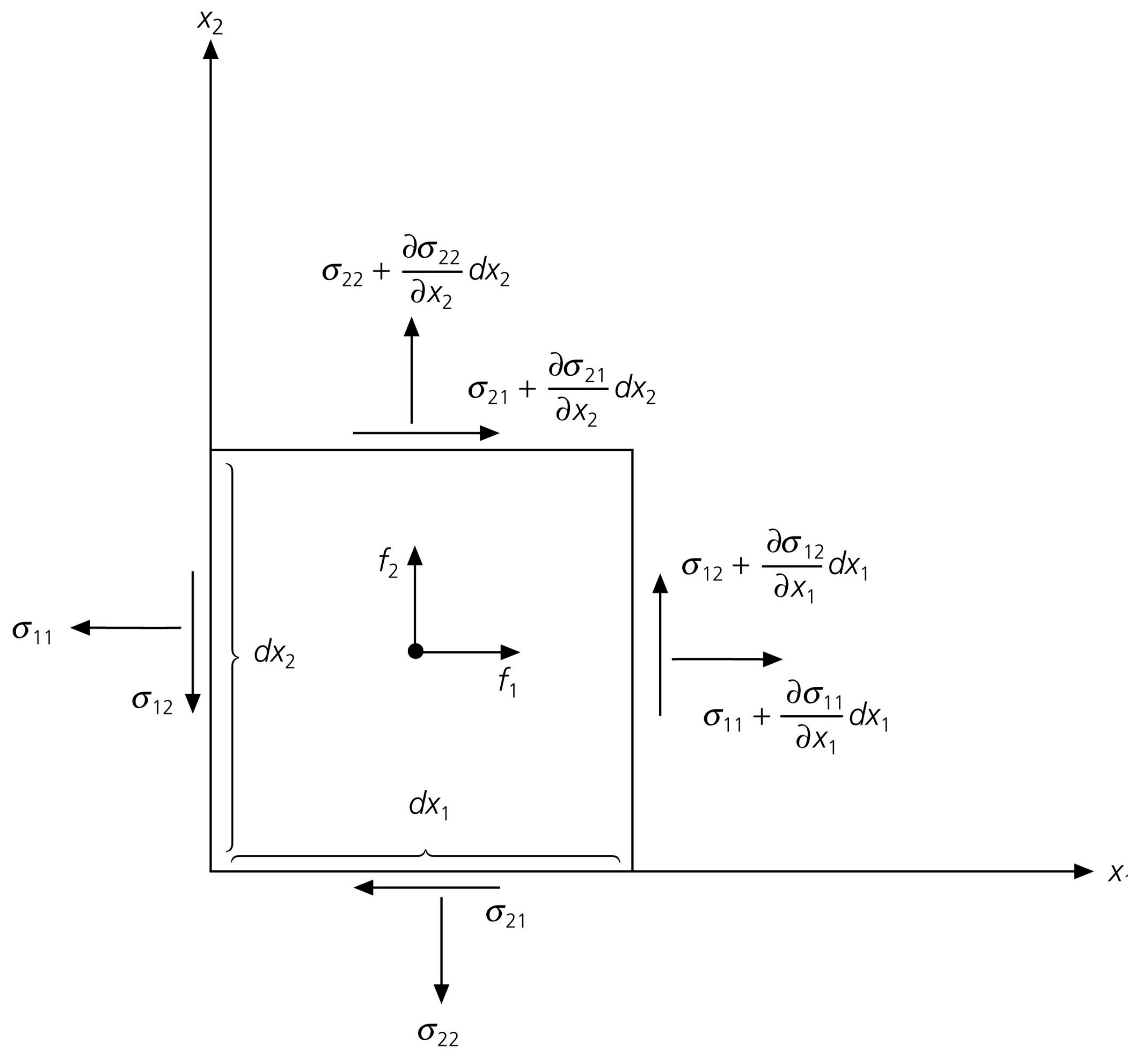
$$T_i = \sum_{j=1}^{3} \sigma_{ij} n_j = \sigma_{ij} n_j$$

(as components)

$$\mathbf{T} = \sigma \hat{\mathbf{n}}$$

(as vectors)

Figure 2.3-5: Torques on a rectangle.



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Stress = force/area

In cgs: 1 bar = 10^6 dyn/cm^2

1 atm = 1.01 bars

In SI: 1 Pascal (Pa) = 1 N/m^2 = 10^{-5} bars

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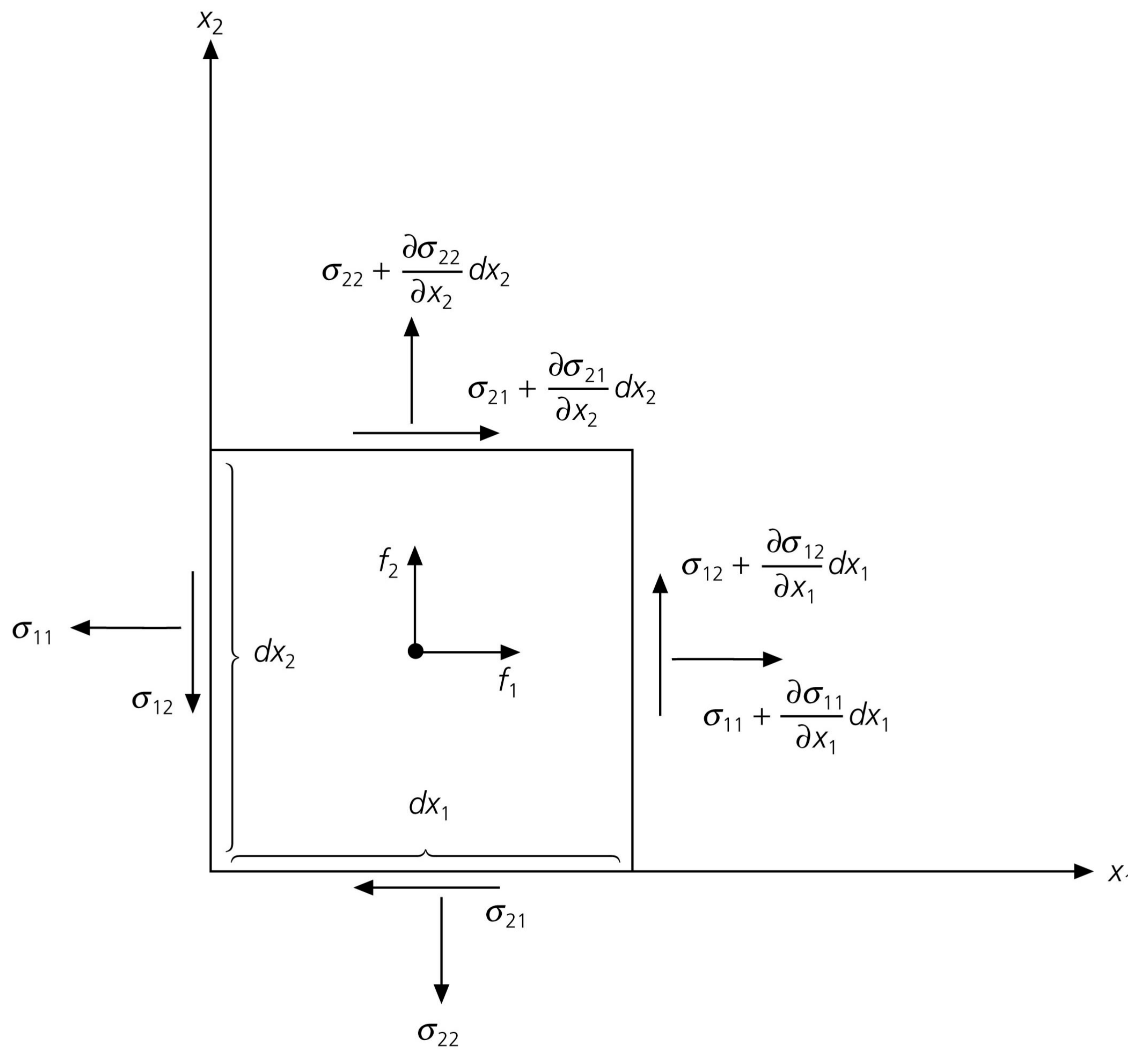

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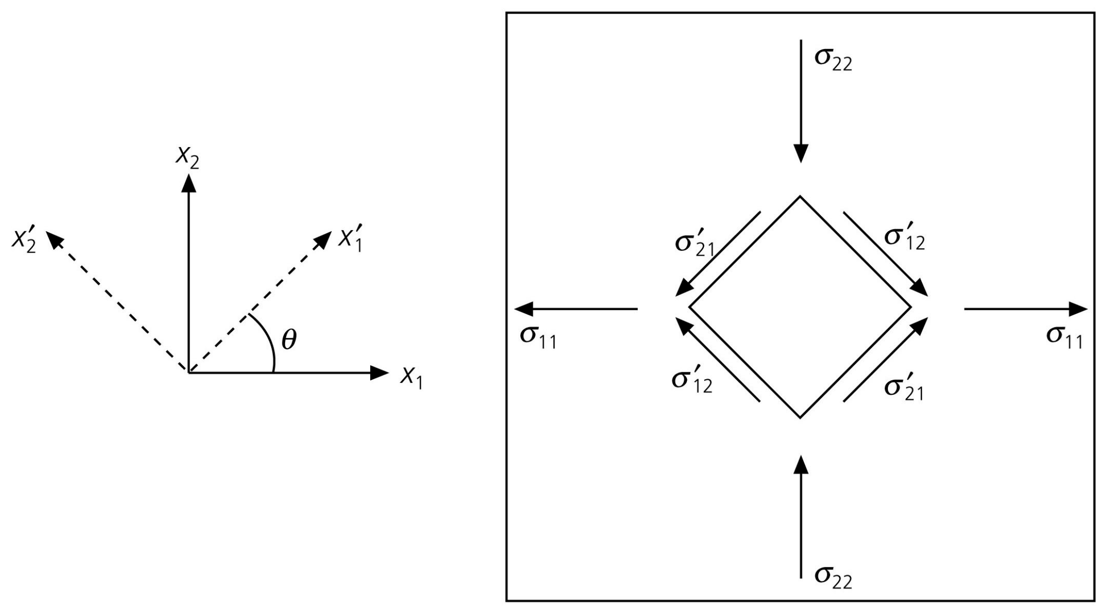
Similarly, σ is a *tensor* (and not just a matrix) because it transforms between coordinates according to:

$$\sigma' = A\sigma A^T$$

Example: A block of material with faces perpendicular
to the x_1 and x_2 axes is subject only to normal stresses σ_1 and σ_2, so the stress tensor is diagonal:

$$\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Figure 2.3-6: Different stress components in different coordinate systems.



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$$\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now consider a different block in the SAME PHYSICAL SITUATION, but with rotated sides:

$$\sigma' = A\sigma A^T$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sigma_1\cos^2\theta + \sigma_2\sin^2\theta & (\sigma_2-\sigma_1)\sin\theta\cos\theta & 0 \\ (\sigma_2-\sigma_1)\sin\theta\cos\theta & \sigma_1\sin^2\theta + \sigma_2\cos^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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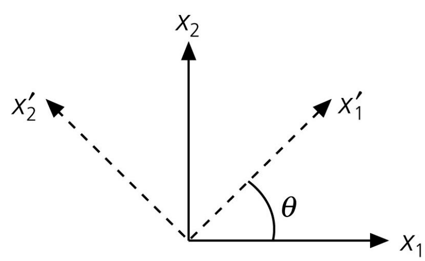

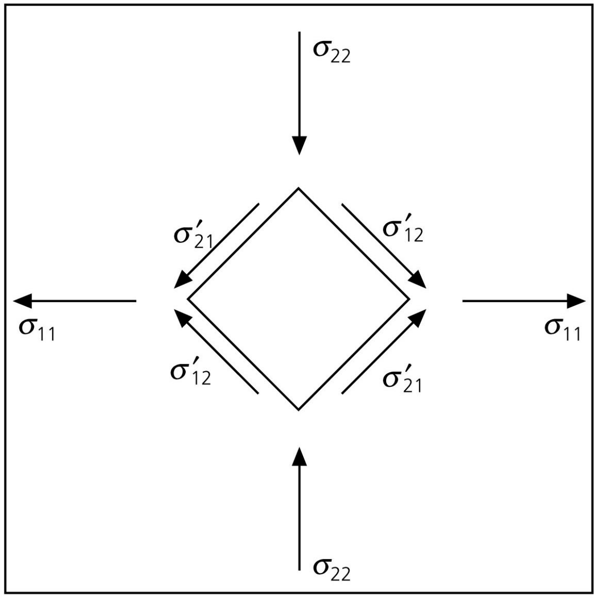

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For example, if $\sigma_1 = 1$, $\sigma_2 = -1$, and $\theta = 45°$,

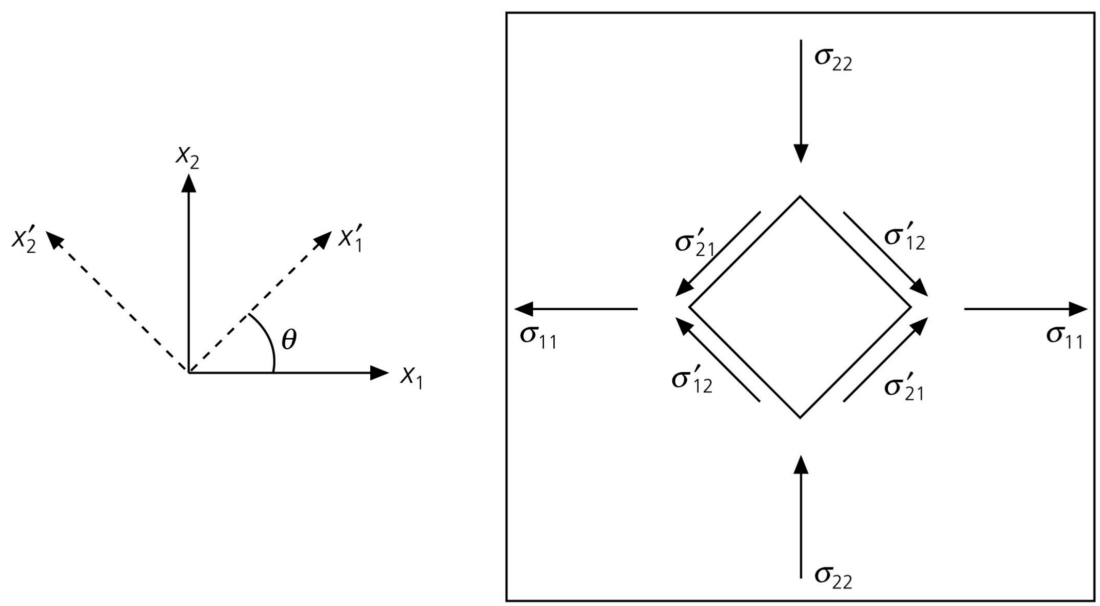
$$\sigma' = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

First block: only normal stresses.

Second block: only shear stresses.

Same state of stress!!
(but different coordinate axes!)

Figure 2.3-6: Different stress components in different coordinate systems.



For any state of stress, a set of coordinate axes can be found that provides only normal stresses (and no shear stresses!).

These axes are called the *principal stress axes* and the normal stresses on these surfaces are called *principal stresses.*

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To find the principal stresses, we use the concepts of eigenvalues and eigenvectors.

The shear components of the traction will be zero if the traction and normal vectors are parallel, such that they differ only by a multiplicative constant, λ,

$$T_i = \sigma_{ij} n_j = \lambda n_i$$

The principal stress axes $\mathbf{\hat{n}}$ are the eigenvectors of the stress tensor.

The principal stresses λ associated with each one are the eigenvalues.

Figure 2.3-9: Stress fields associated with three types of faulting.

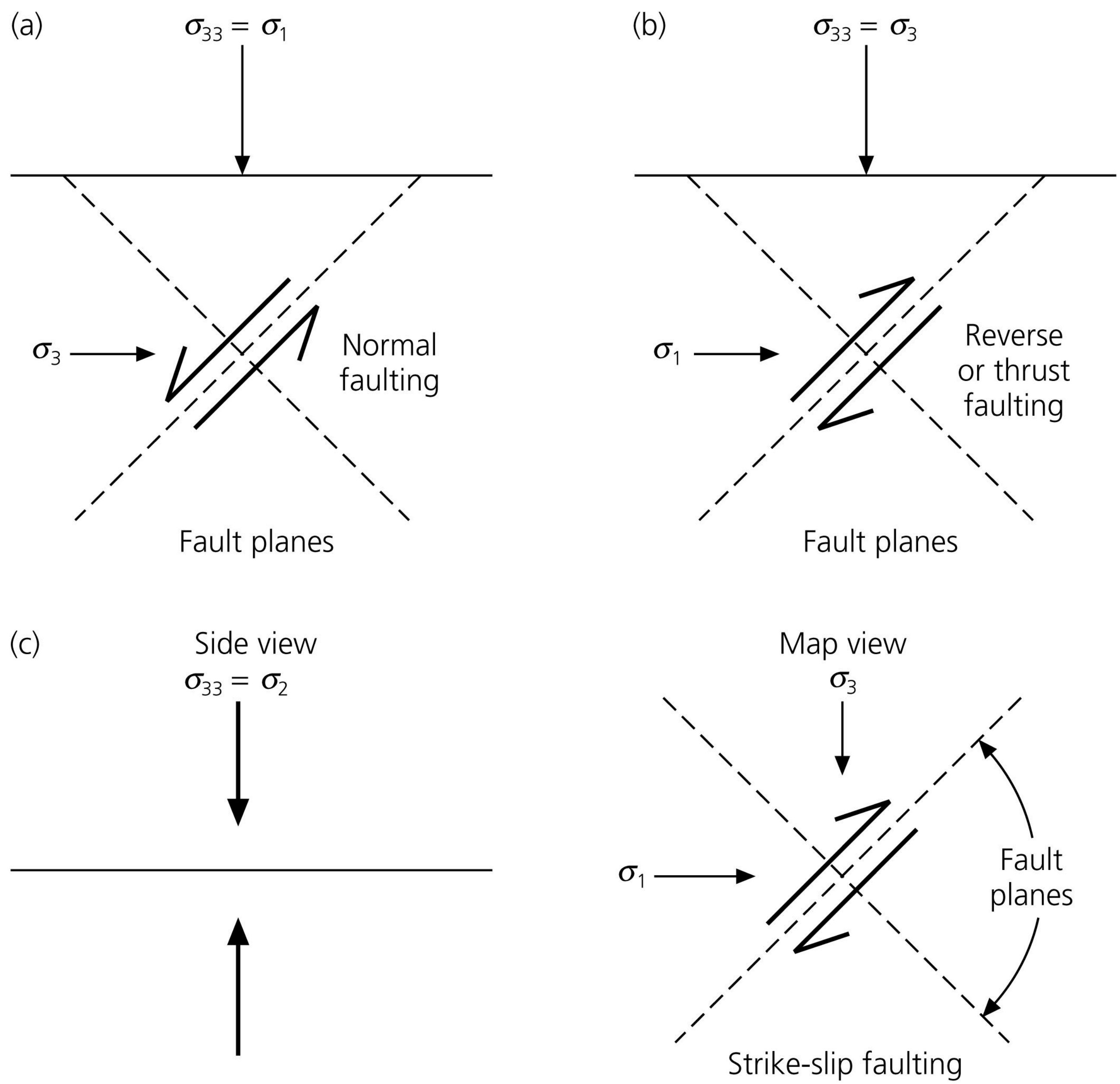
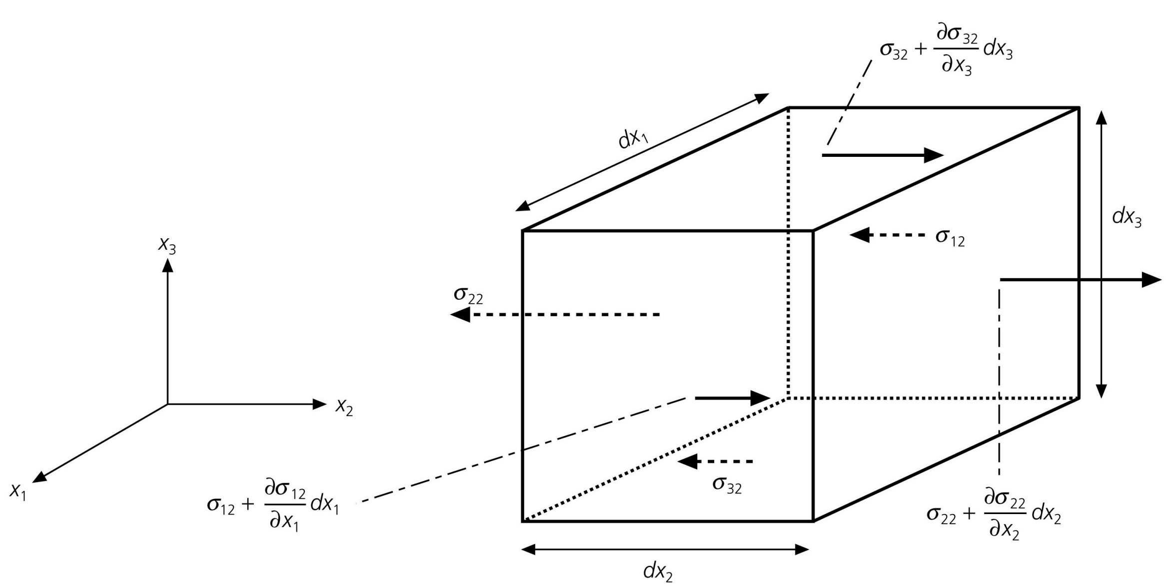


Figure 2.3-10: Stress components contributing to force in the x_2 direction.



Write $\mathbf{F} = m\mathbf{a}$ in terms of body and surface forces for each component.

For example, for the x_2 direction, the terms involving the $\hat{\mathbf{e}}_2$ and $-\hat{\mathbf{e}}_2$ faces (where the area of the faces are $dx_1 dx_3$,) are

$$\left[\sigma_{22}(\mathbf{x} + dx_2\hat{\mathbf{e}}_2) - \sigma_{22}(\mathbf{x})\right] dx_1 dx_3 = \left[\sigma_{22}(\mathbf{x}) + \frac{\partial\sigma_{22}(\mathbf{x})}{\partial x_2} dx_2 - \sigma_{22}(\mathbf{x})\right] dx_1 dx_3 = \frac{\partial\sigma_{22}(\mathbf{x})}{\partial x_2} dx_1 dx_2 dx_3$$

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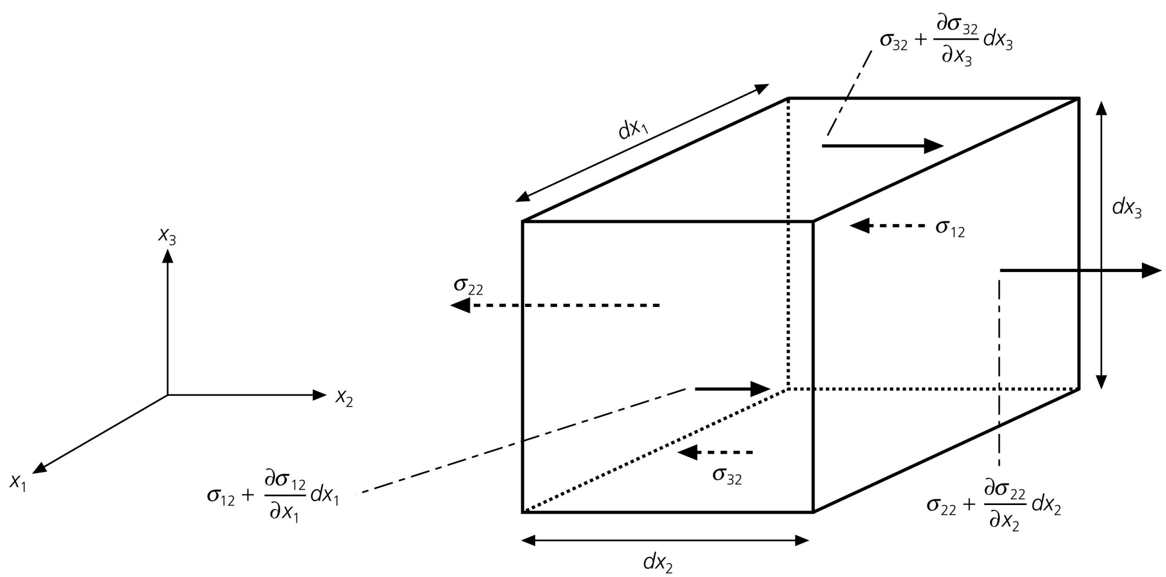

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(Similar for the force in the x_2 direction due to the pairs of faces with normals $\pm\hat{\mathbf{e}}_1$ and $\pm\hat{\mathbf{e}}_3$.)

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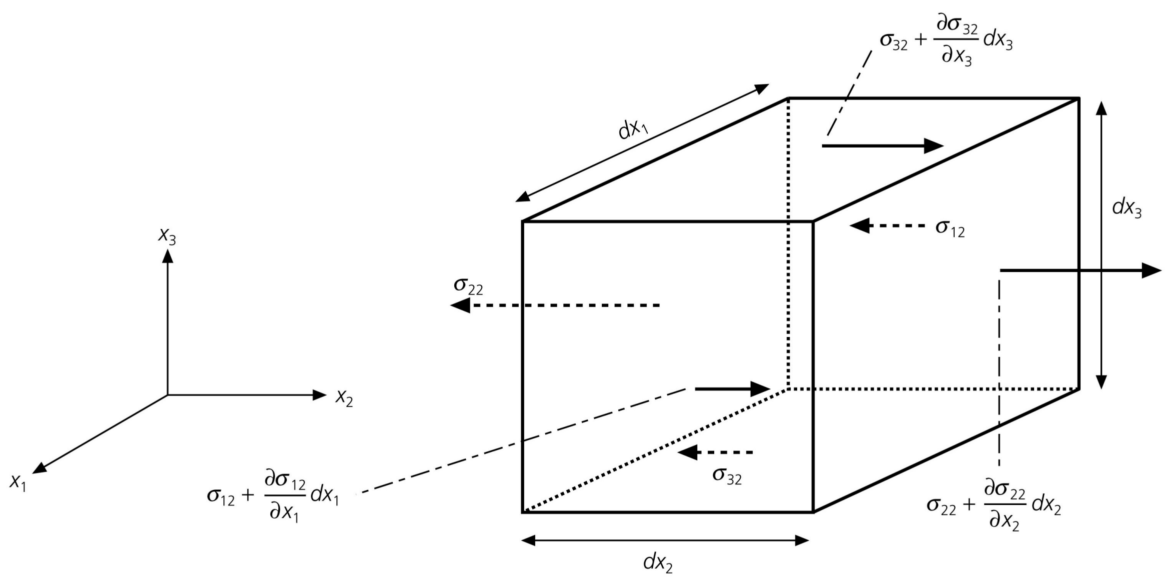

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(Similar for the force in the x_2 direction due to the pairs of faces with normals $\pm\hat{\mathbf{e}}_1$ and $\pm\hat{\mathbf{e}}_3$.)

Summing the three terms, adding the body force component, and equating this net force to the density times this component of the acceleration yields

$$\left[\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3}\right] dx_1 dx_2 dx_3 + f_2 \, dx_1 dx_2 dx_3 = \rho \frac{\partial^2 u_2}{\partial t^2} dx_1 dx_2 dx_3$$

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Similar equations apply for the x_1 and x_3 components of the force and acceleration:

$$\frac{\partial \sigma_{ji}(\mathbf{x}, t)}{\partial x_j} + f_i(\mathbf{x}, t) = \rho \, \frac{\partial^2 u_i(\mathbf{x}, t)}{\partial t^2}$$

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$$\frac{\partial \sigma_{ij}(\mathbf{x}, t)}{\partial x_j} + f_i(\mathbf{x}, t) = \rho \frac{\partial^2 u_i(\mathbf{x}, t)}{\partial t^2}$$

$$\sigma_{ij,j}(\mathbf{x}, t) + f_i(\mathbf{x}, t) = \rho \ddot{u}_i(\mathbf{x}, t)$$

This is the *Equation of motion*, which applies everywhere in a continuous medium.

$$\frac{\partial \sigma_{ij}(\mathbf{x}, t)}{\partial x_j} + f_i(\mathbf{x}, t) = \rho \frac{\partial^2 u_i(\mathbf{x}, t)}{\partial t^2}$$

Equation of equilibrium:
(accelerations are zero, like a static problem such as stresses resulting only from gravity)

$$\sigma_{ij,j}(\mathbf{x}, t) = - f_i(\mathbf{x}, t)$$

$$\frac{\partial \sigma_{ij}(\mathbf{x}, t)}{\partial x_j} + f_i(\mathbf{x}, t) = \rho \frac{\partial^2 u_i(\mathbf{x}, t)}{\partial t^2}$$

Equation of equilibrium:
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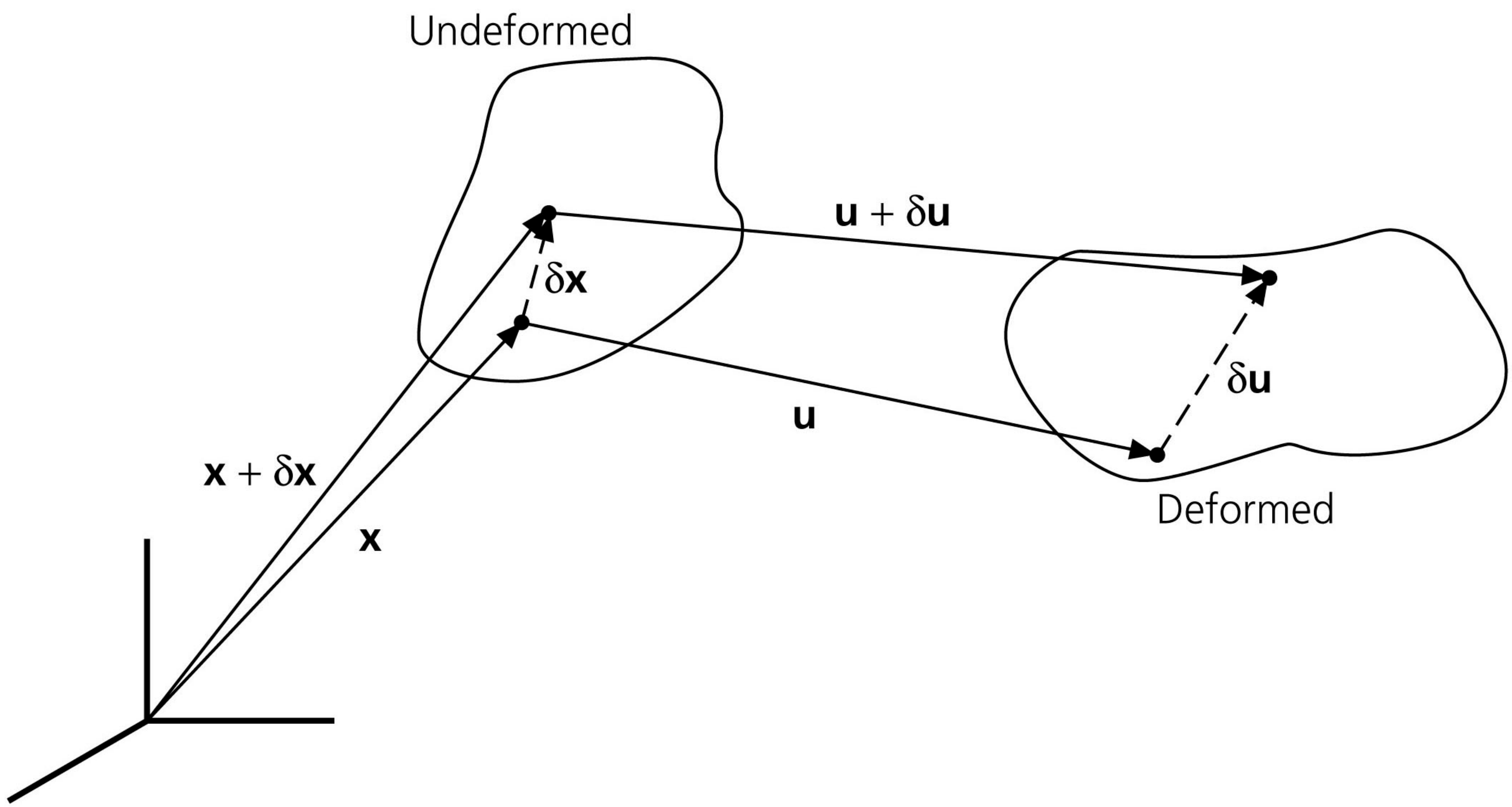
Homogeneous equation of motion:
(with no forces, such as the harmonic oscillation of wave propagation)

$$\sigma_{ij,j}(\mathbf{x}, t) = \rho \frac{\partial^2 u_i(\mathbf{x}, t)}{\partial t^2}$$

KEY Result: Spatial gradients of stress terms are balanced by inertial terms. This is how F=ma manifests in a continuum.

OK, so stresses allow us to express internal forces throughout the deformable (elastic medium). We also need to express the deformations that result. We Will assume small internal deformations (rock can only elastically deform by about 1 part in 10,000 without breaking). Assume INFINITESIMAL STRAIN THEORY.

Figure 2.3-11: Change in relative displacement during deformation.



The *strain tensor* describes the deformation resulting from the differential motion within a body.

$$u_i(\mathbf{x} + \delta\mathbf{x}) \approx u_i(\mathbf{x}) + \frac{\partial u_i(\mathbf{x})}{\partial x_j}\,\delta x_j = u_i(\mathbf{x}) + \delta u_i$$

$$\delta u_i = \frac{\partial u_i(\mathbf{x})}{\partial x_j}\,\delta x_j$$

This assumes the spatial gradient in displacements is a very Small term, approximated by first term of a Taylor Series expansion.

This is INFINITESIMAL STRAIN THEORY; it will be valid for Small deformation with no tearing or fracturing of the medium. It Will break down at a fault there there is a discontinuous displacement (derivative becomes undefined).

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$$\delta u_i = \frac{\partial u_i(\mathbf{x})}{\partial x_j} \delta x_j$$

Although we are interested in deformation that distorts the body, there can also be a rigid body translation or a rigid body rotation, neither of which produces deformation. To distinguish these effects, we add and subtract $\partial u_j / \partial x_i$ and then separate it into two parts

$$\delta u_i = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\delta x_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\delta x_j = (e_{ij} + \omega_{ij})\delta x_j$$

ω_{ij} corresponds to a rigid body rotation without deformation. It is antisymmetric ($\omega_{ij} = -\omega_{ji}$), so the diagonal terms are zero.

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ω_{ij} corresponds to a rigid body rotation without deformation. It is antisymmetric ($\omega_{ij} = -\omega_{ji}$), so the diagonal terms are zero.

Strain tensor:

$$e_{ij} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

Figure 2.3-12: Some possible strains for a two-dimensional element.

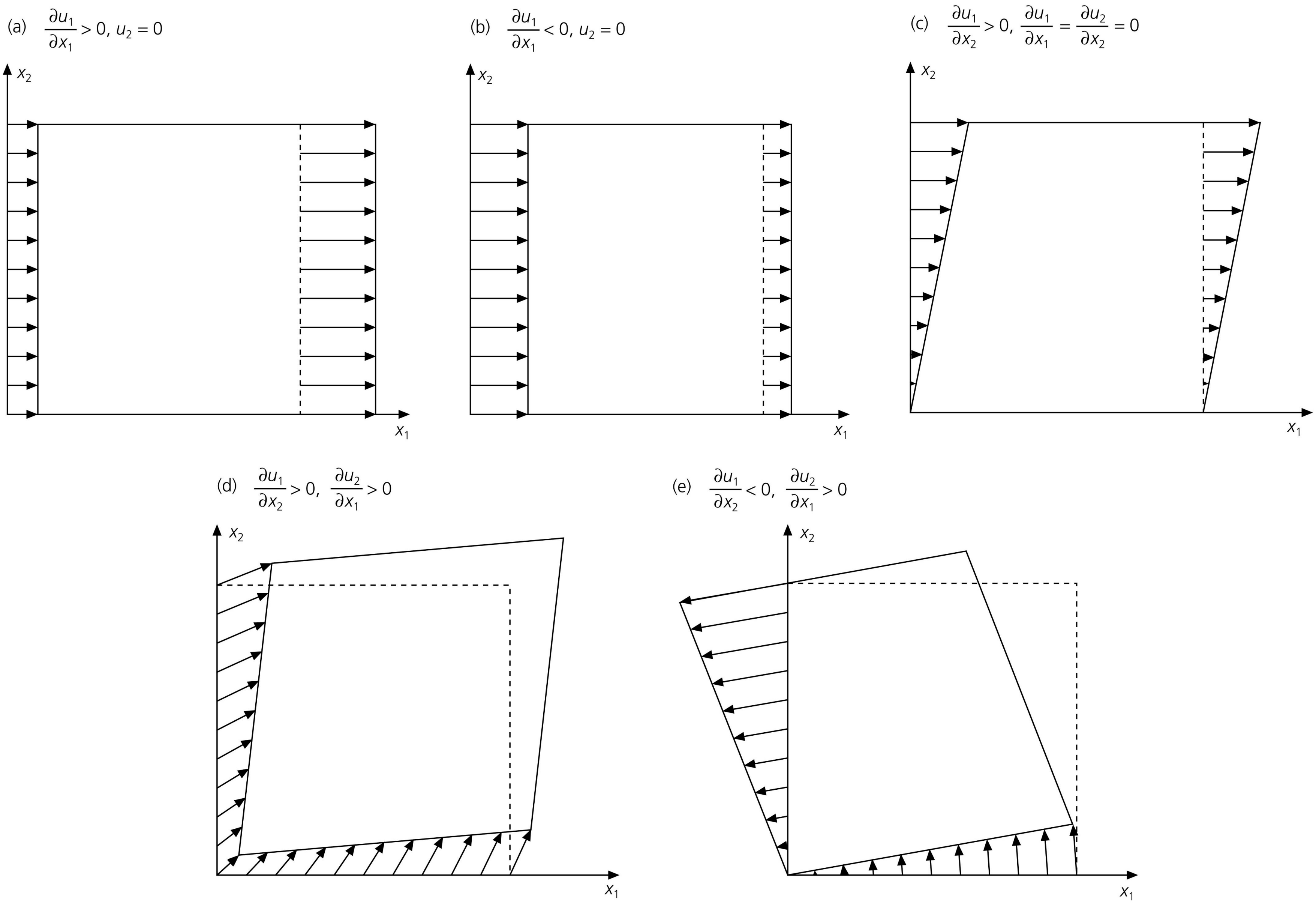
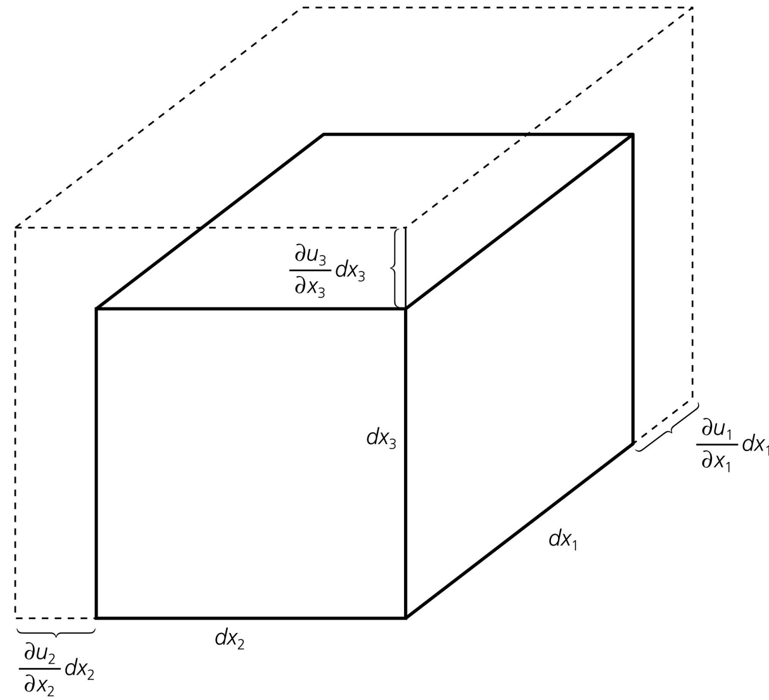


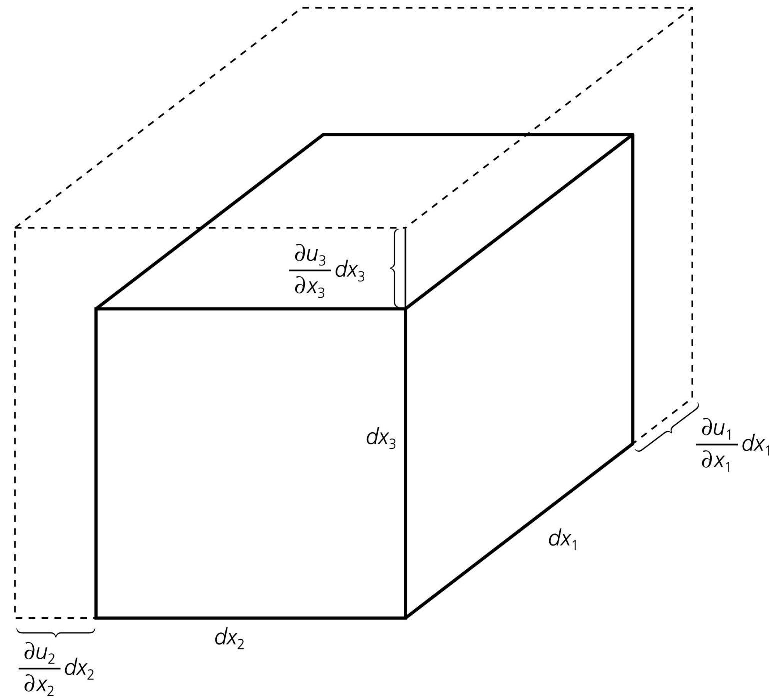
Figure 2.3-13: Change in volume due to principle strains.



The trace or sum of diagonal terms of the strain tensor is the *Dilatation*:
(gives the change in volume per unit volume associated with deformation)

$$\theta = e_{ii} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \nabla \cdot \mathbf{u}$$

Figure 2.3-13: Change in volume due to principle strains.



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$$\theta = e_{ii} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \boldsymbol{\nabla} \cdot \mathbf{u}$$

For initial volume $dx_1 dx_2 dx_3$ the volume after deformation is

$$\left(1 + \frac{\partial u_1}{\partial x_1}\right) dx_1 \left(1 + \frac{\partial u_2}{\partial x_2}\right) dx_2 \left(1 + \frac{\partial u_3}{\partial x_3}\right) dx_3 \approx \left(1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) dx_1 dx_2 dx_3 = (1 + \theta) dx_1 dx_2 dx_3$$

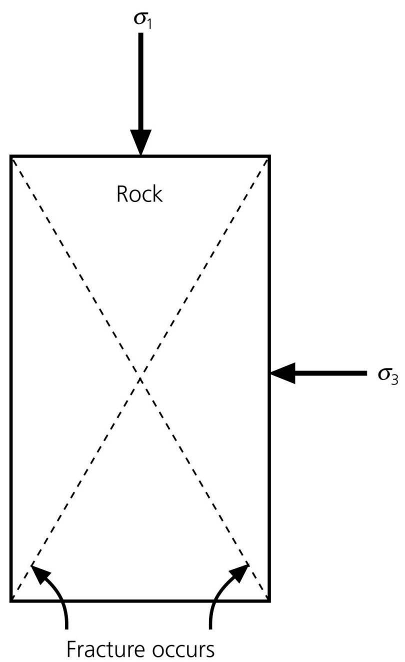
If the initial volume is $V = dx_1 dx_2 dx_3$, the final volume is $V + \Delta V = (1 + \theta)V$,
so $\theta = \Delta V / V$

OK, so now we have a general formulation of internal forces distributed through a medium, for which any choice of coordinate system gives us a specific stress tensor that describes the local state of stress (along with specific rules for how the stress tensor changes if we change the coordinate system). That gives us a local expression for F=ma; the elastic equation of motion, which specifies accelerations in terms of specific spatial gradients of the stress tensor. Then we have a general formulation of how deformations are expressed as linear spatial derivatives of displacements of points in the medium (for infinitesimal strain theory). We would like to relate stress and strain to change the equation of motion to an equation involving just spatial and temporal derivatives of displacement. So, what is the relationship between stress and strain? Next time!

Stress and strain tensors are general representations of internal forces and deformation within an elastic body. They are independent representations, but must be related (F=ma connects forces to accelerations, which are second time derivatives of displacements, and strains are spatial derivatives of displacements.

It would be ‘nice’ to have a first-principle’s theory for a relationship between stress and strain terms, but until very recently we have relied on experimental observation of such behavior for rocks.

Figure 2.3-8: Compression of a rock sample.

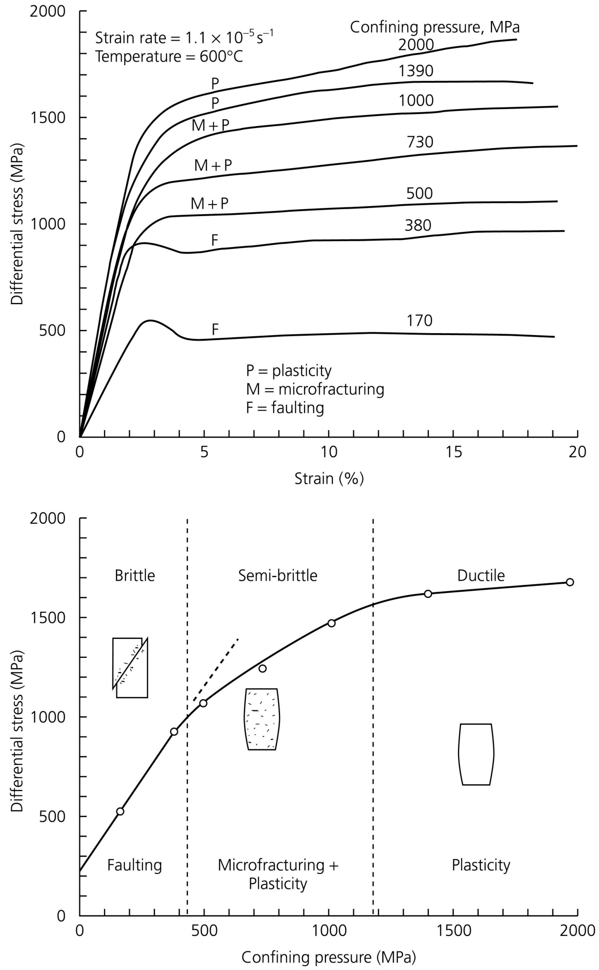


When compressed, all rock materials exhibit an interval of linear relationship between stress and strain that is totally recoverable (elastic).

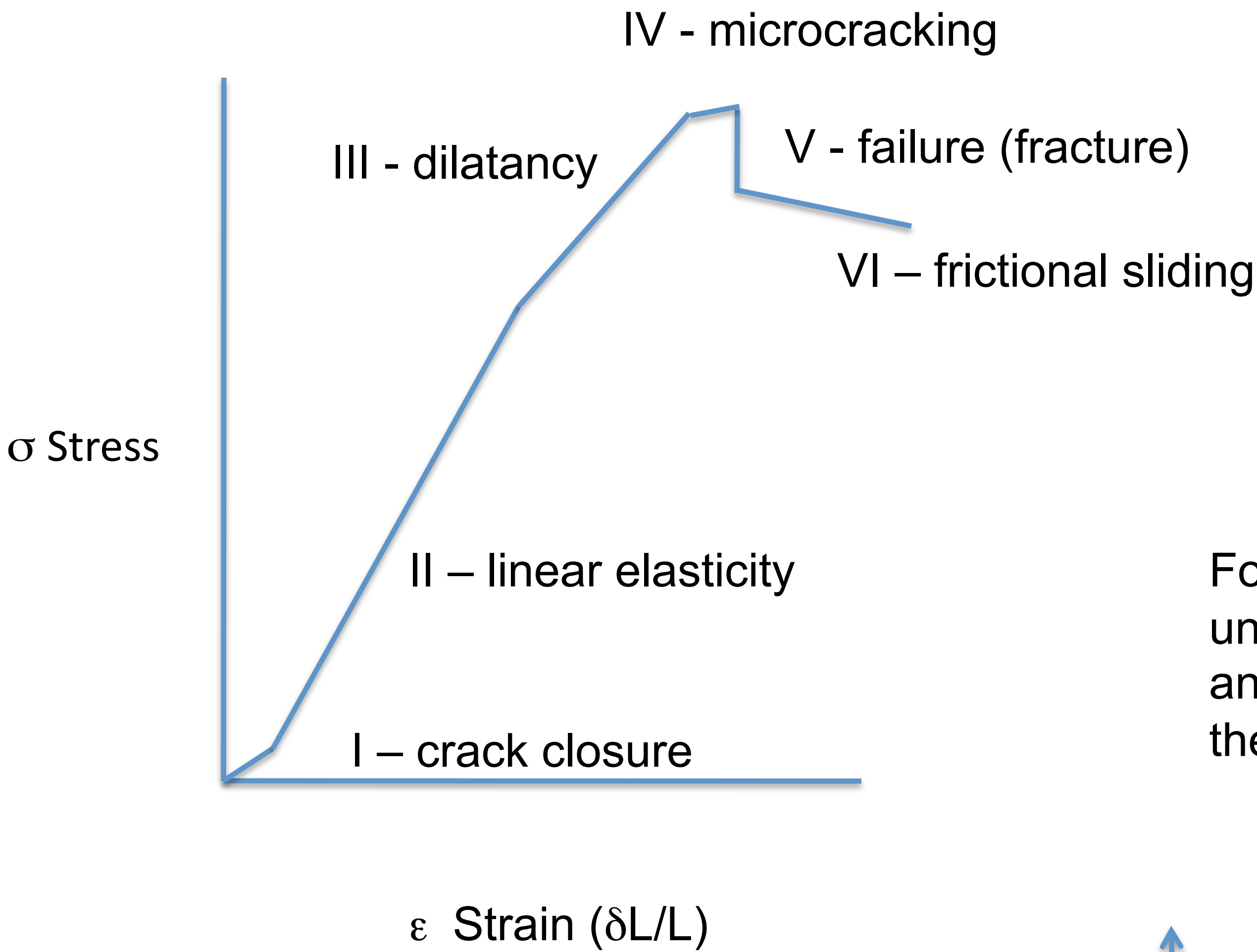
stress = constant x strain

For relatively low pressure (< 400 Mpa, T < 600°C), rocks will fracture as stress Increases to some limit.

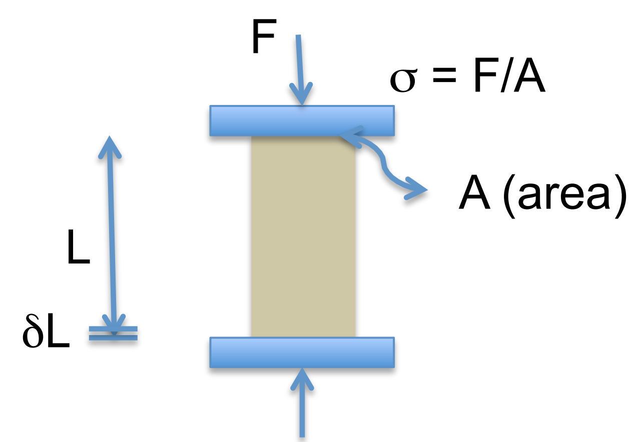
Figure 5.7-3: Rheology of rocks subjected to large compressive stresses.



Constitutive Law: $\varepsilon = f(\sigma)$



For an experiment under fixed pressure and temperature of the rock.



Constitutive Equation – Empirical relationship between stress and strain

Constitutive equations give the relation between stress and strain.

The simplest type of materials are *linearly elastic,* such that there is a linear relation between the stress and strain tensors.

Others could describe viscous (Newtonian and non-Newtonian), viscoelastic, elastic-plastic, etc.

Linearly elastic constitutive equations gives rise to seismic waves.

Linearly elastic material: The constitutive equation is *Hooke's law*:

$$\sigma_{ij} = c_{ijkl} e_{kl}$$

The constants c_{ijkl}, the *elastic moduli,* describe the properties of the material.

Because the subscripts each range from 1 to 3, c_{ijkl} has 3^4, or 81 components.

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The stress and strain tensors are symmetric

$$c_{ijkl} = c_{jikl} \qquad c_{ijkl} = c_{ijlk}$$

(brings the number of independent components to 36)

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(brings the number of independent components to 36)

A further symmetry relation based on the idea of strain energy gives:

$$c_{ijkl} = c_{klij}$$

(...down to 21)

21 independent components are needed to describe general anisotropy.

Isotropy: SOME Material behaves the same way regardless of orientation.

This reduces the number of independent c_{ijkl} to 2!!!!

SOME

Isotropy: ^ Material behaves the same way regardless of orientation.

This reduces the number of independent c_{ijkl} to 2!!!!

One useful pair are the *Lame′ constants* λ and μ:

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}$$

Example:
$\sigma_{11} = \lambda\theta + 2\mu e_{11}$ and $\sigma_{12} = 2\mu e_{12}$.

The *Kronecker delta,* δ_{ij}:

$\delta_{ij} = 0 \quad$ if $i \neq j$

$\quad = 1 \quad$ if $i = j$

So, for example, $\delta_{11} = 1$, but $\delta_{12} = 0$.

$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$.

SOME

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$\sigma_{11} = \lambda\theta + 2\mu e_{11}$ and $\sigma_{12} = 2\mu e_{12}$.

Problem: λ has no physical meaning

More useful: μ and K

The *incompressibility* or *bulk modulus,* K is defined by subjecting a body to a lithostatic pressure dp, such that

$$d\sigma_{ij} = -\, dp\delta_{ij}$$

The resulting strains are $-dp\delta_{ij} = \lambda d\theta\delta_{ij} + 2\mu de_{ij}$

Set $i = j$ and sum ($\delta_{ii} = 3$): $-3dp = 3\lambda d\theta + 2\mu d\theta$ because $\delta_{ii} = 3$.

K is defined as the ratio of the pressure applied to the fractional volume change that results:

$$K = \frac{-dp}{d\theta} = \lambda + \frac{2}{3}\mu$$

The constitutive equation in terms of K and μ:

$$\sigma_{ij} = K\theta\delta_{ij} + 2\mu\,(e_{ij} - \theta\delta_{ij}/3)$$

Two parts: a volume change and a change in shape.

Two other elastic constants are defined by pulling the material along only one axis, leading to a state of stress called *uniaxial tension.* If the tension is applied along the x_1 axis:

$$\sigma_{11} = (\lambda + 2\mu)e_{11} + \lambda e_{22} + \lambda e_{33}$$

$$\sigma_{22} = 0 = \lambda e_{11} + (\lambda + 2\mu)e_{22} + \lambda e_{33}$$

$$\sigma_{33} = 0 = \lambda e_{11} + \lambda e_{22} + (\lambda + 2\mu)e_{33}$$

Subtracting the last two equations shows that $e_{22} = e_{33}$, so

$$e_{22} = e_{33} = \frac{-\lambda}{2(\lambda + \mu)} e_{11} = -\nu e_{11}$$

This defines Poisson's ratio, ν, which gives the ratio of the contraction along the other two axes to the extension along the axis where tension was applied.

Substituting this into the equation for σ_{11}:

$$\frac{\sigma_{11}}{e_{11}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} = E$$

E is called *Young's modulus,* the ratio of the tensional stress to the resulting extensional strain.

Now, we can combine the generalized Hookes' s Law with the equation of motion and the relationships between strain and displacement to combine everything into a single equation involving only constants and displacement.

Start with equations of motion:

$$\sigma_{ij,j}(\mathbf{x},t) = \rho \frac{\partial^2 u_i(\mathbf{x},t)}{\partial t^2}$$

For the x component:

$$\frac{\partial \sigma_{xx}(\mathbf{x},t)}{\partial x} + \frac{\partial \sigma_{xy}(\mathbf{x},t)}{\partial y} + \frac{\partial \sigma_{xz}(\mathbf{x},t)}{\partial z} = \rho \frac{\partial^2 u_x(\mathbf{x},t)}{\partial t^2}$$

Use the constitutive law for an isotropic elastic medium:

$$\sigma_{ij} = \lambda \theta \delta_{ij} + 2\mu e_{ij}$$

Strains in terms of displacements:

$$\sigma_{xx} = \lambda\theta + 2\mu e_{xx} = \lambda\theta + 2\mu \frac{\partial u_x}{\partial x}$$

$$\sigma_{xy} = \mu e_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\sigma_{xz} = \mu e_{xz} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right).$$

Take derivatives of the stress components:

$$\frac{\partial \sigma_{xx}}{\partial x} = \lambda \frac{\partial \theta}{\partial x} + 2\mu \frac{\partial^2 u_x}{\partial x^2}$$

$$\frac{\partial \sigma_{xy}}{\partial y} = \mu \left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial y \partial x} \right)$$

$$\frac{\partial \sigma_{xz}}{\partial z} = \mu \left(\frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial z \partial x} \right)$$

Use definitions of dilation $\theta = \boldsymbol{\nabla} \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$

and Laplacian $\nabla^2(u_x) = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}$

gives x-component displacements:

$$(\lambda + \mu) \frac{\partial \theta}{\partial x} + \mu \nabla^2(u_x) = \rho \frac{\partial^2 u_x}{\partial t^2}$$

Combining all three components, in terms of displacements:

$$(\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}(\mathbf{x}, t)) + \mu \boldsymbol{\nabla}^2 \mathbf{u}(\mathbf{x}, t) = \rho \frac{\partial^2 \mathbf{u}(\mathbf{x}, t)}{\partial t^2}$$

Three equations:
Elastodynamic Equations
for Linear Elastic, Isotropic,
Homogeneous Medium

Problem: Where are P and S waves?

We need to separate this into two different wave equations for P and S.

Use vector identity:

Vector calculus is very useful at this point.

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$$

to get

$$(\lambda + 2\mu)\, \nabla\, (\nabla \cdot \mathbf{u}(\mathbf{x}, t)) - \mu\, \nabla \times (\, \nabla \times \mathbf{u}(\mathbf{x}, t)\,) = \rho\, \frac{\partial^2 \mathbf{u}(\mathbf{x}, t)}{\partial t^2}$$

Define the displacement field in terms of two potential functions, ϕ and $\boldsymbol{\Upsilon}$:

$$\mathbf{u}(\mathbf{x}, t) = \nabla\phi(\mathbf{x}, t) + \nabla \times \boldsymbol{\Upsilon}(\mathbf{x}, t)$$

Helmholtz Theorem:
ANY continuous field can be represented in this form with Potentials (other fields) satisfying specific criteria.

Use vector identities:

$$\nabla \times (\nabla\phi) = 0 \qquad \nabla \cdot (\nabla \times \boldsymbol{\Upsilon}) = 0$$

to get

$$(\lambda + 2\mu)\nabla(\nabla^2\phi) - \mu\nabla \times \nabla \times (\nabla \times \boldsymbol{\Upsilon}) = \rho\, \frac{\partial^2}{\partial t^2}\, (\nabla\phi + \nabla \times \boldsymbol{\Upsilon})$$

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\nabla^2\phi) - \mu\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\Upsilon}) = \rho \frac{\partial^2}{\partial t^2} (\boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{\Upsilon})$$

The second term simplifies to:

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\Upsilon}) = -\boldsymbol{\nabla}^2(\boldsymbol{\nabla} \times \boldsymbol{\Upsilon}) + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\Upsilon})) = -\boldsymbol{\nabla}^2(\boldsymbol{\nabla} \times \boldsymbol{\Upsilon})$$

which separates into:

$$\boldsymbol{\nabla}\left[(\lambda + 2\mu)\, \nabla^2\phi(\mathbf{x}, t) - \rho \frac{\partial^2 \phi(\mathbf{x}, t)}{\partial t^2}\right] = -\boldsymbol{\nabla} \times \left[\mu\boldsymbol{\nabla}^2\boldsymbol{\Upsilon}(\mathbf{x}, t) - \rho \frac{\partial^2 \boldsymbol{\Upsilon}(\mathbf{x}, t)}{\partial t^2}\right]$$

$$\nabla \left[(\lambda + 2\mu)\, \nabla^2 \phi(\mathbf{x}, t) - \rho \frac{\partial^2 \phi(\mathbf{x}, t)}{\partial t^2} \right] = -\nabla \times \left[\mu \boldsymbol{\nabla}^2 \boldsymbol{\Upsilon}(\mathbf{x}, t) - \rho \frac{\partial^2 \boldsymbol{\Upsilon}(\mathbf{x}, t)}{\partial t^2} \right]$$

One solution is for both sides to equal zero all the time:

The scalar potential side:

$$\nabla^2 \phi(\mathbf{x}, t) = \frac{1}{\alpha^2} \frac{\partial^2 \phi(\mathbf{x}, t)}{\partial t^2}$$

with velocity $\alpha = [(\lambda + 2\mu)/\rho]^{1/2}$

This is a wave equation for the P wave – ϕ is the space-time P wavefield

The vector potential side:

$$\boldsymbol{\nabla}^2 \boldsymbol{\Upsilon}(\mathbf{x}, t) = \frac{1}{\beta^2} \frac{\partial^2 \boldsymbol{\Upsilon}(\mathbf{x}, t)}{\partial t^2}$$

with velocity $\beta = (\mu/\rho)^{1/2}$

This is a wave equation for the S waves – φ is the space-time S wavefield

OK, let's take a breath. All this algebra has given a profound result that we want to take stock of.

The general mathematical representation of motions everywhere in the medium satisfying the elastodynamic equations, $\mathbf{U}(\mathbf{x},t)$, is given by simple spatial derivative operations on two space-time functions, $\phi(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$, which are themselves solutions of the three-dimensional wave equation.

$\phi(\mathbf{x},t)$ is the P wavefield
$\psi(\mathbf{x},t)$ is the S wavefield

Physical displacements of the P wave are calculated by taking the gradient of $\phi(\mathbf{x},t)$.

Physical displacements of the S wave are calculated by taking the curl of $\psi(\mathbf{x},t)$.

Because the wavefields satisfy the wave equation, P and S wave motions have basic behavior of waves – this makes the final solution very straightforward: we just need to understand properties of waves.