1. a) Show that $f(x) = \sqrt{x}$ is not a contraction on $(0, +\infty)$.
b) Find an interval $I$, which contains 2, where you can apply Banach's fixed point theorem to verify that $f^n(2) \to 1$.
c) Verify that $f^n(x) \to 1$ for any $x \in (0, +\infty)$.

2. For the Greek method of computing $\sqrt{2}$ we used $f(x) = \frac{x + \frac{2}{x}}{2}$.
a) Prove that $f(x) \geq \sqrt{2}$ for $\forall x > 0$.
b) Prove that if $x \geq \sqrt{2}$ then $f(x) \leq x$.
c) Show that $f$ is not a contraction on $(0, +\infty)$.
d) Show that if $I = [1, 2]$ then $f(I) \subset I$ and $f$ is a contraction on $I$.
e) Prove that $f^n(x) \to \sqrt{2}$ for any $x > 0$.

3. Find a simple dynamical system which can provide you with sequences such that $f^n(x) \to \sqrt[3]{5}$. (Greek method of finding $\sqrt[3]{5}$.)

4. Suppose $(X, \mathcal{B}, \mu, T)$ is a given dynamical system. Without using Birkhoff's ergodic theorem try to give a proof (as elementary as possible) of the fact, that if $f \in L^1(\mu)$ then $\frac{f(T^k x)}{k} \to 0$ for $\mu$ a.e. $x \in X$.

5. Suppose that $(X, \mathcal{B}, \mu, T)$ is a given invertible dynamical system. Suppose $A \in \mathcal{B}$ is invariant in the "almost everywhere" sense, that is $\mu(T^{-1}A \Delta A) = 0$. Show that there is $A' \in \mathcal{B}$ such that $\mu(A' \Delta A) = 0$ and $A'$ is invariant in the stricter sense, that is, $T^{-1}A' = A'$.

6. Suppose that $(X, \mathcal{B}, \mu, T)$ is a given dynamical system. Denote by $\mathcal{S}$ the set consisting of the $T$-invariant sets in $\mathcal{B}$.
a) Show that $\mathcal{S}$ is a $\sigma$-algebra.
b) Let $f$ be in $L^1(\mu)$ and $f^*(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$ (from Birkhoff's ergodic theorem).
Show that $f^*$ is $\mathcal{S}$ measurable and $\int_A f^* d\mu = \int_A f d\mu$ for any $A \in \mathcal{S}$.
(This implies that $f^*$ is the conditional expectation of $f$ on $\mathcal{S}$, denoted by $E_\mu(f|\mathcal{S})$.)

7. Suppose $(X, \mathcal{B}, \mu, T)$ is a given dynamical system. Suppose $f : X \to \mathbb{R}$ is measurable. Show that $\overline{f} = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$ is $T$-invariant, that is $\overline{f} \circ T = \overline{f}$.

8. Suppose $T_\alpha = \{x + \alpha\}$ is an irrational rotation of $\mathbb{T}$.
a) Can you give an example of a measurable function $f : \mathbb{T} \to \mathbb{R}$ for which $\frac{f(T_\alpha^k x)}{k} \not\to 0$ for $\mu$ a.e. $x \in X$?
b) Can you give an example of a rotation $T_\alpha$ and of a measurable function $f : \mathbb{T} \to \mathbb{R}$, $f \notin L^1(\mu)$ for which $\frac{f(T_\alpha^k x)}{k} \to 0$ for $\mu$ a.e. $x \in X$?
c) Show that the Lebesgue measure of the set $\{x \in \mathbb{T} : \frac{f(T_\alpha^k x)}{k} \to 0\}$ is either zero, or one.
d) Can you give an example of a rotation $T_\alpha$ and of a measurable function $f : \mathbb{T} \to \mathbb{R}$ such that $\frac{f(T_\alpha^k x)}{k} \to 0$ Lebesgue a.e. but the ergodic averages $\frac{1}{n} \sum_{k=0}^{n-1} f(T_\alpha^k x)$ do not converge almost everywhere?