Arithmetic Nullstellensatz and Applications

Problem sheet 2

1. Let $A = K[X_1, \ldots, X_s]$. Let $F = A_m X_0^m + \cdots + A_0$ and $G = B_n X_0^n + \cdots + B_0$ be polynomials in $A[X_0]$ with A_j homogeneous of degree $m-j$, and B_k homogeneous of degree $n-k$, for every $0 \leq j \leq m$ and every $0 \leq k \leq n$. Prove that

$$\mathrm{Res}(F, G; X_0)[t\underline{X}] = t^{mn}\,\mathrm{Res}(F, G; X_0)[\underline{X}].$$

2. Let K be an infinite field, and $f \in K[X_0, \ldots, X_n]$ be an homogeneous polynomial such that $f(1, a_1, \ldots, a_n) = 0$ for every $a_1, \ldots, a_n \in K$. Prove that $f = 0$.

3. Let $f = a_m X^m + \cdots + a_0 \in \mathbb{C}[X]$ be a polynomial with complex rooots $\alpha_1, \ldots, \alpha_m$. We define its *Mahler measure* by

$$M(f) = |a_m| \cdot \prod_i \max\{1, |\alpha_i|\}$$

(a) Prove *Landau's inequality*:

$$M(f) \leq \|f\|_2,$$

where $\|f\|_2 = \sqrt{\sum_i |a_i|^2}$.

(b) Deduce a bound for $M(f)$ in terms of $\max\{|a_i|\}_i$.

4. Deduce the general case of Extension Theorem (i.e. for an ideal I generated by s polynomials) from the case proved in the lecture (i.e. for an ideal I generated by two polynomials).

Hint: If $I = \langle f_1, \ldots, f_s \rangle$, consider the polynomials $f = f_1, g = U_2 f_2 + \cdots + U_s f_s$, where $U_2, \ldots, U_s$ are indeterminates.