GEOMETRIC FLOWS IN COMPLEX GEOMETRY

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ABSTRACT. These are notes for lectures delivered at the Hefei Advanced School on PDEs in Geometry and Physics June30th-July 11th 2014.

1. Preliminaries

These are notes for the Hefei Advanced School on PDEs in Geometry and Physics, June 30th-July 11th 2014. We assume familiarity with (almost) complex manifolds, vector bundles, connections, curvature, torsion and characteristic classes. Many good sources exist for obtaining familiarity with this material, one example is [51]. The purpose of these lectures is to motivate and develop the theory of geometric evolution equations in the context of almost-Hermitian geometry, and the material is based on the following papers, listed in chronological order:

(1) Streets, J.; Tian, G. *Hermitian curvature flow*, arXiv:0804.4109
(2) Streets, J.; Tian, G. *A parabolic flow of pluriclosed metrics* arXiv:0903.4418
(3) Streets, J.; Tian, G. *Regularity results for pluriclosed flow* arXiv:1008.2794
(4) Streets, J.; Tian, G. *Symplectic curvature flow* arXiv:1012.2104
(5) Streets, J.; Tian, G. *Generalized Kähler geometry and the pluriclosed flow* arXiv:1109.0503
(6) Streets, J. *Generalized geometry, T-duality and renormalization group flow* arXiv:1310.5121
(7) Streets, J. *Pluriclosed flow on generalized Kähler manifolds with split tangent bundle* arXiv:1405.0727

Our discussion will be largely expository, focusing on guiding philosophy, broad themes, conjectures, and open problems. We will discuss some proofs, but will mostly refer the reader to the original papers for complete proofs. The six lectures will be divided as follows:

(1) Overview of Kähler geometry/Kähler Ricci flow
(2) Introduction to pluriclosed flow
(3) Pluriclosed flow as a gradient flow
(4) Pluriclosed flow and generalized Kähler geometry
(5) T-duality and geometric flows
(6) Symplectic curvature flow

2. Review of Kähler-Ricci flow

2.1. **Uniformization Theorem.** The classification of Riemann surfaces is closely related to the classical *uniformization theorem*

Theorem 2.1. *(Uniformization of Riemann Surfaces) Every simply connected Riemann surface is conformally equivalent to either the open unit disc, the complex plane, or the Riemann sphere.*

Using this, a classification of compact Riemann surfaces follows. In particular, since any covering space of a Riemann surface is again a Riemann surface, lifting to the universal cover and applying the theorem above yields

Theorem 2.2. *Every compact, connected Riemann surface is a quotient by a free, properly discontinuous action of a group on the unit disc, the complex plane, or the Riemann sphere. In particular, it admits a Riemannian metric of constant (scalar) curvature.*

Remark 2.3. It is possible to prove the theorem above using *Ricci flow.* In particular, fix a Riemann surface (M^2, g, J) with compatible metric. We can ask the (apparently) slightly different question: does there exist a conformally related metric $e^{2u} g$ which has *constant curvature?* The Ricci flow attempts to construct such a metric using a parabolic equation:

$$\frac{\partial}{\partial t} g = \; -2 \operatorname{Rc}.$$

Since the dimension $n = 2$, the Ricci tensor can be expressed as $\operatorname{Rc} = \frac{1}{2} R g$, and then the flow reduces to a flow on the conformal factor alone. The work of many authors [] leads to the statement that, after volume normalization, the solution exists for all time and converges to a constant scalar curvature metric. This is then a new proof of the uniformization theorem.

A fundamental question which drives much research in complex geometry is:

> *Can we use geometric flows to prove geometric/topological classification theorems for complex manifolds in higher dimensions?*

Our inspiration and guiding philosophy for answering this question comes from the Kähler-Ricci flow, which we now briefly recall.

2.2. **Kähler-Ricci flow.**

Definition 2.4. Let (M^{2n}, J) be a compact complex manifold. A Riemannian metric g on M is *Kähler* if

(1) g is compatible with J, i.e.: $g(J\cdot, J\cdot) = g(\cdot, \cdot)$
(2) Setting $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$, we have that $d\omega = 0$.

Remark 2.5. In the above definition, $\omega \in \Lambda^{1,1}_{\mathbb{R}}$ and $[\omega] \in H^{1,1}_{\mathbb{R}}$ is called the *Kähler class.*

Lemma 2.6. *($\partial\bar{\partial}$-Lemma) Let (M^{2n}, g, J) be a compact Kähler manifold. Suppose g' is another metric on M such that $[\omega'] = [\omega]$. Then there exists a unque $f \in C^\infty(M)$ such that $\int_M f dV_g = 0$ and*

$$\omega = \omega' + \sqrt{-1} \partial \bar{\partial} f.$$

Definition 2.7. Given (M^{2n}, J, g) a Kähler manifold, we let Rm denote the curvature tensor of the Levi-Civita connection, which coincides with the Chern connection on $T^{1,0}$. Moreover, we say that

$$\rho_{i\bar{j}} = g^{\bar{l}k} R_{i\bar{j}k\bar{l}}$$

is the *Ricci form* of g. It follows from easy curvature calculations that $\rho \in \Lambda^{1,1}_{\mathbb{R}}$ and moreover $d\rho = 0$ by the Bianchi identity. Alternatively, ρ is the curvature of the induced connection on the determinant line bundle $\Lambda^{n,0}$, and it then follows that $[\rho] = c_1(M, J)$, the *first Chern class of* (M, J).

Definition 2.8. Let (M^{2n}, J, ω_0) be a compact Kähler manifold. We say that a one-parameter family of Kähler metrics ω_t is a solution to *Kähler-Ricci flow with initial condition* ω_0 if

$$\begin{aligned}\frac{\partial \omega}{\partial t} =& \ -\rho(\omega_t),\\ \omega(0) =& \ \omega_0.\end{aligned}$$

Remark 2.9. In general for a Kähler metric one has the identity $\mathrm{Rc}(J\cdot, \cdot) = \rho(\cdot, \cdot)$, and therefore given a solution to Kähler-Ricci flow the associated Riemannian metrics satisfy the Ricci flow equation:

$$\frac{\partial g}{\partial t} = \ -\mathrm{Rc}\,.$$

Given that solutions to the Ricci flow are unique, it follows that *Ricci flow preserves the Kähler condition.*

2.3. **Tian-Zhang's sharp local existence theorem.**

Definition 2.10. Let (M^{2n}, J) be a compact Kähler manifold. Let

$$\mathcal{K} = \{[\phi] \in H^{1,1}_{\mathbb{R}} |\ \exists \omega \in [\phi], \omega > 0\}.$$

Remark 2.11. The set $\mathcal{K}$ is an open cone in the finite dimensional vector space $H^{1,1}_{\mathbb{R}}$.

Now let (M^4, ω_t, J) be a solution to Kähler-Ricci flow. Observe that there is an associated ODE

$$\frac{\partial}{\partial t}[\omega] = -c_1.$$

Certainly, if the boundary of $\mathcal{K}$ is reached along this ODE, the flow must have generated a singularity of some kind. One can ask the natural question: is this the ONLY way that KRF encounters singularities? The answer is yes:

Theorem 2.12. *(Tian-Zhang) Let (M^{2n}, ω_0, J) be a compact Kähler manifold. Let*

$$T = \sup\{t \in \mathbb{R} | [\omega_0] - tc_1 \in \mathcal{K}\}.$$

Then the solution to Kähler-Ricci flow exists smoothly on $[0, T)$, and this solution is maximal.

3. Introduction to pluriclosed flow

The Kähler-Ricci flow is certainly an equation of central importance in Kähler geometry. One could easily fill several courses discussing it alone. However, our purpose in this course is to tell the story of new equations which aim to extend the applicability of the techniques and ideas os Kähler-Ricci flow into the world of complex, non-Kähler manifolds. To begin let us recall the first known example of such a manifold, the *Hopf surface*, which plays a central role in our discussion.

Example 3.1. Consider $\mathbb{C}^2 - (0,0)$. Fix complex numbers α, β, $|\alpha| \geq |\beta| > 1$, and let

$$\Gamma = \langle \gamma \rangle, \qquad \gamma(z_1, z_2) = (\alpha z_1, \beta z_2).$$

The action of Γ is free and properly discontinuous, therefore we may construct the smooth manifold

$$M_{\alpha,\beta} := \frac{\mathbb{C}^2 - (0,0)}{\Gamma}.$$

As it turns out, $M_{\alpha,\beta} \cong S^3 \times S^1$. Moreover, since Γ acts by biholomorphisms, this manifold inherits a complex structure. However, since $H^2(M, \mathbb{R}) \cong 0$, it follows that M cannot admit a Kähler metric. In the case $|\alpha| = |\beta|$, this manifold inherits a metric relevant to us later, specifically consider on $\mathbb{C}^2 - (0,0)$,

$$\omega = \frac{\sqrt{-1}}{\mu^2} \partial \overline{\partial} \mu^2,$$

where $\mu = \sqrt{\left|z_1\right|^2 + \left|z_2\right|^2}$. This metric is certainly invariant under the action of Γ, and so descends to the quotient.

3.1. Integrability conditions for Hermitian metrics, Gauduchon's Theorem.

Remark 3.2. As we discussed earlier, every Riemann surface is in fact a *Kähler* manifold, and in fact every Hermitian metric on a Riemann surface is Kähler. This will no longer be the case in higher dimensions. As inevitably one has $d\omega \neq 0$, there are various natural conditions which can be placed on Hermitian, non-Kähler metrics. As it turns out, in complex dimension $n = 2$, there is really only one integrability condition for non-Kähler metrics.

Definition 3.3. Let (M^{2n}, g, J) be a Hermitian manifold with Kähler form ω. The metric is said to be

1. *Balanced* if $d\omega^{n-1} = 0$
2. *Gauduchon*, or *standard* if $\partial\overline{\partial}\omega^{n-1} = 0$.
3. *pluriclosed*, or *strong Kähler with torsion*, if $\partial\overline{\partial}\omega = 0$.

Remark 3.4. These do not represent all possible "integrability conditions" for Hermitian metrics. However, observe that, trivially, when $n = 2$ a metric is Kähler if and only if it is balanced and is pluriclosed if and only if it is Gauduchon. In this case these do represent the only natural (i.e. diffeomorphism invariant) conditions one can place on a Hermitian metric.

Question 3.5. Is there a natural geometric flow that preserves the "balanced" condition? As we will see, the fact that the pluriclosed condition is *linear* makes it possible to make an educated guess at a natural flow. In the case of the balanced condition, which is nonlinear, it is less clear.

Theorem 3.6. *(Gauduchon,* [9]*) Given (M^{2n}, g, J) a connected compact Hermitian manifold, there exists a unique $\phi \in C^\infty(M)$ such that $\widetilde{g} = \phi g$ is a Gauduchon metric and $\int_M \phi dV_g = 1$.*

3.2. **Pluriclosed Flow.** We now ask a more refined version of the question from the introduction: is it possible to prove classification results in higher dimensions for complex (non-Kähler) manifolds using geometric evolution equations? The rest of the course is devoted to making progress on this difficult question. Our philosophy will be guided by some basic principles. In particular, we hope that our flows:

- Preserve (almost) Hermitianness
- Preserve as much additional structure as possible ((almost)-Kähler, pluriclosed, etc.)
- Have "canonical" fixed points
- Are as close to Ricci flow as possible.

Given the previous discussion, it is natural to ask if there is a geometric flow preserving the pluriclosed condition. One way to guess the answer is by finding the "local generality" of pluriclosed metrics. To begin we first recall the "local generality" of Kähler metrics.

Lemma 3.7. *Let $U \subset \mathbb{C}^n$ be an open subset homeomorphic to a ball, and suppose $\omega \in \Lambda^{1,1}_{\mathbb{R}}$ is a Kähler form on U. There exists $f \in C^\infty(U)$ such that $\omega = \sqrt{-1}\partial\overline{\partial} f$.*

Proof. Since U has trivial cohomologies, since ω is a real closed $(1,1)$ form there exists $\alpha \in \Lambda^1_{\mathbb{R}}(M)$ such that $d\alpha = \omega$. Decomposing $\alpha = \alpha^{0,1} + \alpha^{1,0}$, we observe that $\overline{\partial}\alpha^{0,1} = 0$, and so there exists $f \in C^\infty(M, \mathbb{C})$ such that

$$\alpha^{0,1} = \overline{\partial} f.$$

But since $\overline{\alpha^{0,1}} = \alpha^{1,0}$ we conclude that $\alpha^{1,0} = \partial\overline{f}$. Plugging in this yields

$$\begin{aligned}
\omega &= d\alpha \\
&= \partial\alpha^{0,1} + \overline{\partial}\alpha^{1,0} \\
&= \partial\overline{\partial} f + \overline{\partial}\partial\overline{f} \\
&= \partial\overline{\partial}\left(f - \overline{f}\right) \\
&= \sqrt{-1}\partial\overline{\partial} 2\Im f.
\end{aligned}$$

□

Remark 3.8. It is because of this simple fact that one expects Kähler-Ricci flow to only "depend on one function," which is borne out by computations. Indeed, our original formula for ρ shows that it only depends on the volume form of the metric, which is locally given by a single function.

Lemma 3.9. *Let $U \subset \mathbb{C}^n$ be an open subset homeomorphic to a ball, and suppose $\omega \in \Lambda^{1,1}_{\mathbb{R}}$ is a pluriclosed form on U. There exists $\alpha \in \Lambda^{0,1}$ such that*

$$\omega = \partial\alpha + \overline{\partial}\overline{\alpha}.$$

Proof. Since the form $\overline{\partial}\omega$ is d-closed, so by the local $\partial\overline{\partial}$ lemma we obtain $\beta \in \Lambda^{0,1}$ such that

$$\overline{\partial}\omega = \overline{\partial}\partial\beta$$

Now consider the form

$$\gamma := \omega - \partial\beta - \overline{\partial\beta}.$$

Note that

$$\overline{\partial}\gamma = \overline{\partial}\omega - \overline{\partial}\partial\beta = 0, \qquad \partial\gamma = \partial\omega - \partial\overline{\partial\beta} = 0$$

Since $\gamma \in \Lambda^{1,1}_{\mathbb{R}}$ is d-closed, it follows again by the $\partial\overline{\partial}$-lemma that there exists $f \in C^\infty(M, \mathbb{R})$ such that $\gamma = \sqrt{-1}\partial\overline{\partial}f$. Finally, set

$$\alpha = \beta + \frac{\sqrt{-1}}{2}\overline{\partial}f.$$

We then directly compute

$$\partial\alpha + \overline{\partial\overline{\alpha}} = \partial\beta + \overline{\partial\beta} + \sqrt{-1}\partial\overline{\partial}f = \partial\beta + \overline{\partial\beta} + \gamma = \omega.$$

□

Remark 3.10. With this point of view, it is natural to define a flow of pluriclosed metrics using a second order closed $(1,1)$-form and a first-order $(0,1)$-form. Since we want our flow to reduce to Kähler-Ricci flow, it is natural to let the closed form be the Chern curvature form. For the first order $(0,1)$-form, only one option really presents itself, which is $\partial^*_\omega\omega$.

Definition 3.11. Let (M^{2n}, J) be a compact complex manifold. A one-parameter family of Kähler forms ω_t is a solution to *pluriclosed flow* if

$$\frac{\partial}{\partial t}\omega = \partial\partial^*_\omega\omega + \overline{\partial\partial^*_\omega}\omega + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\det g.$$

Proposition 3.12. *Let (M^{2n}, J) be a complex manifold, and let $\mathcal{P}$ denote the space of Hermitian metrics on M. Given $\omega \in \mathcal{P}$, the operator*

$$\Phi : \mathcal{P} \to \Lambda^{1,1}$$

$$\Phi(\omega) := -\partial\partial^*_\omega\omega - \overline{\partial\partial^*_\omega}\omega - \frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\det g$$

is strictly elliptic.

Proof. First we require a coordinate formula for $\partial^*_\omega\omega$. An exercise shows that

$$(\partial^*_\omega\omega)_{\overline{j}} = \frac{\sqrt{-1}}{2}g^{\overline{q}p}\left[g_{p\overline{j},\overline{q}} - g_{p\overline{q},\overline{j}}\right].$$

Taking conjugates we have

$$(\overline{\partial}^*_\omega\omega)_i = \frac{\sqrt{-1}}{2}g^{\overline{q}p}\left[g_{p\overline{q},i} - g_{i\overline{q},p}\right]$$

It follows that

$$(\partial\partial^*_\omega\omega)_{i\overline{j}} = \frac{\sqrt{-1}}{2}g^{\overline{q}p}\left[g_{p\overline{j},\overline{q}i} - g_{p\overline{q},\overline{j}i}\right] + l.o.t$$

$$(\partial\partial^*_\omega\omega)_{i\overline{j}} = \frac{\sqrt{-1}}{2}g^{\overline{q}p}\left[g_{i\overline{q},p\overline{j}} - g_{p\overline{q},i\overline{j}}\right] + l.o.t$$

We have already determined a formula for the remaining term,

$$\left(\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\det g\right)_{i\bar{j}} = \frac{\sqrt{-1}}{2}g^{\bar{q}p}g_{p\bar{q},i\bar{j}} + l.o.t$$

It follows that

$$\begin{aligned}\Phi(\omega) &= \frac{\sqrt{-1}}{2}g^{\bar{q}p}\left[g_{p\bar{j},\bar{q}i} + g_{i\bar{q},p\bar{j}} - g_{p\bar{q},i\bar{j}}\right]\\ &= \frac{\sqrt{-1}}{2}g^{\bar{q}p}g_{i\bar{j},p\bar{q}},\end{aligned}$$

where the last line follows from applying the pluriclosed condition

$$0 = (\partial\overline{\partial}\omega)_{i\bar{j}k\bar{l}} = g_{k\bar{l},i\bar{j}} - g_{k\bar{j},i\bar{l}} - g_{i\bar{l},k\bar{j}} + g_{i\bar{j},k\bar{l}}.$$

□

Theorem 3.13. *Let (M^{2n}, ω_0, J) be a compact manifold with a pluriclosed metric. There exists $\epsilon > 0$ and a unique solution ω_t to pluriclosed flow on $[0, \epsilon)$ with initial condition ω_0. Moreover, if ω_0 is Kähler, then ω_t is a solution to Kähler-Ricci flow.*

Question 3.14. A basic question is: can we classify the fixed points of this flow? There is a complete answer in dimension $n = 2$ provided by priori work of Gauduchon-Ivanov.

Theorem 3.15. *Let (M^4, g, J) be a solution of $\Phi(\omega) = \lambda\omega$. Then either (M^4, J) is Kähler-Einstein, or it is locally isometric to $\mathbb{R} \times S^3$ with the standard product metric. The universal cover of (M, J) is biholomorphic to $\mathbb{C}^2 \setminus \{(0, 0)\}$, and M admits a finite sheeted cover $\widetilde{M}$ with fundamental group $\mathbb{Z}$, specifically*

$$\pi_1(\widetilde{M}) \cong \mathbb{Z} = \langle (z_1, z_2) \to (\alpha z_1, \beta z_2)\rangle \tag{3.1}$$

where $\alpha, \beta \in \mathbb{C}$, $1 < |\alpha| = |\beta|$.

Question 3.16. Are there "soliton solutions" on the Hopf surfaces with $|\alpha| \neq |\beta|$? More generally, what is the long time behavior of the pluriclosed flow on these surfaces?

3.3. **The formal existence time of pluriclosed flow.**

3.3.1. *The positive cone.*

Definition 3.17. Let (M^{2n}, J) be a complex manifold. Let

$$H^{1,1}_{\partial+\overline{\partial}} := \frac{\left\{\psi \in \Lambda^{1,1}_{\mathbb{R}} | \partial\overline{\partial}\psi = 0\right\}}{\left\{\partial\alpha + \overline{\partial}\overline{\alpha} | \alpha \in \Lambda^{0,1}\right\}}$$

This is referred to as the $(1, 1)$-Aeppli cohomology, defined in []. It was shown in [] that this space is always finite dimensional on a compact manifold. Next, in analogy with the Kähler cone, we next define the cone of classes in $H^{1,1}_{\partial+\overline{\partial}}$ which admit pluriclosed metrics.

Definition 3.18. Let (M^{2n}, J) be a complex manifold. Let

$$\mathcal{P} := \left\{[\psi] \in H^{1,1}_{\partial+\overline{\partial}} \mid \exists\ \omega \in [\psi], \omega > 0\right\}.$$

The space $\mathcal{P}$ is an open cone in $\mathcal{H}^{1,1}_{\partial+\overline{\partial}}$, which is nonempty if and only if M admits pluriclosed metrics. We refer to $\mathcal{P}$ as the *positive cone.*

3.3.2. *The formal existence time.* Observe that, if ω_t is a solution to pluriclosed flow, then $[\omega_t]$ defines a path in $\mathcal{P}$. Moreover, since $H^{1,1}_{\mathbb{R}} \subset H^{1,1}_{\partial + \bar{\partial}}$, we can interpret the first Chern class of (M, J) as an element of $H^{1,1}_{\partial + \bar{\partial}}$. With this point of view we can solve the ODE

$$[\omega_t] = [\omega_0] - tc_1.$$

Certainly the flow cannot exist smoothly if the boundary of $\mathcal{P}$ is reached along this ODE. We state this for emphasis.

Lemma 3.19. *Let (M^{2n}, J) be a compact complex manifold, and suppose ω_0 is a pluriclosed metric on M. Let*

$$\tau^* := \sup\{t | [\omega_0] - tc_1 \in \mathcal{P}\}.$$

If T denotes the maximal existence time of the solution to pluriclosed flow with initial condition ω_0, then $T \leq \tau^*$.

The main guiding conjecture behind our study of pluriclosed flow is that in fact reaching the boundary of the cone is the *only* way to have a singularity.

Conjecture 3.20. Weak existence conjecture: *Let (M^{2n}, g_0, J) be a compact complex manifold with pluriclosed metric. Let*

$$\tau^* := \sup_{t \geq 0}\{t | [\omega_0 - tc_1] \in \mathcal{P}_{\partial + \bar{\partial}}\}.$$

Then the solution to pluriclosed flow with initial condition g_0 exists on $[0, \tau^*)$.

3.3.3. *Characterization of τ^*.* In this subsection we exhibit some cases where the formal existence time τ^* can be computed explicitly. First, in [49] the authors gave a characterization of $\mathcal{P}$ in the case of non-Kähler surfaces, which we record here for convenience. First, recall the Bott-Chern cohomology

$$H^{1,1}_{\mathrm{BC}} = \frac{\{\operatorname{Ker} d : \Lambda^{1,1}_{\mathbb{R}} \to \Lambda^3_{\mathbb{R}}\}}{i\partial\bar{\partial}\Lambda^0_{\mathbb{R}}}.$$

Also, define the spaces

$$B^{1,1}_{\mathbb{R}} = d\{\Lambda^1_{\mathbb{R}}\} \cap \Lambda^{1,1}_{\mathbb{R}},$$
$$H^{1,1}_{\mathbb{R}} = \frac{\{\operatorname{Ker} d : \Lambda^{1,1}_{\mathbb{R}} \to \Lambda^3_{\mathbb{R}}\}}{B^{1,1}_{\mathbb{R}}}.$$

Lemma 3.21. *If $b_1(M)$ is odd, the space*

$$\Gamma = \frac{B^{1,1}_{\mathbb{R}}}{i\partial\bar{\partial}\Lambda^0_{\mathbb{R}}}$$

is one dimensional, and is identified with $\mathbb{R}$ via the L^2 inner product with ω.

Proof. Let $\mu \in B^{1,1}_{\mathbb{R}}$ such that

$$\int_M \mu \wedge \omega = 0.$$

This implies that $\mathrm{tr}_\omega \mu$ is L^2 orthogonal to the constant functions. So in particular it is in the image of $\mathrm{tr}_\omega \partial\overline{\partial}$. Fix $f \in C^\infty$ such that $\mathrm{tr}_\omega \partial\overline{\partial} f = \mathrm{tr}_\omega \mu$. It follows that $\mu - \sqrt{-1}\partial\overline{\partial} f$ is exact, but also anti-self-dual. Thus it is harmonic and exact, and hence vanishes. $\square$

Let γ_0 denote a positive generator of Γ. Since the space of pluriclosed metrics on M is connected, this orientation of Γ is well-defined.

Theorem 3.22. *(Buchdahl) Let M be a compact complex surface with ω a Gauduchon metric. Let $\rho \in \Lambda^{1,1}$ satisfy $\partial\overline{\partial}\rho = 0$, $\int_M \rho \wedge \rho$, $\int_D \rho > 0$ for all effective divisors ρ, and $\int_M \rho \wedge \omega > 0$. Then there is $u \in C^\infty(M)$ such that $\rho + \sqrt{-1}\partial\overline{\partial} u > 0$.*

Theorem 3.23. *([49] Theorem 5.6) Let (M^4, J) be a complex non-Kähler surface. Suppose $\phi \in \Lambda^{1,1}$ is pluriclosed. Then $[\phi] \in \mathcal{P}_{\partial+\overline{\partial}}$ if and only if*

- $\int_M \phi \wedge \gamma_0 > 0$
- $\int_D \phi > 0$ *for every effective divisor with negative self intersection.*

Proof. One can show that $\phi + A\gamma_0$ satisfies the hypotheses of the Buchdahl theorem above for sufficiently large A. $\square$

4. Pluriclosed flow as a gradient flow

4.1. Ricci flow as a gradient flow.

Remark 4.1. We have by now suggested a conjecture for the long time existence behavior of the pluriclosed flow, and moreover classified the fixed points of the flow on complex surfaces. Now we can ask the question, "is it reasonable to expect the pluriclosed flow to converge?" For heat-type equations, such convergence usually comes from the fact that one has a *gradient flow*. Before beginning our discussion of pluriclosed flow as a gradient flow, we recall Perelman's construction showing that Ricci flow is a gradient flow.

Definition 4.2. Let M^n be a compact manifold, and let g denote a Riemannian metric and $f \in C^\infty(M)$. Let

$$\mathcal{F}(g, f) = \int_M \left[R + |\nabla f|^2\right] e^{-f} dV_g.$$

Furthermore, let

$$\lambda(g) := \inf_{\{f \mid \int_M e^{-f} dV = 1\}} \mathcal{F}(g, f).$$

This quantity λ is the first eigenvalue of the Schrodinger operator $-4\Delta + R$.

Theorem 4.3. *(Perelman) Ricci flow is the gradient flow of λ, suitably interpreted.*

Remark 4.4. This monotonicity is exhibited by in some sense exhibiting infinite dimensional families of monotone quantities. More precisely, one shows that one has monotonicity of $\mathcal{F}$ for arbitrary test functions f evolving by the *conjugate heat equation.*

Lemma 4.5. *Let (M^n, g_t) be a solution to the Ricci flow on a compact manifold. Let $\square = \frac{\partial}{\partial t} - \Delta_{g_t}$ be the time-evolving heat operator. The conjugate operator $\square^*$ with respect to the spacetime L^2 metric is $\square^* = -\frac{\partial}{\partial t} - \Delta + R$.*

Lemma 4.6. *Let M^n be a compact manifold, and let g_t, f_t be one-parameter families of metrics and functions. Then*

$$\frac{\partial}{\partial t}\mathcal{F}(g_t, f_t) = \int_M \left[\left\langle -h, \mathrm{Rc} + \nabla^2 f\right\rangle + (\frac{\mathrm{tr}_g h}{2} - \phi)\left(2\Delta f - \left|\nabla f\right|^2 + R\right)\right] e^{-f} dV.$$

We now suppose that our parameter e^{-f} evolves by the conjugate heat equation. This is equivalent to

$$\frac{\partial}{\partial t} f = -\Delta f + \left|\nabla f\right|^2 - R.$$

Lemma 4.7. *Suppose (M^n, g_t, f_t) is a solution to*

$$\begin{aligned}\frac{\partial}{\partial t} g &= -2(\mathrm{Rc} + \nabla^2 f)\\ \frac{\partial}{\partial t} f &= -\Delta f - R.\end{aligned}$$

Then

$$\frac{\partial}{\partial t}\mathcal{F}(g_t, f_t) = 2\int_M \left|\mathrm{Rc} + \nabla^2 f\right|^2 e^{-f} dV \geq 0.$$

4.2. **Pluriclosed flow as a gradient flow.** In this section we exhibit that pluriclosed flow is the gradient flow of the first eigenvalue of a certain Schrödinger operator associated to the time-dependent metric. What we actually show is that, after pulling back a solution to pluriclosed flow by the one-parameter family of diffeomorphisms generated by the vector field dual to the Lee form, one produces a solution to the renormalization group flow of a nonlinear sigma model arising in string theory (see [22] 108-112). This surprising fact both exhibits a connection between pluriclosed flow and mathematical physics, and from another point of view produces a large class of interesting examples of the renormalization group flow.

Let us recall some notation. Fix (M^{2n}, g, J) a complex manifold with pluriclosed metric. Let ∇ denote the Bismut connection, which is the unique Hermitian connection with skew-symmetric torsion. Explicitly, this connection takes the form

$$\nabla = \nabla^{LC} + \frac{1}{2} g^{-1} H,$$

where

$$H = d^c \omega = -d\omega(J, J, J).$$

Furthermore, let

$$\theta = -J d^* \omega$$

be the Lee form of ω. A detailed curvature calculation leads to:

Proposition 4.8. *Given $(M^{2n}, \omega(t), J)$ a solution to pluriclosed flow, one has*

$$\begin{aligned}\frac{\partial}{\partial t} g &= \left[-\mathrm{Rc}^g + \frac{1}{4}\sum_{i=1}^{2n} g(H(X, e_i), H(Y, e_i)) - \frac{1}{2}\mathcal{L}_{\theta^\sharp} g\right],\\ \frac{\partial}{\partial t} H &= \frac{1}{2}\left[\Delta_{LB, g(t)} H - \mathcal{L}_{\theta^\sharp} H\right],\end{aligned}$$

where $\theta^\sharp$ is the vector field dual to θ, taken with respect to the time varying metric.

Theorem 4.9. *Let $(M^{2n}, \widetilde{\omega}(t), J)$ be a solution to pluriclosed flow. Let $X(t) = \frac{1}{2}\widetilde{\theta}^\sharp$, where $\sharp$ means the vector dual taken with respect to the time-varying metric, and let ϕ_t denote the one parameter family of diffeomorphisms generated by $X(t)$. Let $\widetilde{H}$ denote the torsion of the time-varying Bismut connections. Let $(g(t), H(t)) = (\phi^*(\widetilde{g})(t)), \phi_t^*(\widetilde{H})(t))$. Then*

$$\begin{aligned} \frac{\partial}{\partial t} g =& -\operatorname{Rc}^g + \frac{1}{4}\mathcal{H} \\ \frac{\partial}{\partial t} H =& \frac{1}{2}\Delta_{LB} H. \end{aligned} \tag{4.1}$$

where $\mathcal{H}_{ij} = g^{kl}g^{mn}H_{ikm}H_{jln}$.

Proof. This follows from a standard calculation using Proposition 4.8. □

As noted above, the system of equations (4.1) arises naturally in physics as the renormalization group flow of a nonlinear sigma model. By extending Perelman's energy functional ([20]) to this coupled system, Oliynyk, Suneeta, and Woolgar showed that (4.1) is the gradient flow of a nonlinear Schrödinger operator ([19]). To discuss this let us generalize the notation slightly. As in the introduction, let (M^n, g) be a Riemannian manifold, and let H denote a three-form on M. Let

$$\mathcal{F}(g, H, f) = \int_M \left[R - \frac{1}{12}\left|H\right|^2 + \left|\nabla f\right|^2 \right] e^{-f} dV.$$

Furthermore set

$$\lambda(g, H) = \inf_{\{f \mid \int_M e^{-f} dV = 1\}} \mathcal{F}(g, H, f).$$

Proposition 4.10. *([19] Proposition 3.1) The gradient flow of λ is*

$$\begin{aligned} \frac{\partial}{\partial t} g =& -2\operatorname{Rc} + \frac{1}{2}\mathcal{H} - 2\nabla^2 f, \\ \frac{\partial}{\partial t} H =& \Delta_{LB} H - d(\nabla f \lrcorner H), \end{aligned} \tag{4.2}$$

where f satisfies the conjugate heat equation

$$\frac{\partial}{\partial t} f = -\Delta f - R + \frac{1}{4}\left|T\right|^2. \tag{4.3}$$

Furthermore, in [7] Feldman, Ilmanen and Ni gave a generalization of Perelman's steady and shrinking entropies to an entropy modeled on expanding solitons. This expanding entropy has an extension to (4.1), as shown by the first named author. Define

$$\begin{aligned} \mathcal{W}_+(g, H, u, \sigma) =& \int_M \left[\sigma \left(\frac{\left|\nabla u\right|^2}{u} + Ru - \frac{1}{12}\left|H\right|^2 u \right) + u \log u \right] dV \\ =& \int_M \left[\sigma \left(\left|\nabla f_+\right|^2 + R - \frac{1}{12}\left|H\right|^2 \right) - f_+ + n \right] u dV \end{aligned}$$

where f_+ is defined by

$$u = \frac{e^{-f_+}}{(4\pi\sigma)^{\frac{n}{2}}}.$$

Theorem 4.11. *([24] Theorem 6.2) Let $(M^n, g(t), T(t))$ be a solution to (4.1) on $[t_1, t_2]$ and suppose $u(t)$ is the solution to (4.3). Let*

$$v_+ = \left[(t-t_1)(2\Delta f_+ - |\nabla f_+|^2 + R - \frac{1}{12}|H|^2) - f_+ + n\right] u.$$

Then

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \Delta - R + \frac{1}{4}|T|^2\right) v_+ \\
&\qquad = 2(t-t_1)\left(\left|\operatorname{Rc} - \frac{1}{4}\mathcal{H} + \nabla^2 f_+ + \frac{g}{2t}\right|^2 + \frac{1}{4}|d^* H - \nabla f_+ \lrcorner H|^2\right) u + \frac{1}{6}|H|^2 u.
\end{aligned}$$

Corollary 4.12. *Let $(M^n, g(t), H(t))$ be a solution to (4.1) on $[t_1, t_2]$ and suppose $u(t)$ is a solution to the conjugate heat equation. Then*

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{W}_+(g(t), H(t), u(t), t - \tau_1) = &\int_M 2u\left[(t-t_1)\left|\operatorname{Rc} - \frac{1}{4}\mathcal{H} + \nabla^2 f_+ + \frac{g}{2(t-t_1)}\right|^2\right. \\
&\left. + \frac{1}{4}(t-t_1)|d^* H - \nabla f_+ \lrcorner H|^2 + \frac{1}{12}|H|^2\right] dV.
\end{aligned}$$

We can derive further corollaries from these results, akin to the "ruling out of breathers" statements discovered by Perelman ([20]). First recall two definitions.

Definition 4.13. We say that a solution to (4.1) is a *breather* if there are times $t_1 < t_2$, a constant $\alpha > 0$ and a diffeomorphism ϕ such that $\alpha g(t_1) = \phi^* g(t_2)$. The breather is *steady*, *shrinking* or *expanding* if $\alpha = 1, \alpha < 1$, or $\alpha > 1$, respectively.

Definition 4.14. We say that a solution to (4.1) is a *gradient soliton* if there is a function f and a constant λ so that

$$\begin{aligned}
0 &= \operatorname{Rc} - \frac{1}{4}\mathcal{H} + \nabla^2 f - \lambda g \\
0 &= \Delta_{LB} T - d(\nabla f \lrcorner T)
\end{aligned}$$

The soliton is *steady*, *shrinking* or *expanding* if $\lambda = 0, \lambda > 0$, or $\lambda < 0$, respectively.

Corollary 4.15. *Any solution to (4.1) on a compact manifold which is a steady breather is a steady gradient soliton. Any solution to (4.1) on a compact manifold which is an expanding breather is an Einstein metric with $H \equiv 0$.*

Proof. The first statement follows immediately from Proposition 4.10. For the second, we note that Theorem 4.11 clearly implies that an expanding breather is an expanding soliton, and moreover $H \equiv 0$. Thus $g(t)$ is an expanding *Ricci* soliton, which are known to be negative constant Einstein metrics, a result originally due to Hamilton ([12]). □

5. Pluriclosed flow and generalized Kähler geometry

5.1. Introduction to generalized Kähler geometry.

Definition 5.1. A *generalized Kähler manifold* is a Riemannian manifold (M^{2n}, g) together with two complex structures J_+, J_-, each compatible with g, further satisfying

$$\begin{aligned} d^c_+ \omega_+ = -d^c_- \omega_- &= H, \\ dH &= 0. \end{aligned} \tag{5.1}$$

This concept first arose in the work of Gates, Hull, and Roček [8], in their study of $N = (2, 2)$ supersymmetric sigma models. Later these structures were put into the rich context of Hitchin's generalized geometric structures [42] in the thesis of Gualtieri [39] (see also [40]).

Theorem 5.2. *Pluriclosed flow preserves generalized Kähler geometry, suitably interpreted.*

Proof. Consider the Hermitian manifold (M^{2n}, g, J_+). By (5.1), this is a pluriclosed structure, i.e.

$$dd^c_+ \omega_+ = 0. \tag{5.2}$$

By ([48] Theorem 1.2), there exists a solution to pluriclosed flow with initial condition ω_+ on $[0, T)$ for some maximal $T \leq \infty$. Call this one-parameter family of Kähler forms $\omega_+(t)$, and define $\omega_-(t)$ analogously as the solution to pluriclosed flow on the complex manifold (M, J_-) with initial condition ω_-. Next consider the time-dependent vector fields

$$X^{\pm} = \left(-J_{\pm} d^*_{g_{\pm}} \omega_{\pm}\right)^{\sharp_{\pm}}, \tag{5.3}$$

and let $\phi_{\pm}(t)$ denote the one-parameter family of diffeomorphisms of M generated by $X^{\pm}$, with $\phi_0^{\pm} = \mathrm{Id}$. Theorem 1.2 in [49] implies that $(\phi_+(t)^* g_+(t), \phi_+(t)^*(d^c_+ \omega_+(t)))$ is a solution to (4.1) with initial condition $(g, d^c_+ \omega_+)$. Likewise, we have a solution $(\phi_-(t)^* g_-(t), \phi_-(t)^*(d^c_- \omega_-(t)))$ to (4.1) with initial condition $(g, d^c_- \omega_-)$. However, if we let $(\widetilde{g}(t), \widetilde{H}(t))$ denote this latter solution, we observe that

$$\begin{aligned} \frac{\partial}{\partial t} \widetilde{g}_{ij} = &- 2\widetilde{\mathrm{Rc}}_{ij} + \frac{1}{2} \widetilde{H}_{ipq} \widetilde{H}_j{}^{pq} = -2\widetilde{\mathrm{Rc}}_{ij} + \frac{1}{2} \left(-\widetilde{H}_{ipq}\right) \left(-\widetilde{H}_j{}^{pq}\right) \\ \frac{\partial}{\partial t} \left(-\widetilde{H}\right) &= \Delta_d \left(-\widetilde{H}\right), \end{aligned} \tag{5.4}$$

i.e. $(\widetilde{g}(t), -\widetilde{H}(t))$ is a solution to (4.1) with initial condition $(g, -d^c_- \omega_-)$. By (5.1), we see that $(\phi_+(t)^* g_+(t), \phi_+(t)^*(d^c_+ \omega_+(t)))$ and $(\phi_-(t)^* g_-(t), -\phi_-(t)^*(d^c_- \omega_-(t)))$ are two solutions of (4.1) with the same initial condition. Using the uniqueness of solutions of (4.1) ([47] Proposition 3.3), we conclude that these two solutions coincide, and call the resulting one-parameter family $(g(t), H(t))$.

Next we want to identify the two complex structures with which g remains compatible. We observe by that for arbitrary vector fields X, Y,

$$
\begin{aligned}
g\left(\phi_{\pm}(t)^* J_{\pm} X, \phi_{\pm}(t)^* J Y\right) &= g\left(\phi_{\pm}(t)_*^{-1} \cdot J_{\pm} \cdot \phi_{\pm}(t)_* X, \phi_{\pm}(t)_*^{-1} \cdot J_{\pm} \cdot \phi_{\pm}(t)_* Y\right)\\
&= \left[\phi_{\pm}(t)^{-1,*} g\right]\left(J_{\pm} \cdot \phi_{\pm}(t)_* X, J_{\pm} \cdot \phi_{\pm}(t)_* Y\right)\\
&= g_{\pm}\left(J_{\pm} \cdot \phi_{\pm}(t)_* X, J_{\pm} \cdot \phi_{\pm}(t)_* Y\right)\\
&= g_{\pm}\left(\phi_{\pm}(t)_* X, \phi_{\pm}(t)_* Y\right)\\
&= \left[\phi_{\pm}(t)^* g_{\pm}\right](X, Y)\\
&= g(X, Y).
\end{aligned} \tag{5.5}
$$

Therefore $g(t)$ is compatible with $\phi_{\pm}(t)^* J_{\pm}(t)$. Denote these two time dependent complex structures by $\widetilde{J}_{\pm}$. It follows that $\widetilde{\omega_{\pm}} = \phi_{\pm}(t)^* \omega_{\pm}$. Next we note by naturality of d that

$$
\begin{aligned}
\widetilde{d^c_{\pm}} \widetilde{\omega_{\pm}}(X, Y, Z) &= -\left[d \widetilde{\omega_{\pm}}\right]\left(\widetilde{J}_{\pm} X, \widetilde{J}_{\pm} Y, \widetilde{J}_{\pm} Z\right)\\
&= -\left[d \phi_{\pm}(t)^* \omega_{\pm}\right]\left(\phi_{\pm}(t)_*^{-1} \cdot J_{\pm} \cdot \phi_{\pm}(t)_* X, \cdots\right)\\
&= \left[\phi_{\pm}(t)^*\left(-d \omega_{\pm}\right)\right]\left(\phi_{\pm}(t)_*^{-1} \cdot J_{\pm} \cdot \phi_{\pm}(t)_* X, \cdots\right)\\
&= -d \omega_{\pm}\left(J_{\pm} \cdot \phi_{\pm}(t)_* X, \cdots\right)\\
&= d^c_{\pm} \omega_{\pm}\left(\phi_{\pm}(t) X, \cdots\right)\\
&= \phi_{\pm}(t)^*\left(d^c_{\pm} \omega_{\pm}\right)(X, Y, Z)\\
&= \pm H(X, Y, Z).
\end{aligned} \tag{5.6}
$$

It follows that

$$
\widetilde{d^c_{+}} \widetilde{\omega_{+}} = -\widetilde{d^c_{-}} \widetilde{\omega_{-}} = H, \qquad dH = 0, \tag{5.7}
$$

showing that the triple $(g(t), \widetilde{J}_{+}(t), \widetilde{J}_{-}(t))$ is generalized Kähler for all time. $\square$

Remark 5.3. This theorem points out an important point regarding (4.1). Based on physical intuition, one hopes that the system (4.1) preserves generalized Kähler geometry. Theorem 5.2 says that this is true, *only if we allow the complex structures to flow as well.* We discovered this via the use of pluriclosed flow, but it is also interesting to express this purely using (4.1), by augmenting it with a flow of complex structures.

Proposition 5.4. *Let $(M^{2n}, \widetilde{g}(t), J)$ be a solution to the pluriclosed flow. Let ϕ_t be the one parameter family of diffeomorphisms generated by $\left(-J d^*_{\widetilde{g}} \widetilde{\omega}\right)^{\sharp}$ with $\phi_0 = \mathrm{Id}$, and let $g(t) = \phi_t^*(\widetilde{g}(t)), J(t) = \phi_t^*(J)$. Then*

$$
\begin{aligned}
\frac{\partial}{\partial t} J_k^l = &\left(\Delta J\right)_k^l - \left[J, g^{-1} \operatorname{Rc}\right]_k^l\\
&- J_k^p D^s J_i^l D_p J_s^i - J_i^l D^s J_k^p D_p J_s^i + J_s^p D^s J_i^l D_p J_k^i + J_i^l D^s J_s^p D_p J_k^i\\
&- J_p^l D_k J_t^p D^s J_s^t + J_k^p D_p J_t^l D^s J_s^t - J_t^p D^s J_s^t D_p J_k^l.
\end{aligned} \tag{5.8}
$$

With this proposition in hand we can add an equation to the B-field flow system to yield a new system of equations which preserves the generalized Kähler condition. Specifically, given a Riemannian manifold (M^n, g) and $J \in \mathrm{End}(TM)$, let

$$\begin{aligned}
\mathcal{R}(J)_k^l &= [J, g^{-1}\,\mathrm{Rc}]_k^l \\
\mathcal{Q}(DJ)_k^l &= -J_k^p D^s J_i^l D_p J_s^i - J_i^l D^s J_k^p D_p J_s^i + J_s^p D^s J_i^l D_p J_k^i + J_i^l D^s J_s^p D_p J_k^i \\
&\quad - J_p^l D_k J_t^p D^s J_s^t + J_k^p D_p J_t^l D^s J_s^t - J_t^p D^s J_s^t D_p J_k^l.
\end{aligned} \tag{5.9}$$

Now consider the system of equations for an a priori unrelated Riemannian metric g, three-form H, and tangent bundle endomorphisms $J_\pm$:

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -2\,\mathrm{Rc}_{ij} + \frac{1}{2} H_{ipq} H_j{}^{pq} \\
\frac{\partial}{\partial t} H &= \Delta_d H, \\
\frac{\partial}{\partial t} J_\pm &= \Delta J_\pm + \mathcal{R}(J_\pm) + \mathcal{Q}(DJ_\pm).
\end{aligned} \tag{5.10}$$

Remark 5.5. Adapting Theorem 5.2, it is clear then that with the appropriate identifications, the system (5.10) preserves generalized Kähler structure.

6. T-duality and pluriclosed flow

In this section we present an interesting symmetry of the B-field RG flow encountered above. We recall the basic setup here. Let (M^n, g) be a Riemannian manifold and let $H_0 \in \Lambda^3(T^*M)$, $dH_0 = 0$. Given this setup and $b \in \Lambda^2(M)$ we set $H = H_0 + db$. The *B-field renormalization group flow* is the system of equations

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -2\,\mathrm{Rc}_{ij} + \frac{1}{2} H_{ipq} H_j{}^{pq}, \\
\frac{\partial}{\partial t} b &= -d_g^* H.
\end{aligned} \tag{6.1}$$

For the sequel we require a gauge-fixed version of this flow. In particular, given the above setup and a one-parameter family of functions f_t, consider

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -2\,\mathrm{Rc}_{ij} + \frac{1}{2} H_{ipq} H_j{}^{pq} + (L_{\nabla f} g)_{ij}, \\
\frac{\partial}{\partial t} b &= -d_g^* H + i_{\nabla f} \lrcorner H.
\end{aligned} \tag{6.2}$$

6.1. **Topological T-duality.** In this section we recall some background on the topological aspect of T-duality. Our discussion here follows closely the work of Cavalcanti-Gualtieri [35].

Definition 6.1. Let M, $\overline{M}$ be principal T^k bundles over a common base manifold B, and let $H \in \Omega^3_{T^k}(M)$ and $\overline{H} \in \Omega^3_{T^k}(\overline{M})$ be invariant closed forms, and finally let θ and $\overline{\theta}$ denote connection 1-forms on M and $\overline{M}$. Consider $M \times_B \overline{M}$ the fiber product of M and $\overline{M}$, with projection maps $p : M \times_B \overline{M} \to M, \overline{p} : M \times_B \overline{M} \to \overline{M}$. We say that (M, H, θ) and $(\overline{M}, \overline{H}, \overline{\theta})$ are *topologically T-dual* if

$$p^* H - \overline{p}^* \overline{H} = d(p^*\theta \wedge \overline{p}^*\overline{\theta}). \tag{6.3}$$

Remark 6.2. While as written this definition requires specific choices of H and $\overline{H}$, the definition only depends on the cohomology classes $[H]$ and $[\overline{H}]$. Specifically, if (M, H, θ) and $(\overline{M}, \overline{H}, \overline{\theta})$ are T-dual, and we set $H' = H + db$, with $b \in \Omega^2_{T^k}(M)$, there exists a new connection θ' on M and also $\overline{H}', \overline{\theta}'$ on $\overline{M}$ such that for the quadruple $(H', \theta', \overline{H}', \overline{\theta}')$ the relation (6.3) holds. In particular, as a corollary of Lemma 6.13 we may choose any S^1-invariant metric g whose induced connection 1-form is θ and then take the T-dual data to (g, b) provides the requisite data.

Theorem 6.3. *([30] Theorem 3.1) If (M, H) and $(\overline{M}, \overline{H})$ are T-dual with $p^*H - \overline{p}^*\overline{H} = dF$, then*

$$\tau : (\Omega_{T^k}(M), d_H) \to (\Omega_{T^k}(\overline{M}), d_{\overline{H}}), \quad \tau(\rho) = \int_{T^k} e^F \wedge \rho \tag{6.4}$$

is an isomorphism of differential complexes, where the integration is along the fibers of $M \times_B \overline{M} \to \overline{M}$.

Remark 6.4. The map τ is a map on the Clifford module of T^k-invariant forms. To show that it is an isomorphism of Clifford modules we require an isomorphism $\phi : (TM \oplus T^*M)/T^k \to (T\overline{M} \oplus T^*\overline{M})/T^k$, which we define next.

Definition 6.5. Given $(X + \xi) \in (TM \oplus T^*M)/T^k$, choose the unique lift $\hat{X}$ of X to $T(M \times \overline{M})$ such that

$$p^*\xi(Y) - F(\hat{X}, Y) = 0, \qquad \text{for all } Y \in \mathfrak{t}^k_M$$

Due to this condition the form $p^*\xi - F(\hat{X}, \cdot)$ is basic for the bundle determined by $\overline{p}$, and can therefore be pushed forward to $\overline{M}$. We define a map

$$\phi(X + \xi) = \overline{p}_*(\hat{X}) + p^*\xi - F(\hat{X}, \cdot).$$

Lemma 6.6. *The map ϕ defined above depends only on $[H]$ and $[\overline{H}]$.*

Proof. Following the discussion in Remark 6.2, if $H' = H + dB$ then

$$p^*H' - \overline{p}^*\overline{H} = d(F + p^*B) =: dF'$$

Moreover, the action of p^*B on $\mathfrak{t}^k_M \otimes \mathfrak{t}^k_{\overline{M}}$ is trivial. Hence when lifting vectors to the configuration space as in Definition 6.5, using either F or F' yields the same result, and so the lemma follows. □

6.2. **Geometric T-duality.** In this section we present the notion of T-duality for generalized metrics. We take as background data topologically T-dual S^1-bundles (M, H, θ) and $(\overline{M}, \overline{H}, \overline{\theta})$. The metric data then consists of an S^1-invariant metric g on M and an S^1-invariant two-form b on M. In [32], [33] Buscher discovered a way to transform this data, as well as an auxiliary dilaton, to the manifold $\overline{M}$ in such a way that fixed points of (6.1) on M are transformed into fixed points of (6.2) with a particular choice of f_t on $\overline{M}_t$. The content of Theorem 6.17 is to show that this behavior persists for general solutions of (6.1).

Definition 6.7. Let (M, H, θ) and $(\overline{M}, \overline{H}, \overline{\theta})$ be T-dual. Given Γ a generalized metric on $(TM \oplus T^*M)/T^k$, the *dual metric* is

$$\overline{\Gamma} := \phi\Gamma\phi^{-1} \tag{6.5}$$

Remark 6.8. The simplicity of this definition illustrates the value of adopting the viewpoint of Courant algebroids. Indeed, using the map ϕ it is possible to easily define T-duality transformations for other natural objects such as generalized complex structures. By working out the induced map on (g, b) one recovers the famous "Buscher rules," [32], [33], which we now record.

Given (M, H, θ) and $(\overline{M}, \overline{H}, \overline{\theta})$ T-dual bundles with connections θ and $\overline{\theta}$, recall that an S^1-invariant generalized metric Γ is determined by an S^1 invariant pair (g, b) of metric and two-form potential on M, which can be expressed as

$$\begin{aligned} g &= g_0 \theta \odot \theta + g_1 \odot \theta + g_2 \\ b &= b_1 \wedge \theta + b_2 \end{aligned} \tag{6.6}$$

where g_i and b_i are basic forms of degree i.

Lemma 6.9. *(**Buscher Rules**) Suppose (M, H, θ) and $(\overline{M}, \overline{H}, \overline{\theta})$ are topologically T-dual. Given Γ an S^1-invariant generalized metric on $TM \oplus T^*M$ and $\overline{\Gamma} = \phi\Gamma\phi^{-1}$ the dual metric on $T\overline{M} \oplus T^*\overline{M}$, if the pair (g, b) associated to Γ is given by (6.6), then the pair $(\overline{g}, \overline{b})$ determined by Γ takes the form*

$$\begin{aligned} \overline{g} &= \frac{1}{g_0}\overline{\theta} \odot \overline{\theta} - \frac{b_1}{g_0} \odot \overline{\theta} + g_2 + \frac{b_1 \odot b_1 - g_1 \odot g_1}{g_0} \\ \overline{b} &= -\frac{g_1}{g_0} \wedge \overline{\theta} + b_2 + \frac{g_1 \wedge b_1}{g_0}. \end{aligned} \tag{6.7}$$

For the calculations to come later, it will be fruitful to give yet another version of the T-duality relationship explicitly in terms of the canonical decomposition of an S^1-invariant pair (g, b) on a principal bundle which we now record.

Lemma 6.10. *A S^1-invariant metric on a principal bundle with canonical vector field e_θ is uniquely determined by a base metric, a family of fiber metrics, and a connection. More precisely, g may be uniquely expressed*

$$g = \phi\theta \otimes \theta + h$$

where

$$\begin{aligned} \phi &= g(e_\theta, e_\theta) \\ \theta &= \frac{g(e_\theta, \cdot)}{g(e_\theta, e_\theta)} \\ h(\cdot, \cdot) &= g(\pi^\theta \cdot, \pi^\theta \cdot), \end{aligned}$$

and here π^θ is the horizontal projection determined by θ, i.e.

$$\pi^\theta(X) = X - \theta(X)e_\theta.$$

Lemma 6.11. *Let M denote the total space of an S^1 principal bundle. Given θ a connection on M, an S^1 invariant two-form b admits a unique decomposition*

$$b = \theta \wedge \eta + \mu$$

where η and μ are basic forms.

Proof. Let $\eta = e_\theta \lrcorner b$. Obviously $\eta(e_\theta) = 0$ and so η is basic. We may then declare

$$\mu = b - \theta \wedge \eta$$

Observe that

$$e_\theta \lrcorner \mu = e_\theta \lrcorner b - e_\theta \lrcorner (\theta \wedge \eta) = \eta - \eta = 0,$$

so that μ is basic as well. □

Proposition 6.12. *Let (M, H, θ) and $(\overline{M}, \overline{H}, \overline{\theta})$ be topologically T-dual, and suppose (g, b) is dual to $(\overline{g}, \overline{b})$. Let θ_g, ϕ_g, h_g denote the connection 1-form, fiber metric, and base metric determined by g via Lemma 6.10. Furthermore, let η_g and μ_g denote the basic 1-form and 2-form associated to b and θ_g via Lemma 6.11. Then if $\theta_{\overline{g}}$, etc. denote the corresponding data associated to $\overline{g}$, one has*

$$\begin{aligned}
\phi_{\overline{g}} &= \frac{1}{\phi_g} \\
\theta_{\overline{g}} &= \overline{\theta} + \eta_g \\
h_{\overline{g}} &= h_g \\
\eta_{\overline{g}} &= \theta_g - \theta \\
\mu_{\overline{g}} &= \mu_g - \eta_g \wedge \eta_{\overline{g}}.
\end{aligned}$$

Proof. First we compute

$$\theta_g = \theta + \frac{g_1}{g_0}.$$

Then we obtain

$$\eta_g = e_\theta \lrcorner b = \ - b_1.$$

Then we may express

$$\begin{aligned}
\mu_g &= b - \theta_g \wedge \eta_g \\
&= b_1 \wedge \theta - \left(\theta + \frac{g_1}{g_0} \right) \wedge (-b_1) + b_2 \\
&= b_2 + \frac{g_1}{g_0} \wedge b_1.
\end{aligned}$$

Furthermore we obtain

$$\overline{\theta}_{\overline{g}} = \overline{\theta} - b_1 = \overline{\theta} + \eta_g$$

Then, according to the Buscher rules,

$$\begin{aligned}\eta_{\overline{g}} &= \overline{e_\theta} \lrcorner b \\ &= \frac{g_1}{g_0} \\ &= \theta_g - \theta.\end{aligned}$$

Then we obtain

$$\begin{aligned}\mu_{\overline{g}} &= \overline{b} - \overline{\theta}_{\overline{g}} \wedge \eta_{\overline{g}} \\ &= -\frac{g_1}{g_0} \wedge \overline{\theta} + b_2 + \frac{g_1 \wedge b_1}{g_0} - \left(\overline{\theta} + \eta_g\right) \wedge (\theta_g - \theta) \\ &= (\overline{\theta} - b_1) \wedge \frac{g_1}{g_0} + b_2 - \overline{\theta}_{\overline{g}} \wedge (\frac{g_1}{g_0}) \\ &= b_2 \\ &= \mu_g - \frac{g_1}{g_0} \wedge b_1 \\ &= \mu_g - \eta_g \wedge \eta_{\overline{g}}.\end{aligned}$$

□

Lemma 6.13. *Let (M, H, θ) and $(\overline{M}, \overline{H}, \overline{\theta})$ be topologically T-dual, and suppose (g, b) is dual to $(\overline{g}, \overline{b})$. Then (6.3) holds for the quadruple $(H_b, \theta_g, \overline{H}_{\overline{b}}, \overline{\theta}_{\overline{g}})$.*

Proof. We directly compute (suppressing the presence of p^* and $\overline{p}^*$) using Proposition 6.12 that

$$\begin{aligned}H_b - \overline{H}_{\overline{b}} &= H + db - \overline{H} - d\overline{b} \\ &= H - \overline{H} + d\left(\theta_g \wedge \eta_g + \mu_g\right) - d\left(\overline{\theta}_{\overline{g}} \wedge \eta_{\overline{g}} + \mu_{\overline{g}}\right) \\ &= d\left(\theta \wedge \overline{\theta} + \theta_g \wedge \eta_g - \overline{\theta}_{\overline{g}} \wedge \eta_{\overline{g}} + \eta_g \wedge \eta_{\overline{g}}\right) \\ &= d(\theta_g \wedge \overline{\theta}_{\overline{g}}).\end{aligned}$$

□

Lemma 6.14. *Given (g, b) and $(\overline{g}, \overline{b})$ T-dual data, if we declare θ_g and $\overline{\theta}_g$ to be the background connections, which is valid by Lemma 6.13, then the pair $(g, 0)$ and $(\overline{g}, 0)$ is T-dual with respect to this background.*

Proof. This follows immediately from Proposition 6.12. □

Lemma 6.15. *If θ denotes a choice of connection, given H an S^1-invariant three-form, H admits a unique decomposition*

$$H = \theta \wedge Y + Z$$

where Y and Z are basic forms.

Proof. Following the proof of Lemma 6.11 we let $Y = e_\theta \lrcorner H$ and $Z = H - \theta \wedge Y$ and this is the required decomposition. □

Next we relate the three-form decomposition of Lemma 6.15 for T-dual structures.

Lemma 6.16. *Let (M, g, b) and $(\overline{M}, \overline{g}, \overline{b})$ be T-dual data. Then*
$$Z = \overline{Z}, \qquad Y = -\overline{F}_{\overline{\theta}}, \qquad \overline{Y} = -F_{\theta}.$$

Proof. Let $\widetilde{e_\theta}$ denote the vector field defining the action of S^1 coming from the bundle M induced on the fiber product $M \times_{S^1} \overline{M}$. Likewise define $\widetilde{\overline{e_\theta}}$. We compute
$$\begin{aligned}
\pi^* Y &= \pi^* \left(e_\theta \lrcorner H \right) \\
&= \widetilde{e_\theta} \lrcorner \pi^* H \\
&= \widetilde{e_\theta} \lrcorner \left(\overline{\pi}^* \overline{H} + d(\theta_g \wedge \overline{\theta}_{\overline{g}}) \right) \\
&= \widetilde{e_\theta} \lrcorner \left(F_\theta \wedge \overline{\theta}_{\overline{g}} - \theta_g \wedge \overline{F}_{\overline{\theta}} \right) \\
&= \ - \overline{F}_{\overline{\theta}}.
\end{aligned}$$
The calculation of $\overline{\pi}^* \overline{Y}$ is identical. Finally we have
$$\begin{aligned}
\pi^* Z &= \pi^* \left(H - \theta \wedge Y \right) \\
&= \overline{\pi}^* \overline{H} + d \left(\theta_g \wedge \overline{\theta}_{\overline{g}} \right) + \theta_g \wedge \overline{F}_{\overline{\theta}}. \\
&= \overline{\pi}^* \overline{H} + F_g \wedge \overline{\theta}_{\overline{g}} \\
&= \overline{\pi}^* \overline{H} + \overline{\theta}_g \wedge F_\theta \\
&= \overline{\pi}^* \overline{H} - \overline{\theta} \wedge \overline{Y} \\
&= \overline{\pi}^* \overline{Z}.
\end{aligned}$$
□

6.3. **Statement of Theorem and examples.**

Theorem 6.17. *Suppose (M^n, H, θ) and $(\overline{M}, \overline{H}, \overline{\theta})$ are topologically T-dual circle bundles (cf. Definition 6.3). Given (g, b) an S^1-invariant pair of metric and two-form, and f_t a one-parameter family of S^1-invariant functions, let (g_t, b_t) be the unique solution to (6.2) with this initial condition. Let $(\overline{g}_t, \overline{b}_t)$ denote the one-parameter family of T-dual pairs to (g_t, b_t). Then $(\overline{g}_t, \overline{b}_t)$ is the unique solution to (6.2) with initial condition $(\overline{g}, \overline{b})$ with $\overline{f}_t = f_t + \log \phi_t$, where $\phi_t = g_t(e_\theta, e_\theta)$ is the function determining the length of the circle fiber on M at each time t.*

Example 6.18. We begin with a simple example to illustrate how T-duality affects solutions to (6.1). Let $M \cong S^3$ and consider the Hopf fibration $S^1 \to S^3 \to S^2$, and let θ denote the connection one form on S^3 satisfying $d\theta = \omega_{S^2}$, where ω_{S^2} denotes the standard area form on S^2, and furthermore let $H = 0$. Next let $\overline{M} \cong S^2 \times S^1$, and consider the trivial fibration $S^1 \to S^1 \times S^2 \to S^2$. Let $\overline{\theta}$ denote the pullback of the canonical line element on S^1 to $\overline{M}$, and let $\overline{H} = -\overline{\theta} \wedge \omega_{S^2}$. Certainly $d\overline{H} = 0$. Moreover, with the notation of §6.1, observe that
$$p^* H - \overline{p}^* \overline{H} = \overline{p}^* \left(\omega_{S^2} \wedge \overline{\theta} \right) = dp^* \theta \wedge \overline{\theta} = d \left(p^* \theta \wedge \overline{p}^* \overline{\theta} \right).$$
Thus (M, H, θ) and $(\overline{M}, \overline{H}, \overline{\theta})$ are topologically T-dual. Let g_{S^2} denote the round metric on S^2 and consider an S^1-invariant metric of the form
$$g = A \theta \otimes \theta + B g_{S^2}.$$

Observe that by applying Proposition 6.12 we obtain that $(g,0)$ is T-dual to $(\overline{g},\overline{b})$ with

$$\begin{aligned} \overline{g} &= \frac{1}{A}\overline{\theta}\otimes\overline{\theta} + Bg_{S^2}, \\ \overline{b} &= 0. \end{aligned} \tag{6.8}$$

The solution to (6.1) with initial condition $(g,0)$ on M is given by the Ricci flow, which takes the form

$$\dot{A} = -\frac{A^2}{B^2}, \qquad \dot{B} = -2 + \frac{A}{B}.$$

Expressing the T-dual data as $\overline{g} = \overline{A\theta}\otimes\overline{\theta} + \overline{B}g_{S^2}$ and using (6.8) we obtain the evolution equation for $\overline{g}$ as

$$\dot{\overline{A}} = \frac{1}{B^2}, \qquad \dot{\overline{B}} = -2 + \frac{A}{B},$$

which, one directly checks is the solution to (6.1). Observe that M shrinks to a round point under the flow, whereas on $\overline{M}$ the S^2 shrinks to a point while the S^1 fiber blows up.

Example 6.19. More generally, we may let $M \cong S^{2n+1}$ and consider the Hopf fibration $S^1 \to S^{2n+1} \to \mathbb{CP}^n$, and let θ denote the connection one form on S^{2n+1} satisfying $d\theta = \omega_{FS}$, where ω_{FS} is the Kähler form of the Fubini-Study metric on $\mathbb{CP}^n$, and furthermore let $H = 0$. Next let $\overline{M} \cong \mathbb{CP}^n \times S^1$, and consider the trivial fibration $S^1 \to S^1 \times \mathbb{CP}^n \to \mathbb{CP}^n$. Let $\overline{\theta}$ denote the pullback of the canonical line element on S^1 to $\overline{M}$, and let $\overline{H} = -\overline{\theta}\wedge\omega_{FS}$. As in the previous example one easily checks that (M,H,θ) and $(\overline{M},\overline{H},\overline{\theta})$ are topologically T-dual.

Now let g_0 denote any metric on S^{2n+1} with positive curvature operator. Consider the solution to (6.1) with initial condition $(g_0,0)$. One observes that by the maximum principle the condition $H_0 \equiv 0$ is preserved by (6.1), and so the solution $(g_t,b_t) = (g_t,0)$, where g_t is the unique solution to Ricci flow with initial condition g_0. By the theorem of Bohm-Wilking [29], we have that g_t exists on some finite time interval $[0,T)$, and converges to a round point as $t \to T$. It follows from Proposition 6.12 that the dual solution $(\overline{g}_t,\overline{b}_t)$ also exists on a finite time interval, asymptotically converging to a solution which homothetically shrinks the $\mathbb{CP}^2$ base and expands the S^1 fiber, analogously to the previous example.

7. Symplectic curvature flow

Much effort is made in recent years fleshing out the analogies/similarities/differences between symplectic geometry and Kähler geometry. Of course Kähler geometry is certainly more rigid, but nonetheless similar ideas play a role in both fields. In this section we outline a method for extending Kähler-Ricci flow into the world of symplectic geometry. As Kähler-Ricci flow of course demands a complex structure, our symplectic curvature flow will require an *almost complex structure*, which we now define.

Definition 7.1. Let M^{2n} be a smooth manifold. An *almost complex structure* on M is an endomorphism of the tangent bundle J covering the identity map satisfying

$$J^2 = -\operatorname{Id}.$$

The pair (M^{2n},J) is called an *almost complex manifold.*

Remark 7.2. The restriction to an even dimensional manifold is of course necessary for the existence of an endomorphism which squares to $-\operatorname{Id}$.

Definition 7.3. Given (M^{2n}, J) an almost complex manifold, the *Nijenhuis tensor of J* is

$$N(X,Y) = [JX, JY] - [X,Y] - J[JX,Y] - J[X,JY].$$

Remark 7.4. Different authors may define the Nijenhuis tensor to be a different multiple of our definition. Observe that the Nijenhuis tensor is a first order differential operator acting on an almost complex structure J.

Definition 7.5. Let (M^{2n}, J) be an almost complex manifold. The almost complex structure J is *integrable* if $N \equiv 0$. In this case we say that (M^{2n}, J) is a *complex manifold.*

Remark 7.6. It follows from the Newlander-Nirenberg Theorem that the vanishing of the Nijenhuis tensor is equivalent to the existence of a complex coordinate atlas, i.e. complex coordinate charts covering the manifold with biholomorphic transition maps.

Definition 7.7. Given M^{2n} a smooth manifold, a *symplectic form* on M is $\omega \in \Lambda^2(M)$ such that $d\omega = 0$ and ω is nondegenerate, i.e. for all $p \in M$, $\omega^n \neq 0$.

Definition 7.8. Let (M^{2n}, ω) be a symplectic manifold. An almost complex structure J is *compatible with ω* if

$$\omega(J,J) = \omega, \qquad \omega(J, \cdot) > 0$$

Proposition 7.9. *(Gromov) Every symplectic structure admits a compatible almost complex structure.*

Definition 7.10. A triple (M^{2n}, ω, J) of a symplectic form with a compatible almost complex structure is an *almost Kähler* structure. Observe that we also have a Riemannian metric defined by $g(X,Y) = \omega(JX, Y)$.

Lemma 7.11. *Let (M^{2n}, ω, J) be an almost Kähler structure. There is a unique linear connection ∇ on TM satisfying*

$$\nabla\omega \equiv 0, \qquad \nabla J \equiv 0, \qquad T^{1,1} \equiv 0,$$

where $T^{1,1}$ is the $(1,1)$-component of the torsion. We will call this the Chern connection.

Remark 7.12. By general principles the Chern connection can be used to generate a representative of the first Chern class, namely

$$P_{ij} := \Omega_{ijk}^{l} J_{l}^{k}.$$

This is a closed form by the Bianchi identity, represents $c_1(M,J)$, and agrees with the usual Ricci form in Kähler geometry if J is integrable. This suggests the evolution equation

$$\frac{\partial}{\partial t}\omega = -P. \tag{7.1}$$

However, in general, $P \notin \Lambda_{\mathbb{R}}^{1,1}$, and therefore this equation will not preserve the compatibility of ω with J. However, a simple solution suggests itself, which is to allow J to flow so as to preserve compatibility.

$$\frac{\partial}{\partial t} J = \, - g^{-1} P^{(2,0)+(0,2)} \tag{7.2}$$

However, after some delicate calculations, one can see that the coupled system of (7.1-7.2) is not even (weakly) parabolic. However, we have not made full use of the J evolution equation. In particular, one can add an arbitrary J-skew symmetric, g-symmetric piece to the evolution equation while still preserving compatibility. Again, after delicate calculations, what is suggested is to use

$$(\mathcal{R})_i^j := J_i^k \operatorname{Rc}_k^j - \operatorname{Rc}_i^k J_k^j$$

Definition 7.13. Let (M^{2n}, ω_0, J_0) be an almost Kähler manifold. We say that a one-parameter family (ω_t, J_t) is a solution to *symplectic curvature flow* with initial condition (ω_0, J_0) if it satisfies

$$\begin{aligned} \frac{\partial}{\partial t}\omega &= -P \\ \frac{\partial}{\partial t}J &= -g^{-1}P^{(2,0)+(0,2)} + \mathcal{R} \end{aligned} \tag{7.3}$$

Theorem 7.14. *(*[50] *Theorem 1.6) Let (M^{2n}, ω_0, J_0) be a compact almost Kähler manifold. There exists $\epsilon > 0$ and a unique one-parameter family of almost Kähler structures $(\omega(t), J(t))$ solving (7.3) for $t \in [0, \epsilon)$. If J_0 is integrable, $J(t) = J(0)$ for all t and $\omega(t)$ is a solution to Kähler Ricci flow.*

Remark 7.15. Since the complex structure is not fixed background data, the evolution equation (7.3) admits an action by the diffeomorphism group, and is therefore only weakly parabolic. The so-called "DeTurck method" succeeds in this case to show short-time existence.

Theorem 7.16. *Let (M^{2n}, ω_0, J_0) be an almost Kähler manifold. There is a unique solution to (7.3) on a maximal time interval $[0, \tau)$. Furthermore, if $\tau < \infty$ then*

$$\limsup_{t \to \tau} \left| \operatorname{Rm} \right|_{C^0} = \infty.$$

Given this starting point, many natural questions of analytic or geometric nature present themselves. Certainly one would want to extend as much of the Perelman theory of Ricci flow to this setting as possible. One fundamental question is:

Question 7.17. Is (7.3) the gradient flow of a natural functional?

Ideally, one would like to use (7.3) to answer questions arising in symplectic topology. There are many questions asked concerning the topological structure of the space of symplectic forms in a fixed cohomology class. Certainly geometric evolution equations are a natural tool for approaching such questions. Here is a representative folklore conjecture:

Conjecture 7.18. *Let M be a closed hyperkähler surface (i.e. a four-torus or a $K3$ surface) and let $a \in H^2(M; \mathbb{R})$ be a cohomology class such that $a^2 > 0$. Then the space $\mathcal{S}_a$ of symplectic forms in the class a is connected.*

A natural approach to this conjecture would be to show that in the situation described in the conjecture, an arbitrary symplectic form with compatible J flows under (7.3) to a "standard" structure. A similar strategy was suggesting using a more specialized flow in [6].

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