ICTP - NLAGA

School on Dynamical Systems and Ergodic Theory

Hyperbolic Isomorphisms in Banach Spaces

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1. Let E be a finite dimensional vector space and $A \in GL(E)$. Show that:
   (a) if $e_1, \dots, e_n$ are vectors in a basis associated to a Jordan block of an eigenvalue λ of A with $|\lambda| < 1$, then $A^n x \to 0$ as $n \to +\infty$ for all $x \in \text{span}\{e_1, \dots, e_n\}$;
   (b) if $e_1, \dots, e_n$ are vectors in a basis associated to a Jordan block of an eigenvalue λ of A with $|\lambda| > 1$, then $A^{-n} x \to 0$ as $n \to +\infty$ for all $x \in \text{span}\{e_1, \dots, e_n\}$;
   (c) if A has no eigenvalues in the unit circle, then there are subspaces E^s and E^u of E such that:
      i. $E = E^s \oplus E^u$;
      ii. $A^n x \to 0$ as $n \to +\infty$ for all $x \in E^s$;
      iii. $A^{-n} x \to 0$ as $n \to +\infty$ for all $x \in E^u$.

2. Let E be a Banach space and $T \in L(E)$. Show that:
   (a) if $\sum_{n=0}^{\infty} T^n$ converges, then $I - T$ is invertible and $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$. Hint: compute $(I - T) \sum_{n=0}^{m} T^n$ and $\left(\sum_{n=0}^{m} T^n\right)(I - T)$ for $m \in \mathbb{N}$;
   (b) if $\|T\| < 1$, then $I - T$ is invertible and $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$;
   (c) if $I - T$ is invertible, then $\|(I - T)^{-1}\| \leq 1/(1 - \|T\|)$.

3. Let E be a complex Banach space and $A, B \in GL(E)$. Show that:
   (a) $\sigma\left(A^{-1}\right) = \{1/\lambda : \lambda \in \sigma(A)\}$.
   (b) $\sigma(BAB^{-1}) = \sigma(A)$.

4. Let $P : E \to E$ be a projection. Show that:
   (a) $I - P$ is a projection;
   (b) $P(E) = \ker(I - P)$;
   (c) $P(E)$ is a closed subspace of E;
   (d) $E = P(E) \oplus (I - P)(E)$.

5. Let E be a real Banach space. Show that:
   (a) if $T \in L(E)$ is bounded, then $T_{\mathbb{C}} \in L(E_{\mathbb{C}})$ is bounded and $\|T_{\mathbb{C}}\|_{\mathbb{C}} = \|T\|$;
   (b) the mapping $L(E) \ni T \longmapsto T_{\mathbb{C}} \in L(E_{\mathbb{C}})$ is continuous.

6. Let $A \in GL(E)$ and let $A_{\mathbb{C}} \in L(E_{\mathbb{C}})$ be the complexification of A. Show that:

   (a) $E^s_{\mathbb{C}} = E^s + iE^s$ and $E^u_{\mathbb{C}} = E^u + iE^u$;

   (b) $A^s_{\mathbb{C}}(x + iy) = A^s x + iA^s y$ for each $x, y \in E^s$;

   (c) $A^u_{\mathbb{C}}(x + iy) = A^u x + iA^u y$ for each $x, y \in E^u$.

7. Let (X, d) a complete metric space, Y a topological space and $f : X \times Y \to X$ continuous for which there is $0 < \lambda < 1$ such that

$$\mathrm{dist}(f(x, y), f(x', y)) \leq \lambda \, \mathrm{dist}(x, x')$$

   for all $y \in Y$ and $x, x' \in X$. Show that there is $p : Y \to X$ continuous such that for each $y \in Y$:

   (a) $p(y)$ is the unique fixed point of $f_y : X \to X$ given by $f_y(x) = f(y, x)$ for all $x \in X$;

   (b) $\lim_{n \to +\infty} f^n_y(x) = p(y)$ for all $x \in X$.

8. Let E be a finite dimensional linear space. Show that:

   (a) $H(E)$ is dense in $GL(E)$;

   (b) if $A \in GL(E)$ is structurally stable, then $A \in H(E)$.

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