LECTURES ON THE HEAT EQUATION

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I will discuss some basic techniques for analyzing parabolic equations $\partial_t u = Lu$, where L is either an operator of the form $\partial_i(a_{ij}\partial_j)$ on $\mathbb{R}^n$ or the Laplacian on a Riemannian manifold. The emphasis will be on simple arguments that can be applied in many different situations.

Prerequisites: We essentially only need the heat kernel on $\mathbb{R}^n$ and the maximum principle for the heat equation. Some knowledge of calculus on Riemannian manifolds (gradient, Hessian, divergence) is helpful but not strictly necessary. I also recommend the following survey by Grigoryan.

1. Uniqueness and nonuniqueness [2, 3]

The solution to the initial value problem for the heat equation in $\mathbb{R}^n$ is not unique in general, but is unique if we assume an exponential growth bound. I will explain Tychonov's counterexamples to uniqueness and give two proofs for uniqueness under an exponential growth bound: via pointwise estimates (maximum principle) and via integral estimates (Carleman inequalities).

2. Analyticity [7, 8]

Solutions to parabolic equations are usually analytic in space for $t > 0$, and are analytic in time under suitable boundary conditions, but are not analytic in time in general. I will prove these facts using Bernstein's derivative estimates. This complements the first lecture because (non)analyticity in time is closely related to (non)uniqueness of solutions to the initial value problem.

3. Classical heat kernel estimates [4, 9]

It is a classical problem to compare the integral kernel of e^{-tL} to the standard heat kernel on $\mathbb{R}^n$. We will prove fairly general upper bounds using the iteration method of Moser and Davies, which is based on Sobolev and Nash type inequalities. Lower heat kernel bounds are a much more delicate question and often depend on strong assumptions on the underlying geometry or on the coefficients of L. I will explain the Li-Yau lower bounds for manifolds with nonnegative Ricci curvature.

4. Logarithmic Sobolev inequalities [5, 6]

A metric space equipped with a probability measure ν satisfies a logarithmic Sobolev inequality if $\int u^2 \log u^2 d\nu \leq 4\tau \int |\nabla u|^2 d\nu$ for some $\tau > 0$ and all Lipschitz functions u with $\int u^2 d\nu = 1$. I will show that this holds on Riemannian manifolds with either (i) $d\nu = e^{-f}d\mathrm{vol}$ and $\mathrm{Ric} + \nabla^2 f \geq \frac{1}{2\tau}g$, or (ii) $d\nu = H_{\tau,x}d\mathrm{vol}$ ($H_{\tau,x}$ = heat kernel based at x) and $\mathrm{Ric} \geq 0$. The case of $\mathbb{R}^n$ with the standard Gaussian probability measure is already very interesting. The proof of these inequalities is closely related to the Li-Yau theory in Lecture 3. I will also mention some applications.

5. The theory of Nash [4, 10]

In 1958 Nash published a very original paper proving upper and lower heat kernel estimates for general elliptic operators $L = \partial_i(a_{ij}\partial_j)$ on $\mathbb{R}^n$ and deducing Hölder continuity of the heat kernel. I will go over some interesting aspects of Nash's proof, including his notion of entropy.

Date: June 20, 2014.

6. Two-point functions [1]

This lecture will introduce a method for proving very sharp gradient or oscillation estimates for solutions to parabolic equations by applying the maximum principle to test functions on $M \times M$. This is the PDE version of the "coupling technique" for Brownian motions in stochastic analysis. As an application, we will get a clean and simple proof of the Zhong-Yang eigenvalue estimate for compact Riemannian manifolds with nonnegative Ricci curvature.

References

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