Arithmetic Nullstellensatz and Applications

Problem sheet 1

1. Let $f \in K[X]$ be a polynomial of degree m, and let $g = X - \beta \in K[X]$. Prove that $\mathrm{Res}_{m,1}(f, g) = (-1)^m \cdot f(\beta)$.

2. Let $f, g \in K[X]$ be polynomials of degree m and n respectively. Using the linear map ϕ defined by

$$\phi : K[X]_{<m} \times K[X]_{<n} \longrightarrow K[X]_{<n+m}, \qquad (s, t) \mapsto sf + tg,$$

prove that

$$\mathrm{Res}_{m,n}(f, g) = 0 \iff \gcd(f, g) \neq 1.$$

3. Let $f, g \in K[X]$ be polynomials of degree m and n respectively.

(a) Prove that if $g = b_n \cdot \prod_i (X - \beta_i)$, then

$$\mathrm{Res}_{m,n}(f, g) = (-1)^{mn} b_n^m \prod_i f(\beta_i).$$

(b) Prove that if, furthermore, $f = a_m \cdot \prod_j (X - \alpha_i)$, then

$$\mathrm{Res}_{m,n}(f, g) = (-1)^{mn} a_m^n b_n^m \prod_{i,j} (\beta_i - \alpha_j).$$

4. Let $f, g \in K[X]$ be polynomials such that $\deg(f) = m$ and $\deg(g) = n'$, where $n' \leq n$. Prove that

$$\mathrm{Res}_{m,n}(f, g) = a_m^{n-n'} \mathrm{Res}_{m,n'}(f, g),$$

where a_m is the leading coefficient of f.

5. Prove that the space of homogeneous polynomials of degree d in $n + 1$ variables over a field K has dimension $\binom{d+n}{n}$.

6. A complex number α is *algebraic* if there exists $f \in \mathbb{Q}[X]$ such that $f(\alpha) = 0$. Using resultants, prove that set of algebraic numbers is a subring of $\mathbb{C}$.

7. Let $f \in \mathbb{Q}[X]$ be a polynomial of degree m, with complex roots $\alpha_1, \ldots, \alpha_m$. We define its *discriminant* by $D(f) = \prod_{i<j} (\alpha_j - \alpha_i)^2$. Prove that

$$a_m^{2n-1} \cdot D(f) = (-1)^{\frac{n(n-1)}{2}} \cdot \mathrm{Res}_{m,m-1}(f, f').$$

where a_m is the leading coefficient of f.