Exercises on Number rings – lecture 1

1. Let it be given that the ring $\mathbf{Z}[\frac{1+\sqrt{-19}}{2}]$ is a principal ideal domain. Prove that $7^3 = 343$ is the only cube in $\mathbf{Z}$ that is of the form $x^2 + 19$ with $x \in \mathbf{Z}$.

2. Find (in the footsteps of Fermat) all integral solutions to the equation

$$x^2 + 2 = y^3.$$

Show that there are infinitely many *rational* solutions to this equation. [Hint: aren't you following a simultaneous course on elliptic curves?]

3. An integer N is called a *triangular number* if it is of the form

$$N = 1 + 2 + 3 + \ldots + m = \frac{1}{2}m(m+1)$$

for some $m \geq 1$. Check that m-th triangular number is a square for the values $m = 1, 8, 49, 288, 1681, 9800, 57121, 332928$. Prove that there are infinitely many m with this property, and that the values above form a complete list of such m below 1,000,000.

4. View $\mathbf{Z}[\sqrt{-3}]$ as a subring of $\mathbf{C}$ by taking for $\sqrt{-3}$ the complex number $i\sqrt{3}$.
   a. Prove that for every $x \in \mathbf{C}$ there exists $r \in \mathbf{Z}[\sqrt{-3}]$ with $|x - r| \leq 1$, and determine for which x the equality sign is needed.
   b. Prove that every fractional $\mathbf{Z}[\sqrt{-3}]$-ideal is either of the form $\mathbf{Z}[\sqrt{-3}]a$ or of the form $\mathbf{Z}[(1+\sqrt{-3})/2]a$, with $a \in \mathbf{Q}(\sqrt{-3})$.
   c. Prove that $\mathbf{Z}[\sqrt{-3}]$ is not a principal ideal domain, and that its Picard group is trivial.
   d. Prove that $\operatorname{Pic} \mathbf{Z}[(1+\sqrt{-3})/2]$ is trivial.

*5. Let $n \geq 2$ be an integer. Show that the equation $x^2 + 1 = y^n$ has no integral solutions with $x \neq 0$.