ELLIPTIC SOLITONS IN OPTICAL MEDIA

Dikandé Alain Moïse
Laboratory of Research on Advanced Materials and Nonlinear Science (LaRAMaNS), Department of Physics, Faculty of Science, University of Buea
(laramans.blogspot.com)

06/11/2015

OUTLINES

Our Department in brief

- Created in 1993 (with the University of Buea)
- First year of M.Sc. degree program: 2004/2005
- We offer a Ph.D degree programme since 2010
- Staff strength: 12 academic + 1 Administrative Assistant
- Number of current postgraduate students: 13 M.Sc. and 10 Ph.D.

Our Department in brief

Research activities:

- Main research themes: Nonlinear dynamical system analysis and theoretical Condensed Matter Physics.
- Structure of research: there is one main research group, LaRAMaNS
- Main sources of funding: contributions from members, research grants
- Research facilities: a small numerical simulation lab (include one HP Proliant DL 360 G7 Server, 08 terminals, one HP Multifonction printer)

Links with ICTP

Links with ICTP are mainly of three types:

- Assistance with literature (ejds)
- ICTP Associateship scheme
- student trainings (Diploma programs)
- Attendances of staff to some ICTP activities

Associateship of the ICTP

Dikande A. M.: "Regular Associate" (1996 to 2012), Moukam Kakmeni: Junior Associate (2007 to 2012).

Woafo (left) and Dikande (right), Summer 1997 (Main Building)

Recent visits and seminars

Most recent visitors of LaRAMaNS:

- Alexander Sunder-Meyer, Director of Physics Department, Xavier University, Louisiana (2 weeks in June 2014)
- Fritz Ahane, President of STEM committee, Academy of Science of South Africa (October 2014)
- Neil Turok, Direcor, Perimeter Institute, Waterloo (2015)
- Cedric Villani, (FIELDS medalist 2010), Institute Pointcaré, France (2015)
- Kenfack Anatole, Free University of Berlin (2015)

Recent visits

Fritz Ahane in Laramans research lab.

Recent visits

Neil Turok and Cedric Villani.

Recent visits

Anatole Kenfack

Recall on soliton theory

In Mathematical Physics, a soliton is the solution to a specific class of wave equations characterized by the competition between:

- dispersion that tends to spread the wave,
- and nonlinearity that tends to keep harmonics all together.

In optics, the propagation of an EM wave $A(z,t)$ in a $1D$ dispersive linear medium can be described by the paraxial wave equation:

$$iA_z = \beta A_{tt} - n_0 A, \tag{1}$$

where β is the dispersion coefficient and n_0, the index of refraction, is constant for a passive (homogeneous or linear) optical medium. Solutions to the dispersive linear wave equation are in general:

$$A(z,t) = A_0 sin[kz \pm \omega_\beta(k)t] \tag{2}$$

Recall on soliton theory

If the medium responds instantaneously to field propagation (optically active media) the refractive index becomes a function of the propagating wave intensity (field-dependent polarization). In this case the wave equation now reads:

$$iA_z = \beta A_{tt} - n(I)A, \quad I = |A|^2. \tag{3}$$

The precize $I - n$ characteristics for a specific nonlinear optical medium is determined from experiments. However, for mathematical tractability we use a Taylor expansion in I where the minimum term accounting for nonlinearity is the Kerr (i.e. linear) term.

Recall on soliton theory

Thus, if we write $n(I) \propto n_0 + n_2 I + 0(I^2)$ and rescale the reflractive index with respect to n_0, eq. (3) is the Nonlinear Schrödinger equation (NLSE):

$$iA_z = \beta A_{tt} - n_2 |A|^2 A. \tag{4}$$

For negative dispersion coefficient and positive n_2, the NLSE admits a modulated, localized envelope wave solution:

$$A(z,t) = A_0 sech[(z - vt)/\lambda] exp[kz \pm \omega_\beta(k)t]. \tag{5}$$

with λ the "hyperbolic secant" envelope width, $\omega_\beta(k)$ and k the modulation frequency and wavector respectively,

Recall on soliton theory

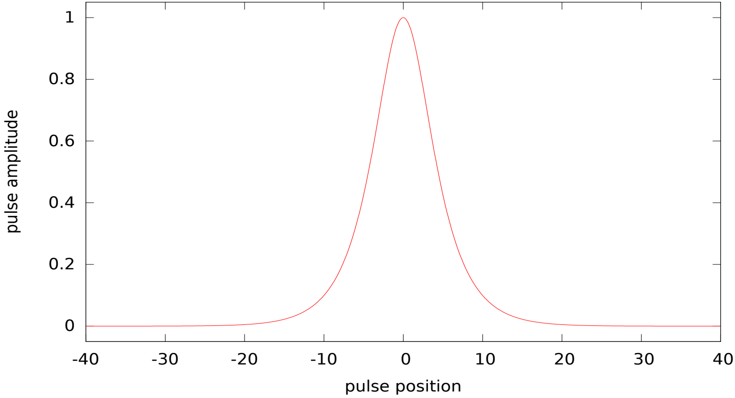


Fig.1: Pulse envelope profile.

Elliptic solitons: characteristic features

While (5) describes single-pulse soliton signal, elliptic solitons are specific nonlinear optical signals characterized essentially by:

- a periodic structure made up of a large number of localized signals (bright or dark-profile solitons), arranged such as to form a "periodic lattice" of pulses,
- a relatively lower amplitude of the constituent single-soliton signals,
- a tunable shape profile, that allows a smooth crossover from cw to pulse regimes of operation.

An elliptic soliton in a fiber laser

Examples of periodic soliton-lattice signals in fiber laser devices:
I. Harmonically mode-locked periodic soliton lattice (Tang et al., Phys. Rev. Lett. 101, p. 153904, 2008)

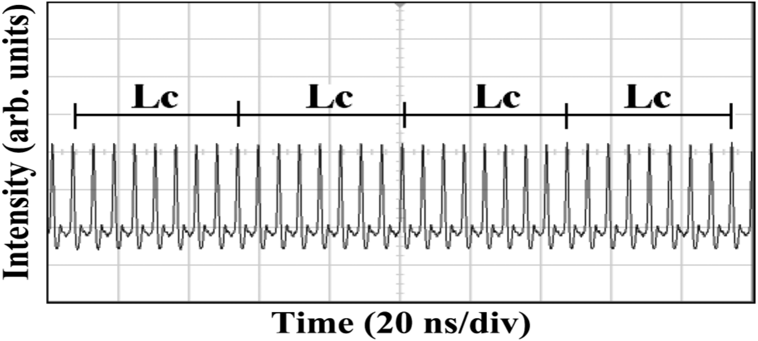


Fig. 2

An elliptic soliton in a fiber laser

II. A soliton crystal and polycrystal in passively mode-locked fiber lasers with saturable absorber (Haboucha et al., Phys. Rev. A78, p. 043806, 2008).

soliton crystal

(a)

soliton polycrystal

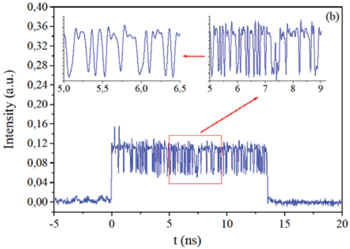


Fig.3: Sequences of soliton crystals (left) and polycristals (right)

Elliptic solitons: generating mechanisms

Elliptic solitons can be generated via two different mechanisms:

- the phenomenon of modulational instability (Benjamin and Feir, J. Fluid Mech. 27, p.59, 1957): a cw signal becomes unstable in a nonlinear optical medium as the input power increases,
- the phenomenon of signal multiplexing, which gives rise to artificial structures consisting of periodically arranged pulses (hence "soliton crystals") with desired profiles, periods, etc. (Wai et al., Jour. Lightwave Tech. 14, p. 1449, 1996).

Optical fibers and fiber laser media

- In optical fibers: cws undergo modulational instability to an n-pulse signal when the input intensity is high (Zhakarov and Ostrovsky, Physica D238, p.540, 2009). In a mono-mode fiber, the MI favors a crystal of pulses.
- In fiber lasers: an active gain process can favor a periodic arrangement of repeatitively pumped single-pulse signals in the cavity. This process is known as harmonic mode-locking, the interaction strength and separations between mode-locked pulses will depend on their powers. Complex structures such as soliton crystals, soliton gas, soliton liquids, soliton rains have been experimentally reported in these optical devices (Amrani et al., Opt. Let. 36, 4239, 2011; Haboucha et al., Phys. Rev. A78, p. 043806, 2008).

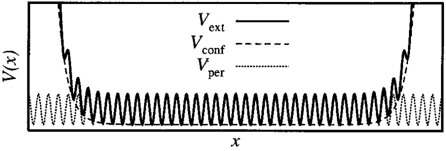
Optical traps in Bose-Einstein condensate systems

In BECs we sometimes need to trap and manipulate shape profiles of pre-formed nonlinear matter waves, from a system of N weakly interacting bosonic particles (Denschlag et al., J. Phys. B35, p.3085, 2002).
For a periodic matter-wave structure the process requires a spatially periodic optical field acting as an external attractive or expulsive potential in the Gross-Pitaevskii equation (J. C. Bronski et al., Phys. Rev. E 64, p. 056615, 2001).

Optical traps in BEC systems

Fig.4 illustrates the BEC phenomenon with a periodic optical trap. A periodic pulse lattice will compel even non-interacting particles to form a periodic structure of coherent bright matter-wave solitons.

Periodic optical lattice



Periodic Matter-wave soliton

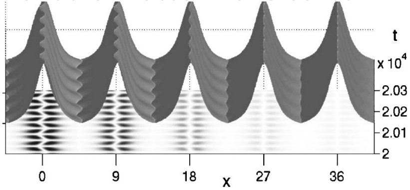


Fig.4: From Gonzalez and Promislow, Phys. Rev. A66, p. 033610, 2002.

Photonic band-gap materials

Photonic crystals: doping (see fig. 5 below) induces an artificial periodic grating affecting optical field transmission (see e.g. Yariv and Yeh, in "Photonics", Oxford Univ. Press, 2007). In nonlinear photonic crystals such artificial periodic grating can "desintegrate" a single pulse to promote a pulse crystal, just like an expulsive harmonic potential "splits" a single bright matter wave into a double-pulse matter wave (Dikandé, J. Math. Phys. 49, p. 073520, 2008).

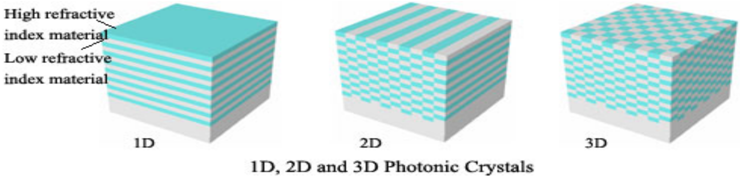


Fig.5: Doped photonic crystal with a periodic bandgap.

Light-guiding light structures

- In some nonlinear optical materials spatial solitons can imprint a waveguide in the medium, a phenomenon experimentally observed in both Kerr and photorefractive media (Steiglitz and Rand, Phys. Rev. A79, p. 021802, 2009).
- Possible observation of the phenomenon of "light-guided light" in an optical waveguide formed by a periodic lattice of spatial solitons, has also been considered (Dikandé, Phys. Rev. A81, p. 013821, 2010).

Mono-mode optical fibers

Optical fiber: a dielectric medium that can guide waves. If χ is the susceptibility of the medium, then any field $\vec{E}$ propagating in the waveguide will induce a polarization:

$$\vec{P} = \chi \vec{E} \tag{6}$$

For linear optical media χ is homogeneous. However, for optically active (i.e. nonlinear media) the medium responds to the propagation through changes of the susceptibility with the field intensity $I = |\vec{E}|^2$. This promotes higher-harmonic components in the polarization, since the susceptibility can be expanded as $\chi(I) \propto \chi_0 + \chi^{(1)} I + \chi^{(2)} I^2 + ...$ The polarization then expresses:

$$\vec{P} = \vec{P}_0 + \chi^{(1)} |\vec{E}|^2 \vec{E} + \chi^{(1)} |\vec{E}|^4 \vec{E} + ... \tag{7}$$

Mono-mode optical fibers

For weak optical response of the dielectrics, we keep only the leading nonlinear term proportional to $|\vec{E}|^2$ (Kerr nonlinearity). For optical media with $1D$ polarization and weak dispersion we pick $E(z,t) = q(z,t)\exp(-ikz)$, where $q(z,t)$ is the envelope of the propagating field and k wavector. From Helmholtz's equation in the paraxial approximation (slow variation of the envelope shape but fast propagation) one finds the propagation equation:

$$i\frac{\partial q(z,t)}{\partial z} - \frac{1}{2}\beta_2\frac{\partial^2 q(z,t)}{\partial t^2} + \gamma|q(z,t)|^2 q(z,t) = 0, \qquad (8)$$

where β_2 ($\propto 1/k$) is the coef. of group-velocity dispersion and γ ($\propto \chi^{(1)}$) the nonlinear coef..

Single-pulse soliton signal

Eq. (8) is the NLSE, by setting $q(z,t) = a(t)\exp(-\beta z)$ and for negative β_2 it can be transformed to the first-integral equation:

$$\left(\frac{da}{dt}\right)^2 = -\frac{2\beta}{\beta_2}a^2 + \frac{\gamma}{\beta_2}a^4 + C, \tag{9}$$

where the constant C determines profiles of the temporal amplitude $a(t)$. For a localized profile we expect a rapid evanescence of the wave outside its time-bandwidth such that the constant C tends to zero. This implies:

$$a(t) = \sqrt{\frac{2\beta}{\gamma}}\; sech\left[\sqrt{\frac{-2\beta}{\beta_2}}(t-t_0)\right]. \tag{10}$$

Single-pulse soliton signal

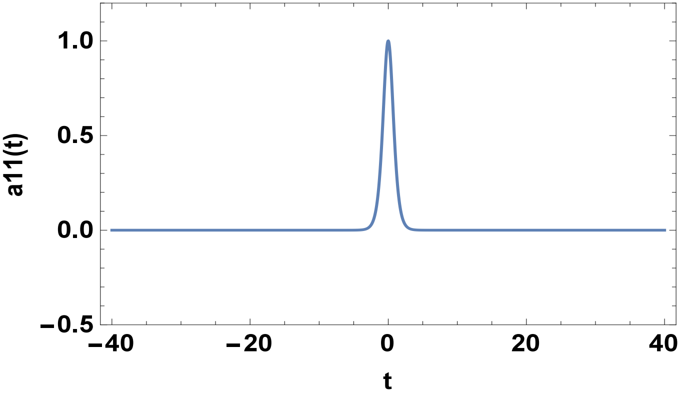


Fig.6: Temporal profile of the single-pulse soliton.

Elliptic-soliton signal

When C is non-zero and negative, the power of wave is reduced and energetic conditions become detrimental to stability of single pulse. However, we can still find nonlinear solutions to the amplitude equation viz:

$$a(t) = \sqrt{\frac{2\beta}{\gamma\,(2-\kappa^2)}}\, dn\left[\sqrt{\frac{-2\beta}{\beta_2(2-\kappa^2)}}(t-t_0), \kappa\right] \tag{11}$$

where dn is the Jacobi elliptic delta function of modulus κ $(0 < \kappa \leq 1)$ and t_0 is some (arbitrary) initial time.

Elliptic-soliton signal

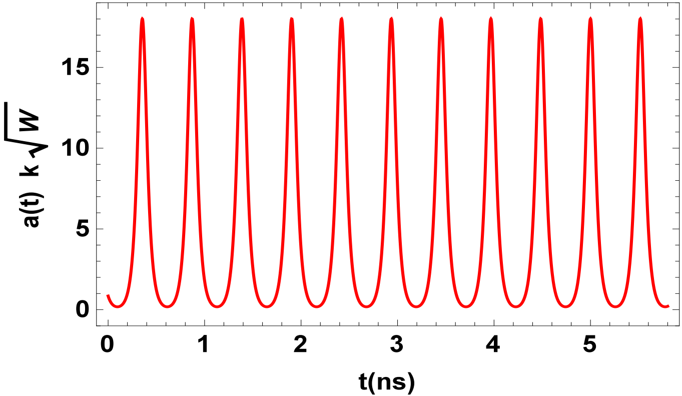


Fig.7: Temporal profile of the elliptic-soliton signal.

An elliptic soliton is a soliton crystal

IS THE ELLIPTIC SOLITON A PERIODIC HARMONIC MODE-LOCKED PULSE TRAIN?

To answer this question we must show that the sum

$$A(t) = \sum_{m=-\infty}^{\infty} \sqrt{\frac{2\beta_m}{\gamma}}\, sech \left[\sqrt{\frac{-2\beta_m}{\beta_2}}(t - t_0 - n\tau_A) \right], \qquad (12)$$

where β_m refers to the propagation constant of the m^{th} pulse mode in the temporal multiplex $A(t)$, is equivalent to (11).

Elliptic soliton is a pulse-crystal signal

For a mono-mode fiber $\beta_m = \beta$, and the sum eq. (12) is exactly (Fandio et al., Phys. Rev. A 2015, in press):

$$A(t) = 2\sqrt{\frac{\beta_2}{\gamma}}\frac{K(\kappa)}{\tau_A}\, dn\left[2K(\kappa)\frac{t}{\tau_A}, \kappa\right]. \qquad (13)$$

where

$$\tau_A = \frac{\pi K(\kappa')}{\lambda K(\kappa)},\ \lambda = \sqrt{\frac{-2\beta}{\beta_2}},\ \kappa' = \sqrt{1-\kappa^2} \qquad (14)$$

Mathematical models for Master-slave systems

When shapes of the optical signal are controlled by a potential we have a master-slave system. The propagation of the field $A(z,t)$ in the potential $V(z)$ is then governed by the generic equation:

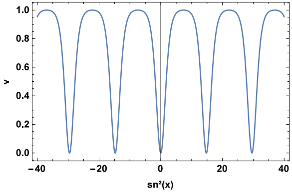
$$i\frac{\partial A(z,t)}{\partial t} = -\frac{1}{2}\beta\frac{\partial^2 A(z,t)}{\partial z^2} + V(z)A(z,t) + \gamma|A(z,t)|^2 A(z,t), \tag{15}$$

$A(z,t)$ is the condensate wavefunction for BEC systems, the envelope of the EM field for optical materials, etc.

Mathematical models for Master-slave systems

The potential that best represents a periodic optical lattice is $V(z) = V_0 sn^2(z/d, \kappa)$ where sn is the Jacobi elliptic function and κ the modulus of sn. V_0 can be positive or negative while d defines the characteristic width of the potential wells.

Expulsive Periodic optical potential



Attractive Periodic optical potential

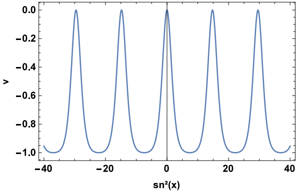


Fig.8: Periodic optical potential

Stationary-state approximation

In the stationary regime $A(z,t) = q(z)\exp(-\Omega t)$, this transforms eq. (15) to:

1. A Lamé-type linear eigenvalue problem when $\gamma = 0$ (Dikandé, Phys. Scripta 60, p. 291, 1999):

$$[-\frac{\partial^2}{\partial z^2} + \frac{2V_0}{\beta} sn^2(z)]q(z) = \frac{2\Omega}{\beta} q(z). \tag{16}$$

2. A nonlinear static Klein-Gordon equation with a periodic external potential, when $\gamma \neq 0$ (J. C. Bronski et al., Phys. Rev. E 64, p. 056615, 2001):

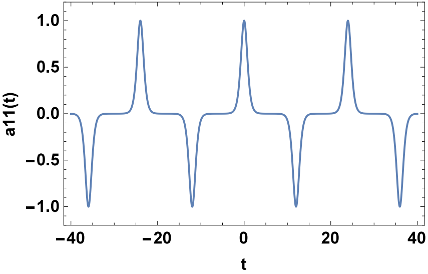
$$\frac{\partial^2 q}{\partial z^2} - \frac{2\gamma}{\beta} q^3 + \frac{2\Omega}{\beta} q = \frac{2}{\beta} V(z) q. \tag{17}$$

Typical solutions to models

Generally solutions to eqs. (16) and (17) are Lamé polynomials (W. Magnus, F. Oberhettinger and F. G. Tricomi, Handbook of Transcendental Functions, McGraw Hill, New York, 1953).
These solutions form a spectrum of boundstates and their number depends on the ratio V_0/β (Dikandé, J. Opt. 13, p. 035203, 2011).
Below we sketch typical profiles of the field as predicted by the two equations.

Mathematical models for Master-slave systems

Periodic soliton



One-soliton structure

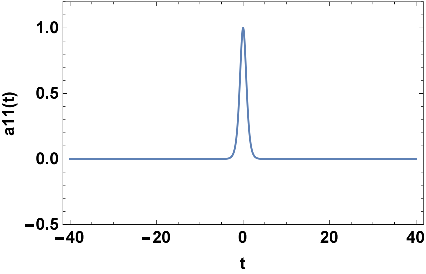
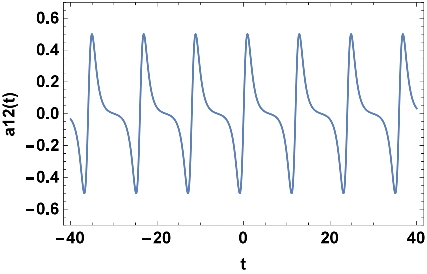


Fig.9: A periodic-soliton solution to eqs. (16) and (17)

Mathematical models for Master-slave systems

Periodic soliton



One-soliton structure

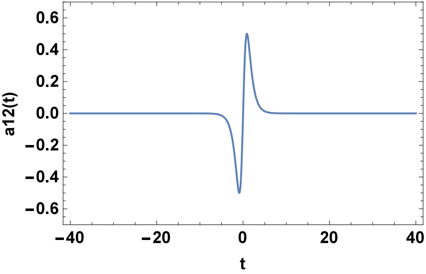


Fig.10: A periodic-soliton solution to eqs. (16) and (17)

Mathematical models for Master-slave systems

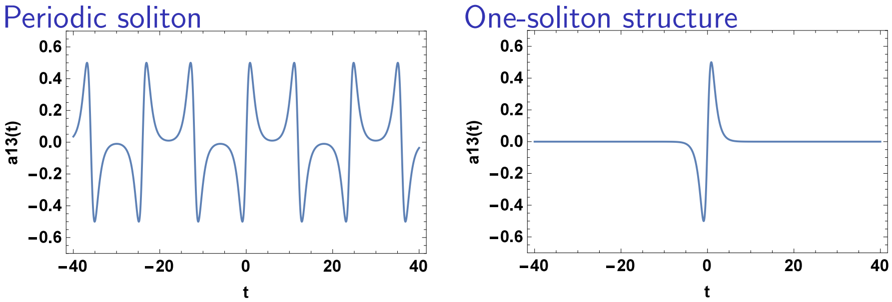


Fig.11: A periodic-soliton solution to eqs. (16) and (17)

Elliptic solitons are periodic structures than can be mathematically represented by Jacobi Elliptic functions. They are naturally generated via the phenomenon of modulational instability, but they can also be created as artificial structures by signal multiplexing, trapping of a single pulse soliton in a periodic optical potential, etc. A relevant issue related to their existence is their stability, this will be discussed in the next talk.