Exercises on Number rings – lecture 2

6. Let R be a number ring and $x \in R$ a non-zero element that is not a unit. Show that x can be written as a finite product of irreducible elements in R. Give an example of x and R where this product is *not* unique up to ordering and multiplication by units.

7. Let R be a number ring and $I \subset R$ an invertible R-ideal. Show that I is a product of prime ideals if and only if all primes $\mathfrak{p} \supset I$ are invertible.

8. Let $\mathfrak{p} = (2, 1 + \sqrt{-19})$ be the singular prime of $R = \mathbf{Z}[\sqrt{-19}]$. Compute the index of $\mathfrak{p}^k$ in R for $k = 1, 2, 3$. Conclude that $R/\mathfrak{p}$ and $\mathfrak{p}/\mathfrak{p}^2$ are not isomorphic as R-modules, and that the principal ideal $2R_\mathfrak{p}$ is not a power of the maximal ideal in $R_\mathfrak{p}$.

9. Let α be a zero of the polynomial $X^3 - X - 1$, and $R = \mathbf{Z}[\alpha]$. Show that R is a Dedekind ring, and determine all prime ideals of index at most 20 in R. Show also that the unit group R^* is infinite.

10. Same questions as in the previous problem for the number ring $\mathbf{Z}[\sqrt[3]{2}]$.

11. Show that $\mathbf{Z}[\sqrt{-5}]$ is a Dedekind domain, and that the identities

$$21 = (4 + \sqrt{-5})(4 - \sqrt{-5}) \qquad \text{and} \qquad 21 = 3 \cdot 7$$

represent two factorizations of 21 into pairwise non-associate irreducible elements. How does the ideal (21) factor into prime ideals in $\mathbf{Z}[\sqrt{-5}]$? Determine the order of the subgroup of $\text{Pic}(\mathbf{Z}[\sqrt{-5}])$ that is generated by the classes of the primes dividing (21). Can you find an ideal in $\mathbf{Z}[\sqrt{-5}]$ whose class is not in this subgroup?