Charge transport in correlated metals – I

I.a The basics

I.b Charge transport in the cuprates – the SCES poster-child

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Charge transport in correlated metals – II

II.a Bad metals

II.b Kadowaki-Woods ratio

The link between transport and thermodynamics

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Outline

- *Introduction*
- *Boltzmann transport theory*
- *Introduction to the cuprates*
- *Charge transport in the cuprates*

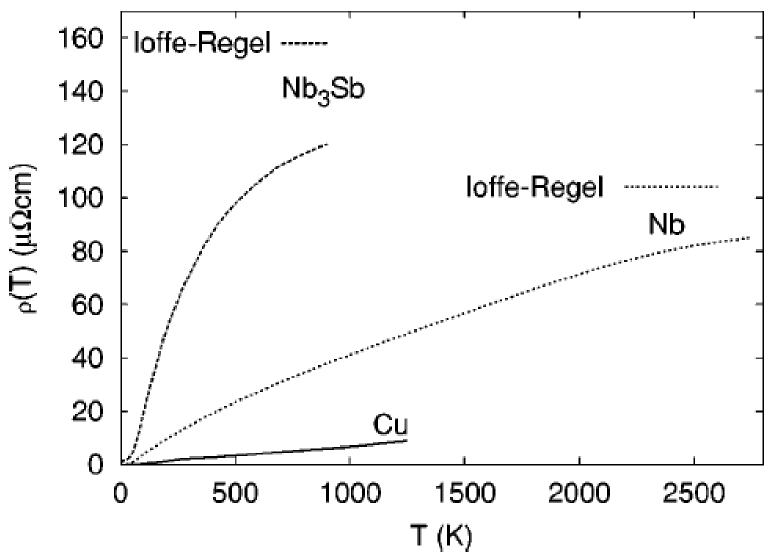
Introduction

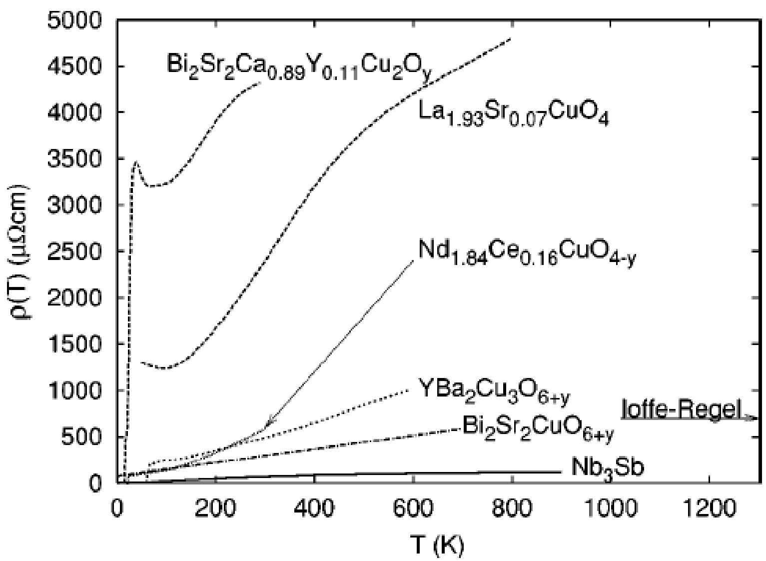
What makes dc transport measurements such an important probe of correlated electron systems?

- Often the first thing to be measured, but the last to be understood…

- “What scatters may also pair”

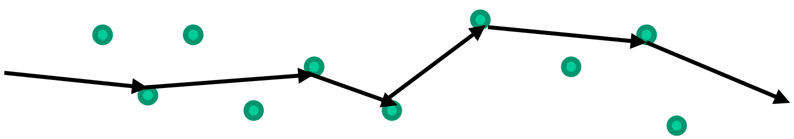
Hence, electrical resistivity is a powerful, albeit coarse, probe of superconductivity





Gunnarsson *et al.*,
RMP **75** 1085 (03)

Drude model



Drude assumed that electrons were scattered by random collisions with the immobile ion cores. He assumed a mean free time τ between collisions so that after a time t, the number of electrons having survived *without* collisions was $n(t) = n_0 \exp(-t/\tau)$

If the electric field **E** has been present for this time t, then an unscattered electron will have achieved a drift velocity of $v = (-eEt/m)$ and have travelled a distance $x = \left(eEt^2/2m\right)$

This gives for the total electronic transport in the direction of the applied field

$$\int_0^\infty x\left(\frac{dn}{dt}\right)dt = \left(-\frac{eEn_0}{2m\tau}\right)\underbrace{\int_0^\infty t^2 \exp(-t/\tau)dt}_{2\tau^3} = \left(-\frac{eEn_0\tau^2}{m}\right)$$

which is equivalent to n_0 electrons having mean drift velocity $v = (-eE\tau/m)$ for time τ. Finally, for a metal containing n electrons / m^3, the current density

$$j = \sigma E = (-nev) = \frac{ne^2 E\tau}{m}$$

with the corresponding electrical conductivity $\boxed{\sigma = \dfrac{ne^2\tau}{m}}$

Drude model

Despite the crude approximations and wildly incorrect assumptions, the Drude expression serves as an excellent, practical way to form simple pictures and rough estimates of properties whose deeper comprehension may require analysis of real complexity.

How come?

Well, some of it is fortuitous. $\rho(300) \approx 2\,\mu\Omega\text{cm} \quad \Rightarrow \quad \tau \approx 1\times 10^{-14}\ \text{sec}$

Drude then estimated the velocity from the kinetic equation

$$\tfrac{1}{2}mv^2 = \tfrac{3}{2}k_B T \quad \Rightarrow \quad v \approx 10^5\,\text{m/s}$$

From this he extracted a mean-free-path of ~1 nm, i.e. the approximate interatomic spacing! Of course, while this seemed reasonable to Drude, it was way off the mark...

The workability of the Drude model reflects the fact that two of the fundamental assumptions (the **action due to the Lorentz force** and the **exponential decay in *n*(*t*)**) are also found to be **equally applicable to Bloch waves and fermionic quasiparticles**.

$$\mathbf{p}(t+\mathrm{d}t) = (1-\mathrm{d}t/\tau)\left(\mathbf{p}(t) + \mathbf{f}(t)\mathrm{d}t + O(\mathrm{d}t)^2\right) \quad \rightarrow \quad \boxed{\frac{\mathrm{d}\mathbf{p}(t)}{\mathrm{d}t} = -\frac{\mathbf{p}(t)}{\tau} + \mathbf{f}(t) \qquad \mathbf{f}(t) = -e\left(\mathbf{E} + [\mathbf{v}\times\mathbf{B}]\right)}$$

Drude model

Hall effect

$$\frac{d\mathbf{p}}{dt} = -\frac{\mathbf{p}}{\tau} - e\left(\mathbf{E} + \left[\frac{\mathbf{p}}{m} \times \mathbf{B}\right]\right)$$

In steady state, current is independent of *t*

p_x and p_y thus satisfy

$$0 = -eE_x - \omega_c p_y - \frac{p_x}{\tau}$$

$$0 = -eE_y + \omega_c p_x - \frac{p_y}{\tau}$$

where $\omega_c = eB/m$ (B//z)

Thus

$$\sigma_0 E_x = \omega_c \tau \, j_y + j_x$$

$$\sigma_0 E_y = -\omega_c \tau \, j_x + j_y$$

Hall field is determined by condition $j_y = 0$

$$\Rightarrow \quad E_y = -\left(\frac{\omega_c \tau}{\sigma_0}\right) j_x = -\left(\frac{B}{ne}\right) j_x$$

Hence, Hall coefficient

$$\boxed{R_H = \frac{E_y}{j_x B} = -\frac{1}{ne}}$$

ac conductivity

$$\frac{d\mathbf{p}}{dt} = -\frac{\mathbf{p}}{\tau} - e\mathbf{E}(t)$$

$$\mathbf{E}(t) = \mathrm{Re}\{\mathbf{E}_0(\omega)\exp(-i\omega t)\}$$

Seek steady state solution of the form

$$\mathbf{p}(t) = \mathrm{Re}\{\mathbf{p}(\omega)\exp(-i\omega t)\}$$

Thus

$$-i\omega \mathbf{p}(\omega) = -\frac{\mathbf{p}(\omega)}{\tau} - e\mathbf{E}_0(\omega)$$

Current density

$$\mathbf{j}(\omega) = \frac{-ne\mathbf{p}(\omega)}{m} = \frac{(ne^2/m)\mathbf{E}_0(\omega)}{(1/\tau) - i\omega} = \sigma(\omega)\mathbf{E}_0(\omega)$$

where

$$\boxed{\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}}$$

Distribution functions

$$\boxed{f_k(\mathbf{r},t)}$$

Local concentration of carriers "occupancy" in the state *k* in the neighbourhood of the point r in space and time *t*

(i) Carriers move in and out of the region r

$$\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{diff}} = -\frac{\partial \mathbf{r}}{\partial t}\cdot\frac{\partial f_k}{\partial \mathbf{r}} = -\mathbf{v}_{\kappa}\cdot\nabla f_{\mathbf{k}}$$

(ii) The k-vector will be changed by external fields

$$\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{field}} = -\frac{\partial \mathbf{k}}{\partial t}\cdot\frac{\partial f_k}{\partial \mathbf{k}} = -\dot{\mathbf{k}}\cdot\frac{\partial f_k}{\partial \mathbf{k}}$$

where

$$\dot{\mathbf{k}} = \frac{e}{\hbar}\left(\mathbf{E}+\left[\mathbf{v}_{\mathbf{k}}\times\mathbf{B}\right]\right)$$

(iii) Carriers are scattered

$$\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{coll}} = \int\left\{f_{\mathbf{k}'}\left(1-f_{\mathbf{k}}\right)-f_{\mathbf{k}}\left(1-f_{\mathbf{k}'}\right)\right\}Q\left(\mathbf{k},\mathbf{k}'\right)\mathrm{d}\mathbf{k}' = -\frac{f_{\mathbf{k}}-f_{\mathbf{k}}^{0}}{\tau}$$

relaxation time approximation

Boltzmann equation – 3 key points

$$\boxed{\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{diff}} + \left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{field}} + \left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{coll}} = 0}$$

Note that this is *steady* state, not *equilibrium* state f_{k}^{0}

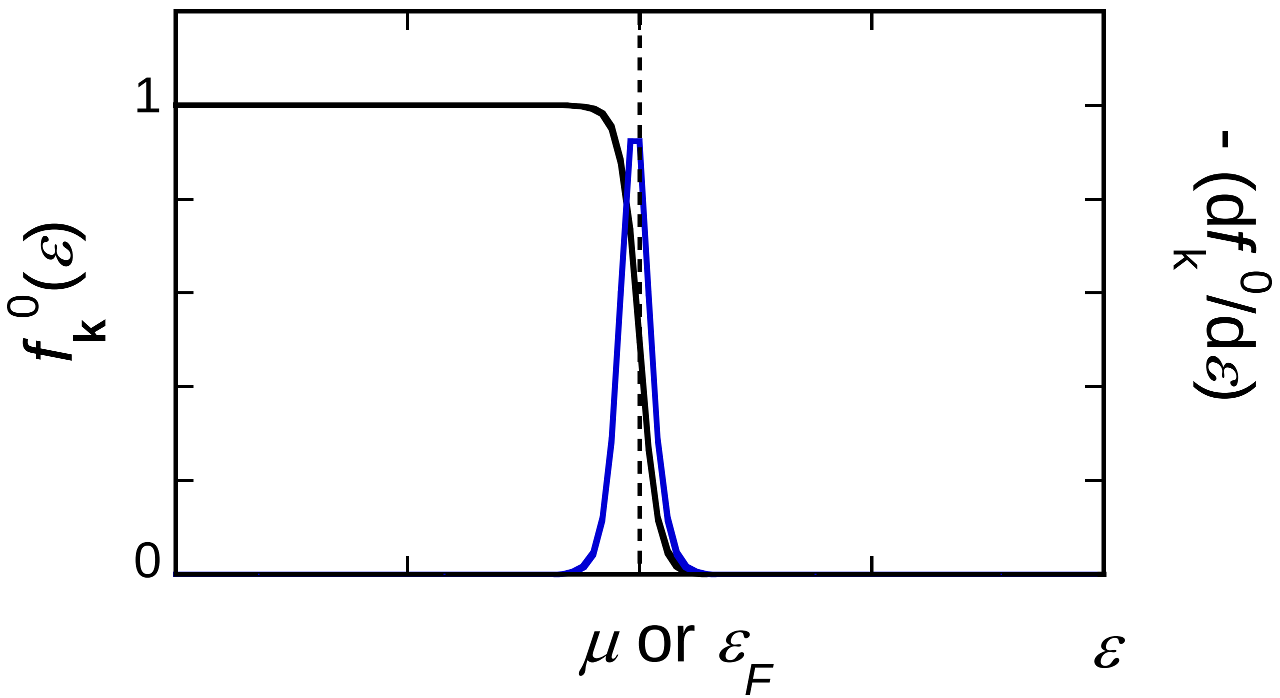
For electrons

$$\boxed{f_{\mathbf{k}}^{\mathbf{0}} = \frac{1}{\exp\left\{\left(\varepsilon_{\mathbf{k}} - \mu\right)/k_B T\right\} + 1}}$$

We are most concerned with small departures from f_{k}^{0}

$$g_{\mathbf{k}} = f_{\mathbf{k}} - f_{\mathbf{k}}^{\mathbf{0}}$$

Fermi distribution



$$g_{\mathbf{k}} = f_{\mathbf{k}} - f_{\mathbf{k}}^{\mathbf{0}} \propto \left(\frac{\partial f_{\mathbf{k}}^{\mathbf{0}}}{\partial \varepsilon} \right)$$

Boltzmann equation – all aboard the Chain Rule!

$$\boxed{\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{diff}} + \left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{field}} + \left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{coll}} = 0}$$

$$\Rightarrow -\mathbf{v}_{\kappa} \cdot \nabla f_{\mathbf{k}} - \frac{e}{\hbar}\left(\mathbf{E} + \left[\mathbf{v}_{\mathbf{k}} \times \mathbf{B}\right]\right) \cdot \frac{\partial f_{\mathbf{k}}}{\partial \mathbf{k}} = -\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{coll}}$$

$$\Rightarrow -\mathbf{v}_{\kappa} \cdot \left\{\frac{\partial f_{\mathbf{k}}}{\partial T}\nabla T + \frac{\partial f_{\mathbf{k}}}{\partial \mu}\nabla \mu\right\} - \frac{e}{\hbar}\left(\mathbf{E} + \left[\mathbf{v}_{\mathbf{k}} \times \mathbf{B}\right]\right) \cdot \frac{\partial f_{\mathbf{k}}}{\partial \mathbf{k}} = -\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{coll}}$$

$$g_{\mathbf{k}} = f_{\mathbf{k}} - f_{\mathbf{k}}^{0} \quad \Rightarrow \quad -\mathbf{v}_{\kappa} \cdot \left\{\frac{\partial f_{\mathbf{k}}^{0}}{\partial T}\nabla T + \frac{\partial f_{\mathbf{k}}^{0}}{\partial \mu}\nabla \mu\right\} - \frac{e}{\hbar}\left(\mathbf{E} + \left[\mathbf{v}_{\mathbf{k}} \times \mathbf{B}\right]\right) \cdot \frac{\partial f_{\mathbf{k}}^{0}}{\partial \mathbf{k}}$$

$$= -\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{coll}} + \mathbf{v}_{\kappa} \cdot \frac{\partial g_{\mathbf{k}}}{\partial \mathbf{r}} + \frac{e}{\hbar}\left(\mathbf{E} + \left[\mathbf{v}_{\mathbf{k}} \times \mathbf{B}\right]\right) \cdot \frac{\partial g_{\mathbf{k}}}{\partial \mathbf{k}}$$

$$\boxed{f_{\mathbf{k}}^{0} = \frac{1}{\exp\left\{\left(\varepsilon_{\mathbf{k}} - \mu\right)/k_B T\right\} + 1}} \quad \Rightarrow \quad \frac{\partial f_{\mathbf{k}}^{0}}{\partial T} = \frac{\left(\varepsilon_{\mathbf{k}} - \mu\right)}{T}\frac{\partial f_{\mathbf{k}}^{0}}{\partial \varepsilon_{\mathbf{k}}} \text{ and } \frac{\partial f_{\mathbf{k}}^{0}}{\partial \mu} = -\frac{\partial f_{\mathbf{k}}^{0}}{\partial \varepsilon_{\mathbf{k}}}$$

Linearized Boltzmann equation

$$\boxed{\left(-\frac{\partial f_k^0}{\partial \varepsilon_{\mathbf{k}}}\right)\mathbf{v}_{\kappa}\cdot\left\{\cancel{\frac{(\varepsilon_{\mathbf{k}}-\mu)}{T}}\nabla T+e\left(\mathbf{E}-\cancel{\frac{1}{e}\nabla\mu}\right)\right\}=-\frac{\partial f_{\mathbf{k}}}{\partial t}\bigg]_{\text{coll}}+\cancel{\mathbf{v}_{\kappa}\cdot\frac{\partial g_k}{\partial \mathbf{r}}}+\cancel{\frac{e}{\hbar}\left[\mathbf{v}_{\mathbf{k}}\times\mathbf{B}\right]}\cdot\frac{\partial g_k}{\partial \mathbf{k}}}$$

Assume "infinite homogeneous medium", constant temperature and zero magnetic field;

$$\Rightarrow\quad\left(-\frac{\partial f_k^0}{\partial \varepsilon_{\mathbf{k}}}\right)\mathbf{v}_{\kappa}\cdot e\mathbf{E}=-\frac{\partial f_{\mathbf{k}}}{\partial t}\bigg]_{\text{coll}}$$

and for simplicity, let us make the phenomenological relaxation time approximation

$$\boxed{\frac{\partial f_{\mathbf{k}}}{\partial t}\bigg]_{\text{coll}}=\frac{\partial\left(g_{\mathbf{k}}+f_{\mathbf{k}}^0\right)}{\partial t}=\frac{\partial g_{\mathbf{k}}}{\partial t}=-\frac{g_{\mathbf{k}}}{\tau}}$$

Hence $g_{\mathbf{k}}(t)=g_{\mathbf{k}}(0)e^{-t/\tau}$

$$\Rightarrow\quad\boxed{g_{\mathbf{k}}=\left(-\frac{\partial f_k^0}{\partial \varepsilon_{\mathbf{k}}}\right)e\tau\mathbf{v}_{\kappa}\cdot\mathbf{E}}$$

$$\boxed{g_{\mathbf{k}}=\int_{-\infty}^{t}\left(-\frac{\partial f_k^0}{\partial \varepsilon_{\mathbf{k}}}\right)e\mathbf{v}_{\kappa}\cdot\mathbf{E}\,e^{-(t-t')/\tau}\,\mathrm{d}t'}$$

Boltzmann vs. Drude

$$\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{diff}} + \left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{field}} + \left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{coll}} = 0$$

$$-\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{diff}} = -\frac{\partial g_{\mathbf{k}}}{\tau} + \left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{field}}$$

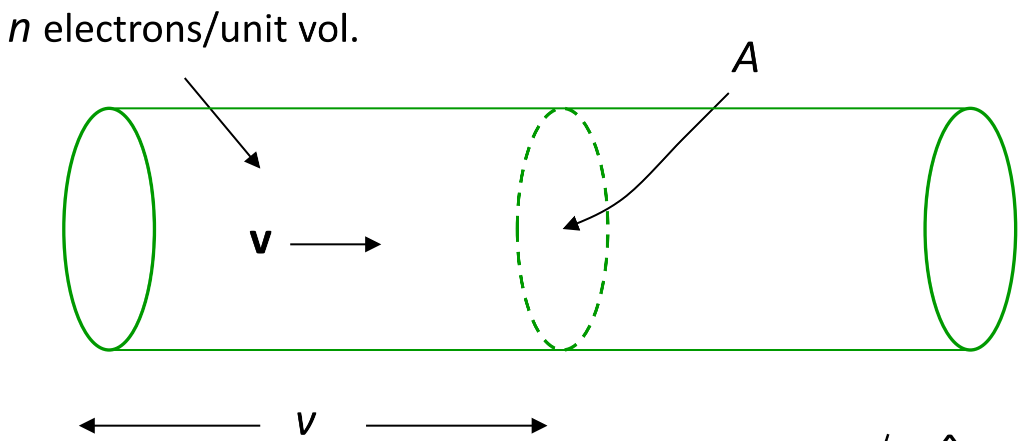
$$-\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{diff}} = -\frac{\partial g_{\mathbf{k}}}{\tau} - \frac{e}{\hbar}\left(\mathbf{E} + \left[v_{\mathbf{k}} \times \mathbf{B}\right]\right) \cdot \frac{\partial f_{\mathbf{k}}}{\partial \mathbf{k}} \qquad g_{\mathbf{k}}(t) = g_{\mathbf{k}}(0) e^{-t/\tau}$$

c.f. Drude

$$\frac{\mathrm{d}\mathbf{p}(t)}{\mathrm{d}t} = -\frac{\mathbf{p}(t)}{\tau} - e\left(\mathbf{E} + \left[\mathbf{v} \times \mathbf{B}\right]\right) \qquad n(t) = n_0 e^{-t/\tau}$$

Thus, despite their extreme starting points, both models consider a steady state solution involving forces acting on charged particles or wave packets with a distribution of velocities.

Current response



$$\mathbf{J} = I/A\,\hat{\mathbf{x}} = neA\mathbf{v}/A = ne\mathbf{v}$$

$$n = \frac{2}{(2\pi)^3}\int f_{\mathbf{k}}\,\mathrm{d}^3k \quad \boxed{\Rightarrow \quad \mathbf{J}_i = \frac{1}{4\pi^3}\int \mathbf{v}_i e g_{\mathbf{k}}\,\mathrm{d}^3k = \sigma_{ij}\mathbf{E}_j}$$

$$\left(\int \mathbf{v}_i e f_{\mathbf{k}}^0\,\mathrm{d}^3k = 0\right)$$

Similarly for thermal current

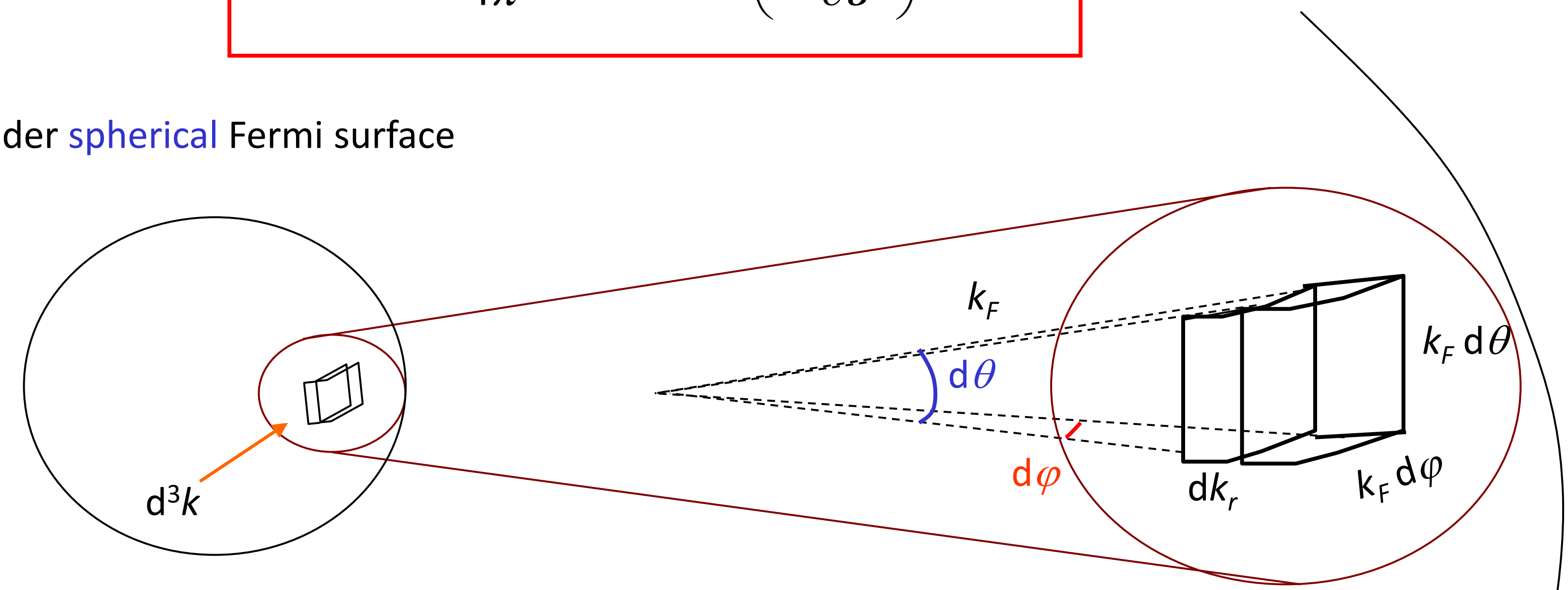
$$\boxed{\mathbf{J}_i = \frac{1}{4\pi^3}\int \mathbf{v}_i(\varepsilon - \mu)g_{\mathbf{k}}\,\mathrm{d}^3k = -\kappa_{ij}\nabla T_j}$$

dc electrical conductivity

$$\mathbf{J}_i = \frac{1}{4\pi^3}\int \mathbf{v}_i e g_{\mathbf{k}}\, \mathrm{d}^3k = \sigma_{ij}\mathbf{E}_j \quad \text{and} \quad g_{\mathbf{k}} = \left(-\frac{\partial f^0}{\partial \varepsilon}\right) e\tau v_{\kappa}\cdot\mathbf{E}$$

$$\boxed{\Rightarrow \quad \sigma_{ij} = \frac{1}{4\pi^3}\int e^2\tau\,\mathbf{v}_i\cdot\mathbf{v}_j\left(-\frac{\partial f^0}{\partial \varepsilon}\right)\mathrm{d}^3k}$$

Consider spherical Fermi surface



$$\int\left(-\frac{\partial f^0}{\partial \varepsilon}\right)\mathrm{d}^3k = \iint\left(-\frac{\partial f^0}{\partial \varepsilon}\right)\mathrm{d}k_r\,\mathrm{d}S_F = \int\left(-\frac{\partial f^0}{\partial \varepsilon}\right)\mathrm{d}k_r\int_0^{2\pi}k_F\,\mathrm{d}\theta\int_0^{\pi}k_F\sin\varphi\,\mathrm{d}\varphi$$

dc electrical conductivity

Recall $$\mathbf{v}_{\mathbf{k}} = \frac{1}{\hbar}\nabla_{\mathbf{k}}\varepsilon_{\mathbf{k}} \quad \Rightarrow \quad \mathrm{d}\varepsilon_{\mathbf{k}} = \hbar\left|\mathbf{v}_F\right|\cos\gamma\,\mathrm{d}k_r$$

where $\cos\gamma$ is the angle between v_F and k_F
= 1 for isotropic 3D FS

And for most metals, $\left(-\frac{\partial f^0}{\partial\varepsilon}\right)$ behaves as a δ-function.

Hence, $$\iint\left(-\frac{\partial f^0}{\partial\varepsilon}\right)\mathrm{d}k_r\,\mathrm{d}S_F = \iint\delta\left(\varepsilon_n(\mathbf{k})-\varepsilon_F\right)\mathrm{d}k_r\,\mathrm{d}S_F = \int\frac{\mathrm{d}S_F}{\hbar\left|\mathbf{v}_F\right|\cos\gamma}$$

Note that this is the same surface integral that appears in the expression for the electronic density of states

$$g_n(\varepsilon) = \int\frac{\mathrm{d}S_F}{4\pi^3}\frac{1}{\left|\nabla\varepsilon_n(\mathbf{k})\right|}$$

dc electrical conductivity

Hence, $$\boxed{\sigma_{ij}^{3D} = \frac{e^2}{4\pi^3\hbar}\int_0^{2\pi}\int_0^{\pi}\tau\frac{\mathbf{v}_i\cdot\mathbf{v}_j}{|v_F|}k_F^2\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta}$$

$$\sigma_{xx} = \frac{e^2}{4\pi^3\hbar}\int_0^{2\pi}\int_0^{\pi}\tau\frac{v_x v_x}{|v_F|}k_F^2\sin\varphi\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{e^2}{4\pi^3\hbar}\int_0^{2\pi}\int_0^{\pi}k_F^2 v_F\tau\sin^3\varphi\cos^2\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

$$(v_x = v_F\sin\varphi\cos\theta)$$

$$\Rightarrow \quad \sigma_{xx} = \frac{e^2k_F^2v_F\tau}{4\pi^3\hbar}\underbrace{\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta}_{\pi}\underbrace{\int_0^{\pi}\sin^3\varphi\,\mathrm{d}\varphi}_{4/3}$$

$$\Rightarrow \boxed{\sigma_{xx} = \frac{e^2}{3\pi^2\hbar}k_F^2\ell}$$

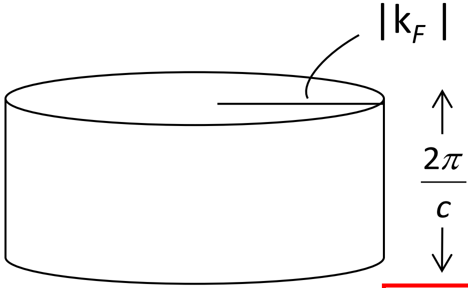
c.f. Drude result: $$\sigma = \frac{ne^2\tau}{m^*} = \frac{2}{(2\pi)^3}\frac{4\pi k_F^3}{3}\left(\frac{e^2\tau}{m^*}\right) = \frac{k_F^3}{3\pi^2}\left(\frac{e^2\tau v_F}{\hbar k_F}\right) = \frac{e^2}{3\pi^2\hbar}k_F^2\ell$$ QED

In-plane conductivity for quasi-2D metal

$$\boxed{\sigma_{ij}^{2D} = \frac{e^2}{4\pi^3\hbar}\int_{-\pi/c}^{\pi/c} \mathrm{d}k_z \int_0^{2\pi} \tau \frac{\mathbf{v}_i \cdot \mathbf{v}_j}{|v_F|\cos\gamma} k_F \mathrm{d}\varphi}$$

$$\sigma_{xx} = \frac{e^2}{4\pi^3\hbar}\int_{-\pi/c}^{\pi/c} \mathrm{d}k_z \int_0^{2\pi} \tau \frac{v_x v_x}{|v_F|\cos\gamma} k_F \mathrm{d}\varphi = \frac{e^2}{4\pi^3\hbar}\left(\frac{2\pi}{c}\right)\int_0^{2\pi} \frac{k_F v_F \tau \cos^2\varphi}{\cos\gamma} \mathrm{d}\varphi$$

Assume, isotropic, cylindrical Fermi surface



$\mathbf{v_F} \,||\, \mathbf{k_F} \quad \Rightarrow \quad \cos\gamma = 1$

$$\Rightarrow \quad \sigma_{xx} = \frac{e^2 k_F v_F \tau}{2\pi^2 \hbar c}\underbrace{\int_0^{2\pi} \cos^2\varphi \, \mathrm{d}\varphi}_{\pi} = \frac{e^2}{2\pi\hbar c} k_F \ell$$

$$\boxed{\sigma_{xx} = \frac{e^2}{2\pi\hbar c} k_F \ell}$$

c.f. Drude result: $\sigma = \frac{ne^2\tau}{m^*} = \frac{2}{(2\pi)^3}\frac{2\pi}{c}\pi k_F^2\left(\frac{e^2\tau}{m^*}\right) = \frac{k_F^2}{2\pi c}\left(\frac{e^2\tau v_F}{\hbar k_F}\right) = \frac{e^2}{2\pi\hbar c} k_F \ell$ QED

Conductivity in a magnetic field

In metals, the effect of a magnetic field is usually to deviate the trajectory of the carriers from their electric-field induced path. The two major corresponding changes to the electrical conductivity tensor are:

(i) the well-known *Hall effect* emerges as a result of cross terms (σ_{xy} etc...)

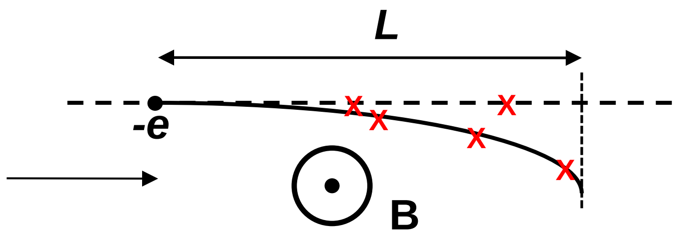
(ii) the longitudinal (diagonal) conductivity also changes - *magnetoresistance*

The Hall effect can be of either sign, depending on the majority or most mobile carrier type.

The magnetoresistance, on the other hand, is almost always positive. Why?

A definition of magnetoresistance

The increase in the longitudinal resistance caused by the additional scattering all mobile carriers experience per unit length *in the direction of the applied electric field* in the presence of an applied magnetic field.



Conductivity in a magnetic field

$$\boxed{\left(-\frac{\partial f_k^0}{\partial \varepsilon_{\mathbf{k}}}\right)\mathbf{v}_\kappa\cdot\left\{\cancel{\frac{(\varepsilon_k-\mu)}{T}}\nabla T + e\left(\mathbf{E}-\cancel{\frac{1}{e}\nabla\mu}\right)\right\} = -\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right]_{\text{coll}} + \cancel{\mathbf{v}_\kappa\cdot\frac{\partial g_k}{\partial \mathbf{r}}} + \frac{e}{\hbar}[\mathbf{v}_\mathbf{k}\times\mathbf{B}]\cdot\frac{\partial g_k}{\partial\mathbf{k}}}$$

Within the relaxation time approximation;

$$\Rightarrow \quad e\mathbf{E}\cdot\mathbf{v}_\kappa\left(\frac{\partial f_k^0}{\partial \varepsilon_{\mathbf{k}}}\right)+\frac{e}{\hbar}[\mathbf{v}_\mathbf{k}\times\mathbf{B}]\cdot\frac{\partial g_k}{\partial\mathbf{k}} = -\frac{g_k}{\tau}$$

Continuous series

Jones-Zener expansion

$$g_k^{(n)} = \left(-\frac{e\tau}{\hbar}[\mathbf{v}_\mathbf{k}\times\mathbf{B}]\cdot\frac{\partial}{\partial\mathbf{k}}\right)^n\left\{e\mathbf{E}\cdot\mathbf{v}_\kappa\tau\left(-\frac{\partial f_k^0}{\partial \varepsilon_{\mathbf{k}}}\right)\right\}$$

Hence

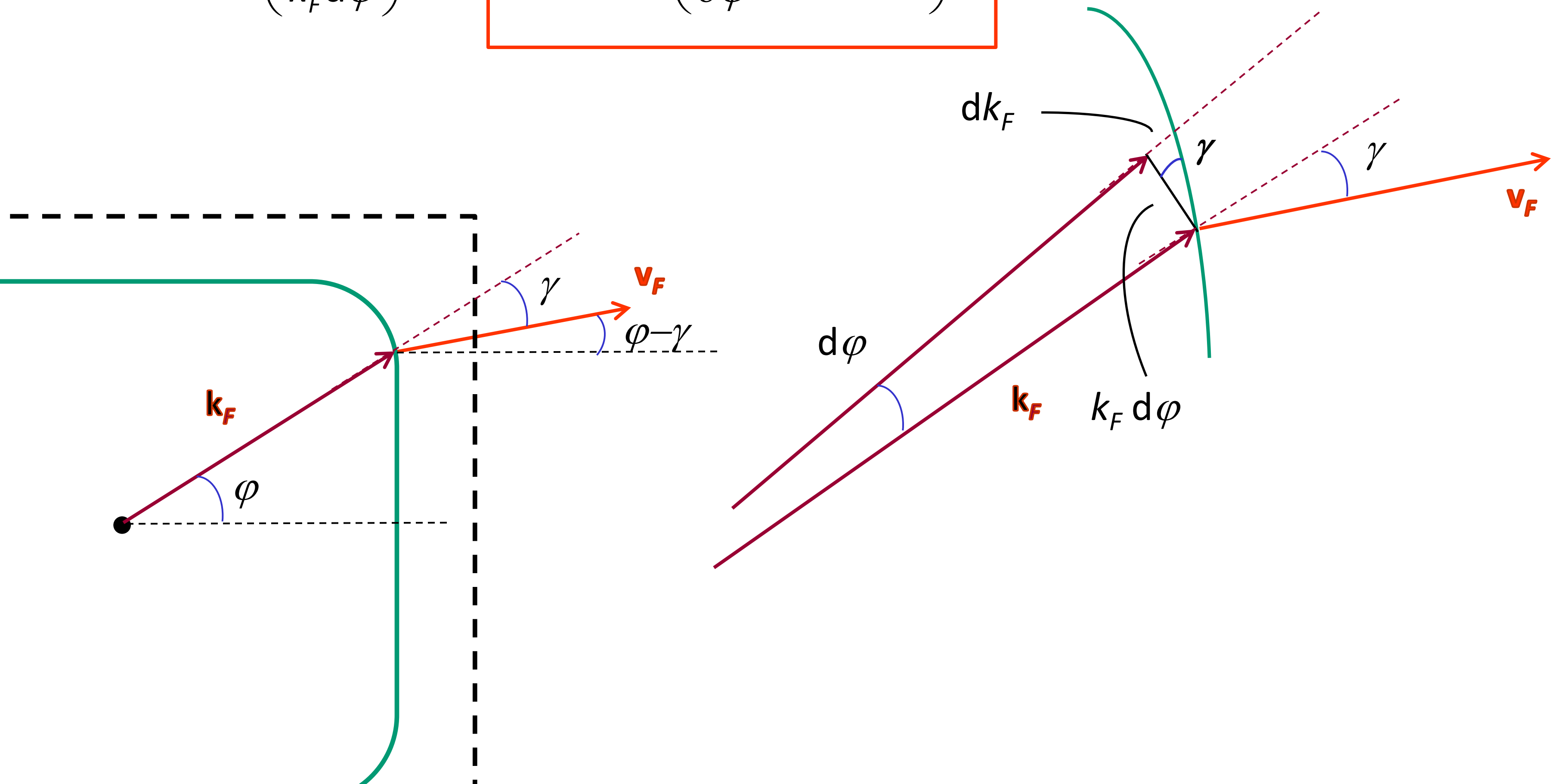
$$\boxed{\sigma_{ij}^{(n)} = \frac{1}{4\pi^3}\int ev_i\left(-\frac{e\tau}{\hbar}[\mathbf{v}_\mathbf{k}\times\mathbf{B}]\cdot\frac{\partial}{\partial\mathbf{k}}\right)^n ev_j\tau\left(-\frac{\partial f_k^0}{\partial \varepsilon_{\mathbf{k}}}\right)\mathrm{d}^3k}$$

e.g. To calculate $\sigma_{xy}^{(1)}$, simply work through with the applied magnetic field **H**//z and the electric field **E**//y and calculate the response J_x with the first-order Jones-Zener equation.

Interlude: Taking into account the shape of quasi-2D FS

The relation between $\mathbf{v}_F$ and $\mathbf{k}_F$

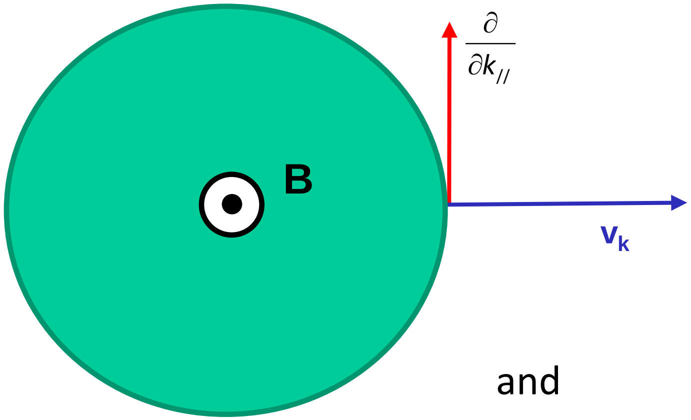
$$\tan\gamma = \left(\frac{\mathrm{d}k_F}{k_F\,\mathrm{d}\varphi}\right) \Rightarrow \boxed{\gamma = \tan^{-1}\left(\frac{\partial}{\partial\varphi}\left[\ln\left(k_F(\varphi)\right)\right]\right)}$$



Hall conductivity in a quasi-2D conductor

$$\sigma_{xy}^{(1)} = \frac{1}{4\pi^3}\int ev_x\left(-\frac{e\tau}{\hbar}[\mathbf{v_k}\times\mathbf{B}]\cdot\frac{\partial}{\partial\mathbf{k}}\right)ev_y\tau\left(-\frac{\partial f_k^0}{\partial\varepsilon_{\mathbf{k}}}\right)\mathrm{d}^3k$$

For B//*c*:



$$[\mathbf{v_k}\times\mathbf{B}]\cdot\frac{\partial}{\partial\mathbf{k}} = v_F B\frac{\partial}{\partial k_{//}}$$

where $\frac{\partial}{\partial k_{//}} = \frac{\cos\gamma}{k_F}\frac{\partial}{\partial\varphi}$

and $$\int\left(-\frac{\partial f^0}{\partial\varepsilon}\right)\mathrm{d}^3k = \int_{-\pi/c}^{\pi/c}\mathrm{d}k_z\int_0^{2\pi}\frac{1}{\hbar|\mathbf{v}_F|\cos\gamma}k_F\,\mathrm{d}\varphi$$

Finally,

$$\boxed{\sigma_{xy}^{(1)} = \frac{-e^3B}{4\pi^3\hbar^2}\frac{2\pi}{c}\int_0^{2\pi}\ell_x\frac{\partial\ell_y}{\partial\varphi}\mathrm{d}\varphi} = \frac{-e^3B}{2\pi^2\hbar^2 c}\int_0^{2\pi}v_F\tau\cos(\varphi-\gamma)\frac{\partial}{\partial\varphi}[v_F\tau\sin(\varphi-\gamma)]\mathrm{d}\varphi$$

NB (i) $\sigma_{xy}^{(1)}$ probes the variation of the mean-free-path around the Fermi surface.
NB (ii) $\sigma_{xy}^{(1)}$ in a quasi-2D metal does NOT depend on the carrier density

In-plane magnetoresistance in a quasi-2D metal

$$\sigma_{xx}^{(2)} = \frac{1}{4\pi^3}\int ev_x\left(-\frac{e\tau}{\hbar}[\mathbf{v_k}\times\mathbf{B}]\cdot\frac{\partial}{\partial\mathbf{k}}\right)\left(-\frac{e\tau}{\hbar}[\mathbf{v_k}\times\mathbf{B}]\cdot\frac{\partial}{\partial\mathbf{k}}\right)ev_x\tau\left(-\frac{\partial f_k^0}{\partial\varepsilon_{\mathbf{k}}}\right)\mathrm{d}^3k$$

$$= \frac{e^4B^2}{2\pi^2\hbar^3c}\int_0^{2\pi}\ell\cos(\varphi-\gamma)\,\frac{\partial}{\partial\varphi}\left\{\frac{\ell\cos\gamma}{k_F}\frac{\partial}{\partial\varphi}[\ell\cos(\varphi-\gamma)]\right\}\mathrm{d}\varphi$$

Consider isotropic cylinder:

$$\sigma_{xx}^{(2)} = \frac{e^4B^2\ell^3}{2\pi^2\hbar^3ck_F}\int_0^{2\pi}\cos\varphi\,\frac{\partial^2(\cos\varphi)}{\partial\varphi^2}\mathrm{d}\varphi = -\frac{e^4B^2\ell^3}{2\pi\hbar^3ck_F}$$

$$\Rightarrow\quad \frac{\sigma_{xx}^{(2)}}{\sigma_{xx}^{(0)}} = \frac{-e^4B^2\ell^3}{2\pi\hbar^3ck_F}\cdot\frac{2\pi\hbar c}{e^2k_F\ell} = \frac{-e^2B^2\ell^2}{\hbar^2k_F^2} = -(\omega_c\tau)^2$$

$$\left[\begin{aligned}\omega_c &= \frac{eB}{m^*}\\ &= \frac{eBv_F}{\hbar k_F}\end{aligned}\right]$$

Thus $\sigma_{xx}{}^{(2)}$ as expected, is negative and scales with B^2

In-plane magnetoresistance in a quasi-2D metal

Magnetoresistance

$$\boxed{\frac{\Delta\rho_{xx}}{\rho_{xx}^{0}} = -\frac{\sigma_{xx}^{(2)}}{\sigma_{xx}^{(0)}} - \left(\frac{\sigma_{xy}^{(1)}}{\sigma_{xx}^{0}}\right)^{2}}$$

$$\sigma_{xx} = \frac{e^{2}}{2\pi\hbar c} k_F \ell$$

$$\sigma_{xy}^{(1)} = \frac{-e^{3}B\ell^{2}}{2\pi^{2}\hbar^{2}c}\int_{0}^{2\pi}\cos\varphi\,\frac{\partial(\sin\varphi)}{\partial\varphi}\mathrm{d}\varphi = \frac{e^{3}B\ell^{2}}{2\pi^{2}\hbar^{2}c}\int_{0}^{2\pi}\underbrace{\cos^{2}\varphi\,\mathrm{d}\varphi}_{\pi} = \frac{e^{3}B\ell^{2}}{2\pi\hbar^{2}c}$$

$$\frac{\Delta\rho_{xx}}{\rho_{xx}^{0}} = +(\omega_c\tau)^{2} - \left(\frac{e^{3}B\ell^{2}}{2\pi\hbar^{2}c}\cdot\frac{2\pi\hbar c}{e^{2}k_F\ell}\right)^{2}$$

$$= +(\omega_c\tau)^{2} - \left(\frac{eB\ell}{\hbar k_F}\right)^{2}$$

$$= +(\omega_c\tau)^{2} - (\omega_c\tau)^{2} = 0$$

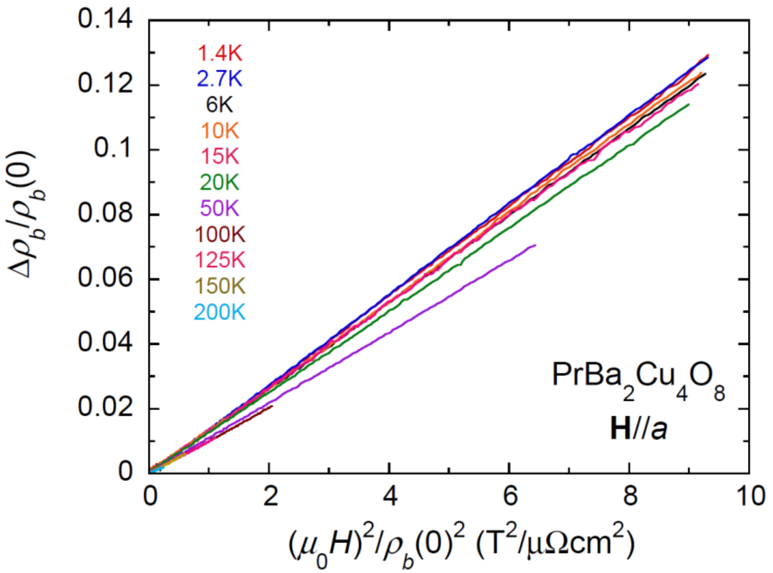
MR is zero in strictly isotropic system even though magnetoconductance is always finite

Kohler's rule

If the only effect of a change of temperature or of a change of purity of the metal is to alter τ_{tr}(k) to $\lambda\tau_{tr}$(k), where λ is not a function of **k**, then $\Delta\rho/\rho$ is unchanged if *B* is changed to *B*/λ. Since $\Delta\rho/\rho \propto (\omega_c\tau)^2$, the product $\Delta\rho.\rho$ (= $\Delta\rho/\rho$. ρ^2) is independent of τ_{tr} and a plot of $\Delta\rho/\rho$ versus $(B/\rho)^2$ is expected to fall on a straight line with a slope that is independent of *T* (provided the carrier concentration remains constant).

Kohler's rule is obeyed in a large number of standard metals, including those with two types of carriers, provided that changes in temperature or purity simply alter τ_{tr}(k) by the same factor.

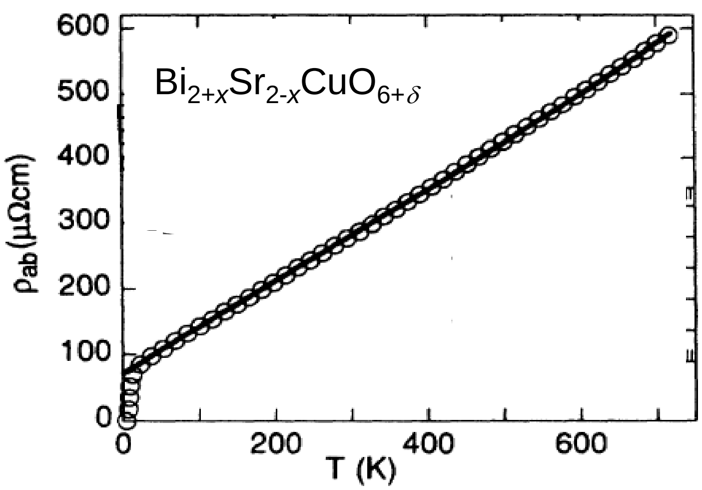
Narduzzo *et al.*,
PRL **98**, 146601 (07)



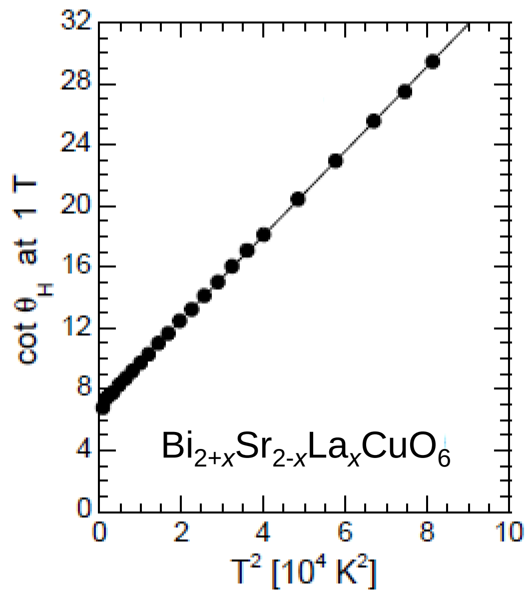
Deviations from Kohler's rule imply a change in carrier content, possibly a change in dimensionality, or most likely, a change in the variation of τ_{tr}(k) around the Fermi surface.

Why are *high temperature superconductors* interesting?

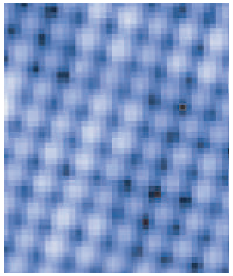
- Their transition temperatures are anomalously high
- Their superconducting order parameter is unconventional (*d*-wave)
- The superconductivity emerges out of a highly correlated insulating state
- Their normal metallic state is unlike anything that has been seen before



Martin *et al.*, *Phys. Rev. B* (1990)



Ando *et al.*, *Phys. Rev. B* (1999)



Towards a complete theory of high T_c

"The metallic state at optimal doping embodies the enlightenment. Rather than being complicated, this 'bad' metal shows a sacred simplicity – symbolized, for example, by its linear resistivity…"

Zaanen

"Most mysterious is the strange optimally doped metal. There simply do not seem to be (m)any good ideas on how to think about it. Yet it is empircally characterized by simply stated laws."

Senthil

"The biggest mystery is the linear resistivity at optimal doping."

Schmalian

"The difficulties lie with the normal state, featuring phenomena like the apparent linear-in-temperature resistivity at optimal doping."

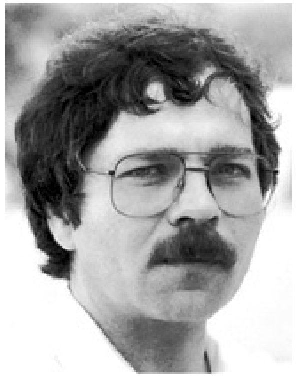
Vojta

"However, many questions about the precise nature of the transition from the superconductor to the Mott insulator, the possibilities of various competing/coexisting ordered states at low doping and, most particularly, the description of the non-Fermi-liquid normal states remain unclear at the present time."

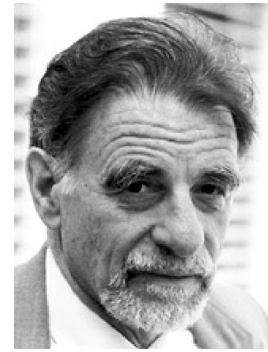
Randeria

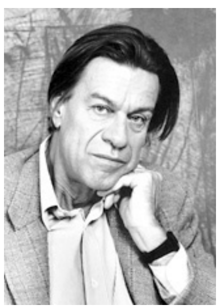
nature physics | VOL 2 | MARCH 2006 | www.nature.com/naturephysics

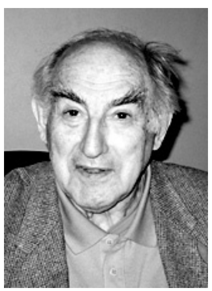
High temperature superconductivity and Nobel Laureates

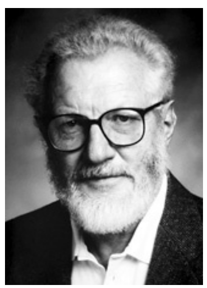


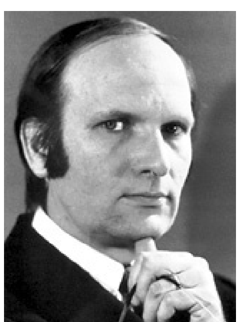
J. G. Bednorz (1987)
K. A. Muller (1987)



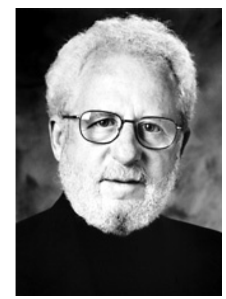


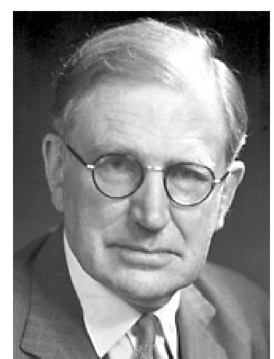


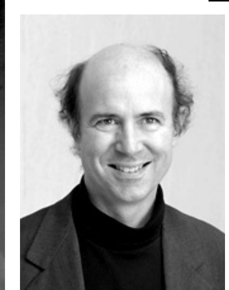




A. A. Abrikosov (2003)
P. W. Anderson (1977)
P. G. de Gennes (1991)
A. W. Geim (2010)
V. L. Ginzburg (2003)
A. J. Heeger (2000)
H. Kroemer (2000)
R. W. Laughlin (1998)
A. J. Leggett (2003)
N. F. Mott (1977)
J. R. Schrieffer (1972)
F. Wilczek (2004)



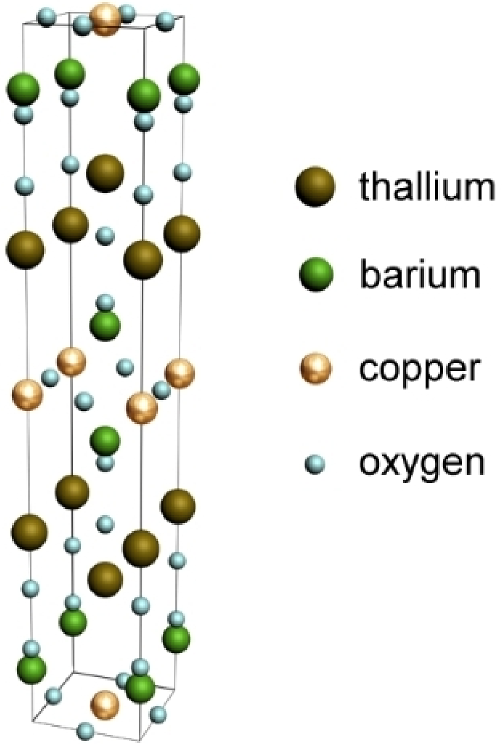




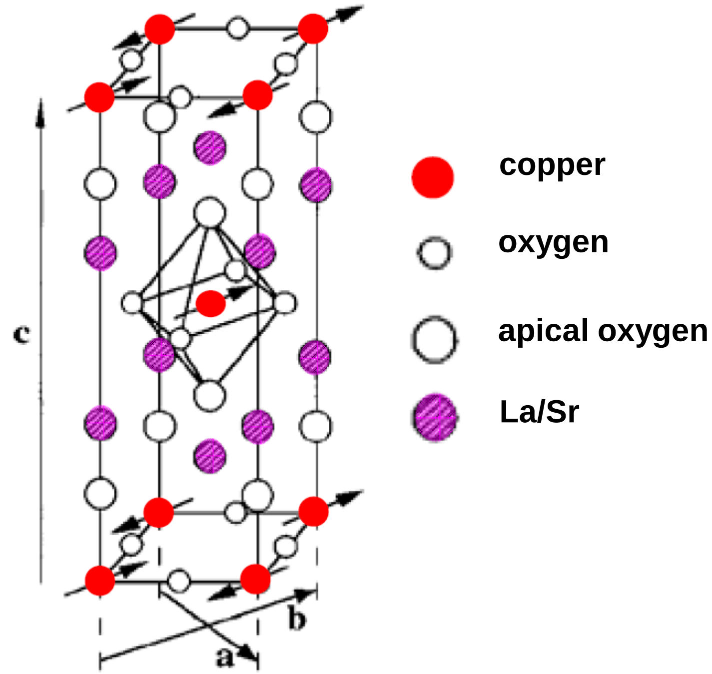
Crystal and electronic structure

The key structural element, the copper-oxide plaquette, appears either in a square planar, or an octahedral arrangement. Hence, the Jahn-Teller effect is not necessarily important in *all* cuprates.

$Tl_2Ba_2CuO_{6+\delta}$



$La_{2-x}Sr_xCuO_4$



Crystal and electronic structure

$(La^{3+})_2Cu^{2+}(O^{2-})_4$

Electronic configuration of Cu

$1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ 3d^9\ 4s^2$

Electronic configuration of Cu^{2+}

$1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ 3d^9$

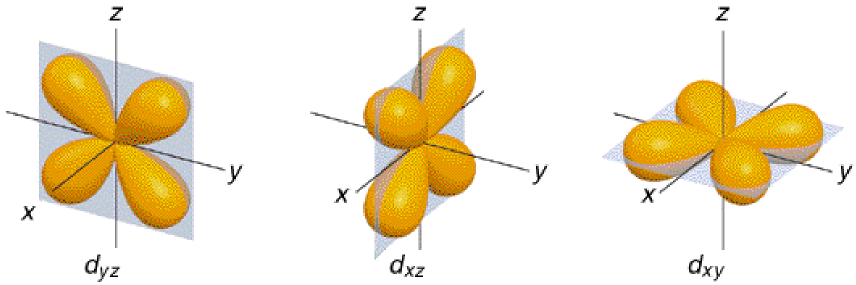
= 1 carrier/unit cell

$(La^{3+})_{2-x}(Sr^{2+})_xCu^{(2-x)+}(O^{2-})_4$

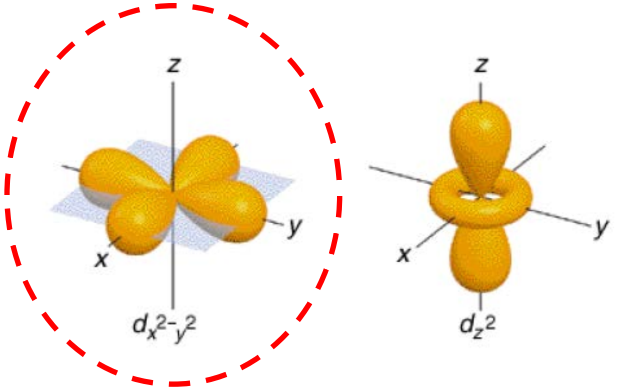
Electronic configuration of $Cu^{(2-x)+}$

$1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ 3d^{9-x}$

= Hole-doped CuO_2 plane

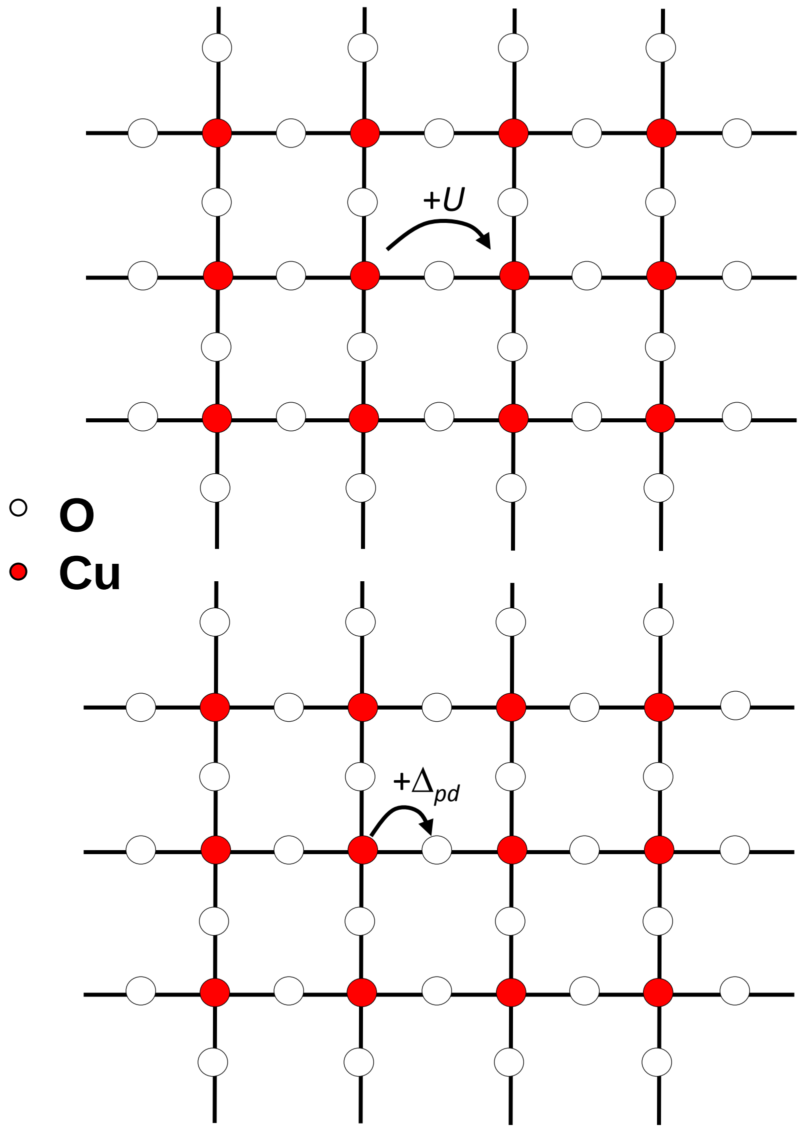


t_{2g} orbitals

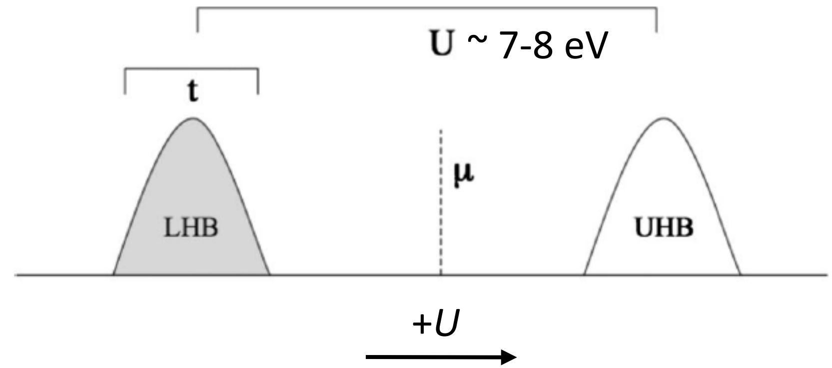


e_g orbitals

At half filling

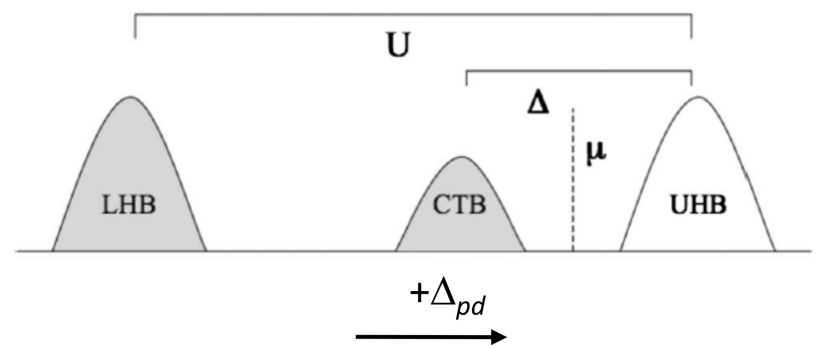


Mott insulator



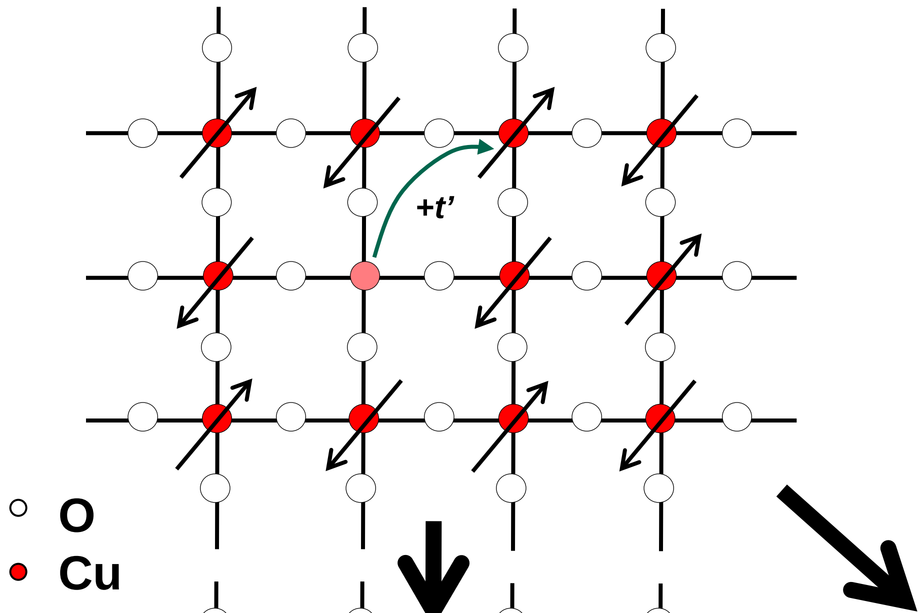
$3d^9\,3d^9 \quad 3d^8\,3d^{10}$

Charge transfer insulator



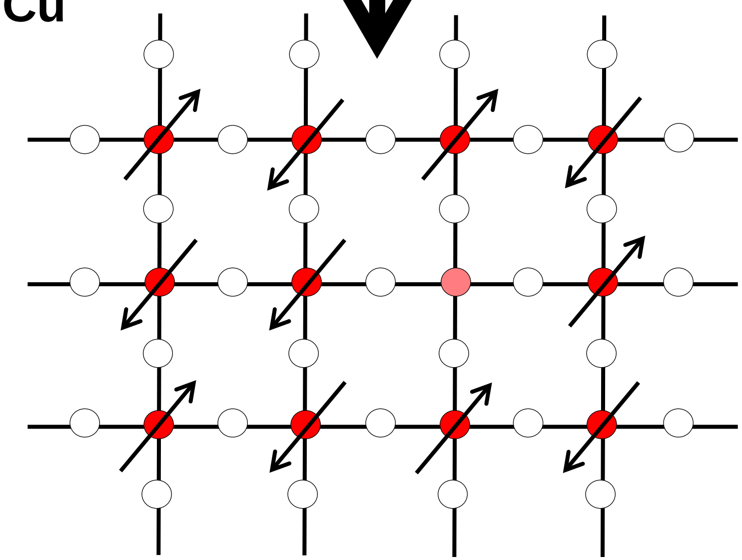
$3d^9 \quad 3d^{10}\,\underline{L}$

Beyond half filling…

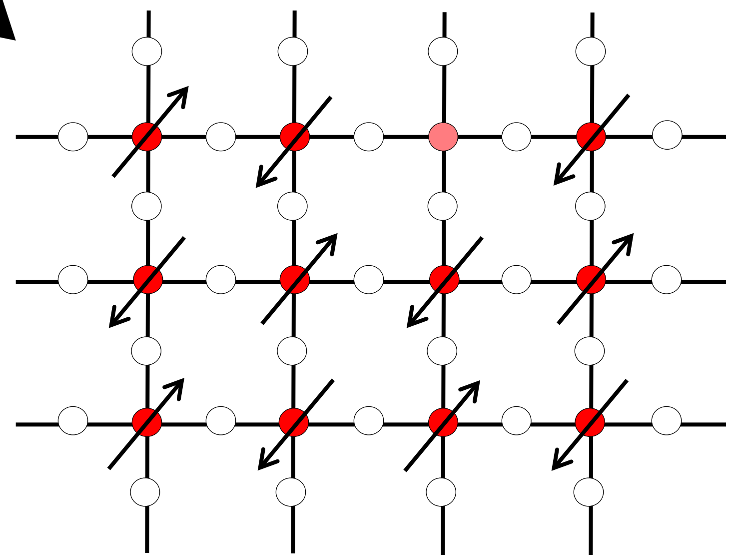


Doping away from half-filling strongly frustrates the spin background – leads to a strong suppression of the AFM ordering temperature.

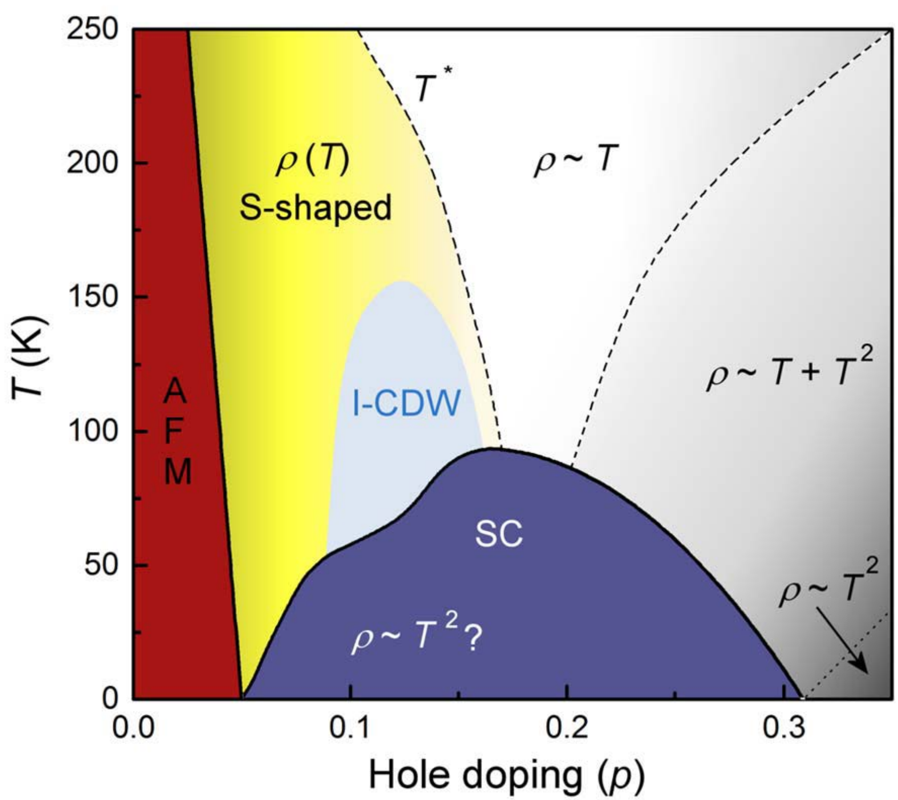
Frustration is significantly greater for spin exchange along the coordination axis than diagonally to it. Inherent in-plane anisotropy within the CuO_2 plane.



“Nodal-anti-nodal dichotomy”



Phase diagram



AFM suppressed with the addition of only a few percent of holes.

Superconductivity emerges at a hole concentration p ~ 0.05.

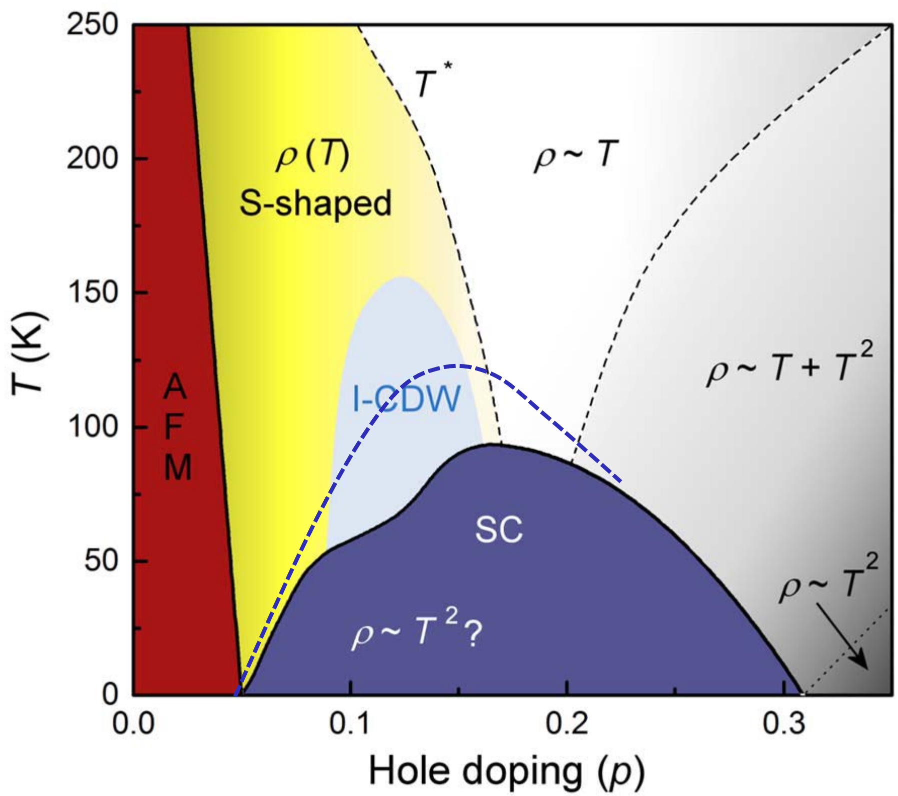
Maximum T_c occurs at an optimal doping of p = 0.16.

Superconductivity vanishes again around p > 0.30.

Dip in T_c also seen around p = 0.125 – so-called 1/8-anomaly.

Normal state phase diagram dominated by strange metallic phase and the pseudogap.

Phase diagram



In underdoped cuprates, possibly an additional temperature scale T_f due to extended region of phase fluctuating superconductivity

AFM suppressed with the addition of only a few percent of holes.

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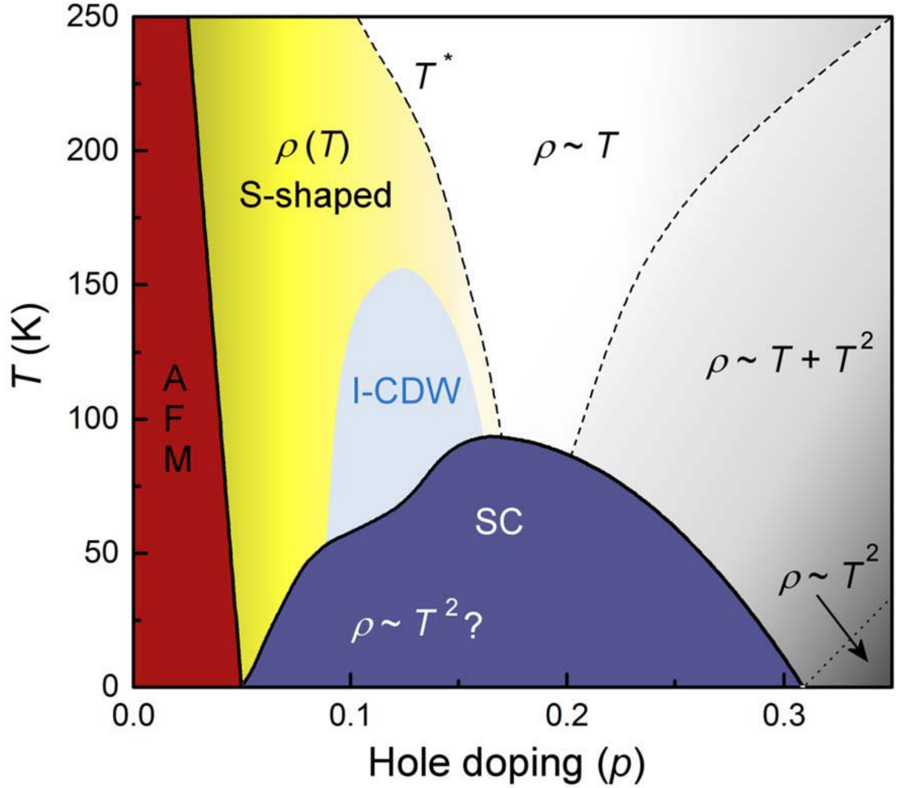
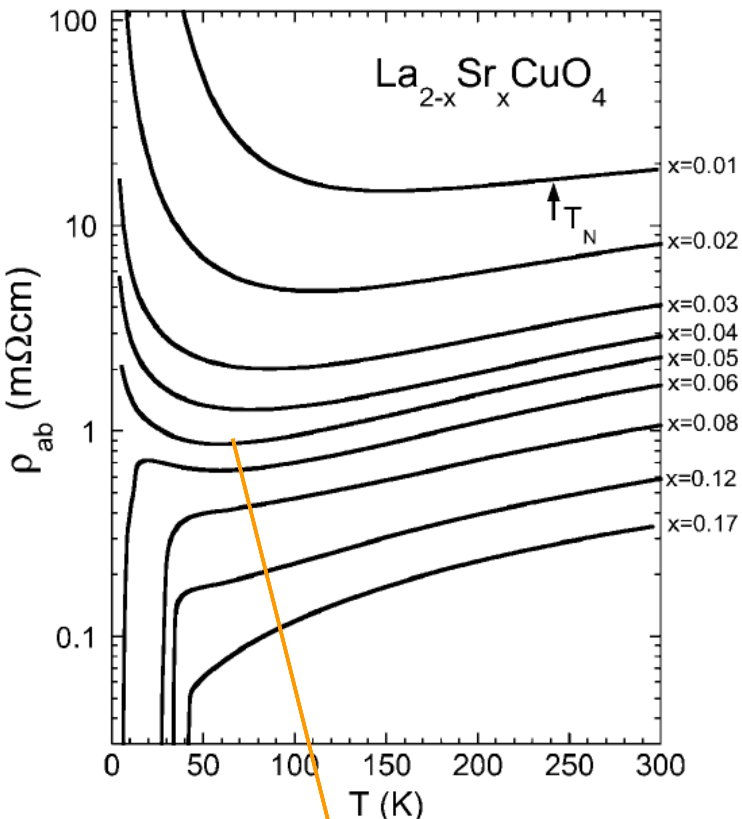
Normal state phase diagram dominated by strange metallic phase and the pseudogap.

Phase diagram – evolution of in-plane resistivity





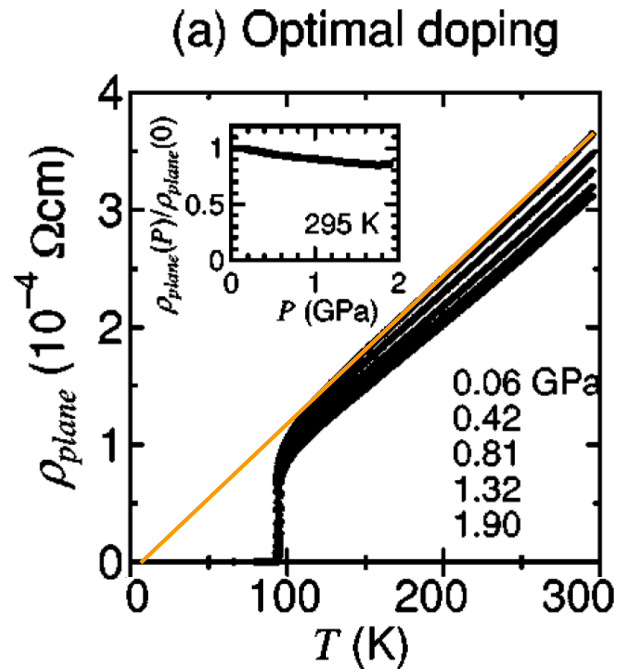
Note almost exponential decrease in ρ_{ab}(300) with increased doping, showing how the mobility of doped holes improves with doping.

At low doping, $\rho_{ab}(T)$ is non-metallic at low T.

Superconductivity develops out of insulating ground state.

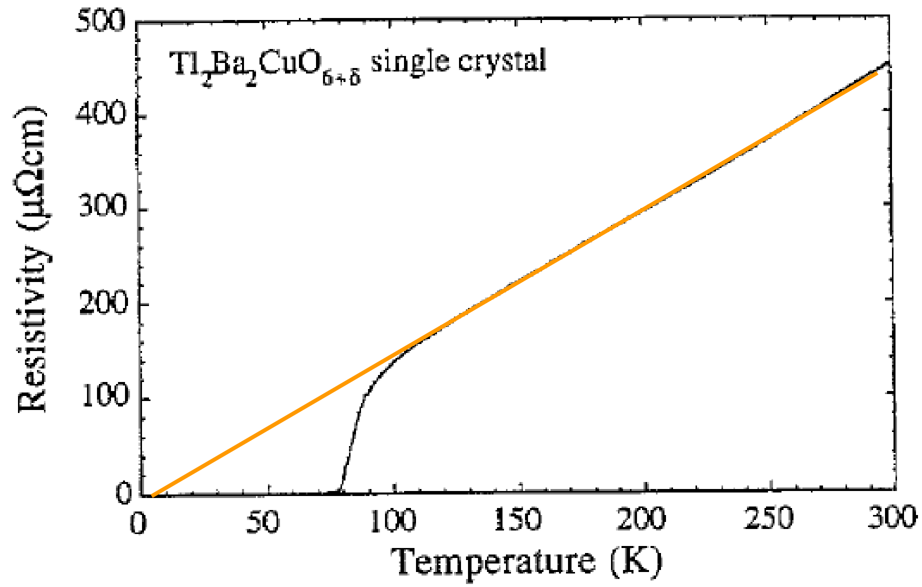
Optimally doped cuprates

$YBa_2Cu_3O_7$



Yoshida *et al.*, *PRB* **61** R15035 (99)

$Tl_2Ba_2CuO_{6+\delta}$



Tyler + Mackenzie, *PhysicaC* **282** 1185 (1997)

Negative intercepts imply that T-linear resistivity cannot continue to absolute zero – there has to be a crossover to a higher exponent of T at the lowest temperature

The problem(s) with phonons

,

1) T-linear resistivity is observed only in a narrow composition range near optimal doping; the sharp crossover to supralinear resistivity on the OD side being more suggestive of electron correlation effects than phonons.

2) The absence of resistivity saturation in OP $La_{2-x}Sr_xCuO_4$ up to 1000K argues against a dominant *e-ph* mechanism.

3) The frequency dependence of $1/\tau_{tr}$, extracted from extended Drude analysis of the in-plane optical conductivity, is inconsistent with an electron-boson scattering response due to phonons. In particular, $\Gamma(\omega)$ does not saturate at frequencies corresponding to typical phonon energies in HTC.

4) It has proved extremely problematic to explain the quadratic T-dependence of the inverse Hall angle $\cot\theta_H(T)$ in a scenario based solely on *e-ph* scattering.

Hall effect in hole-doped cuprates

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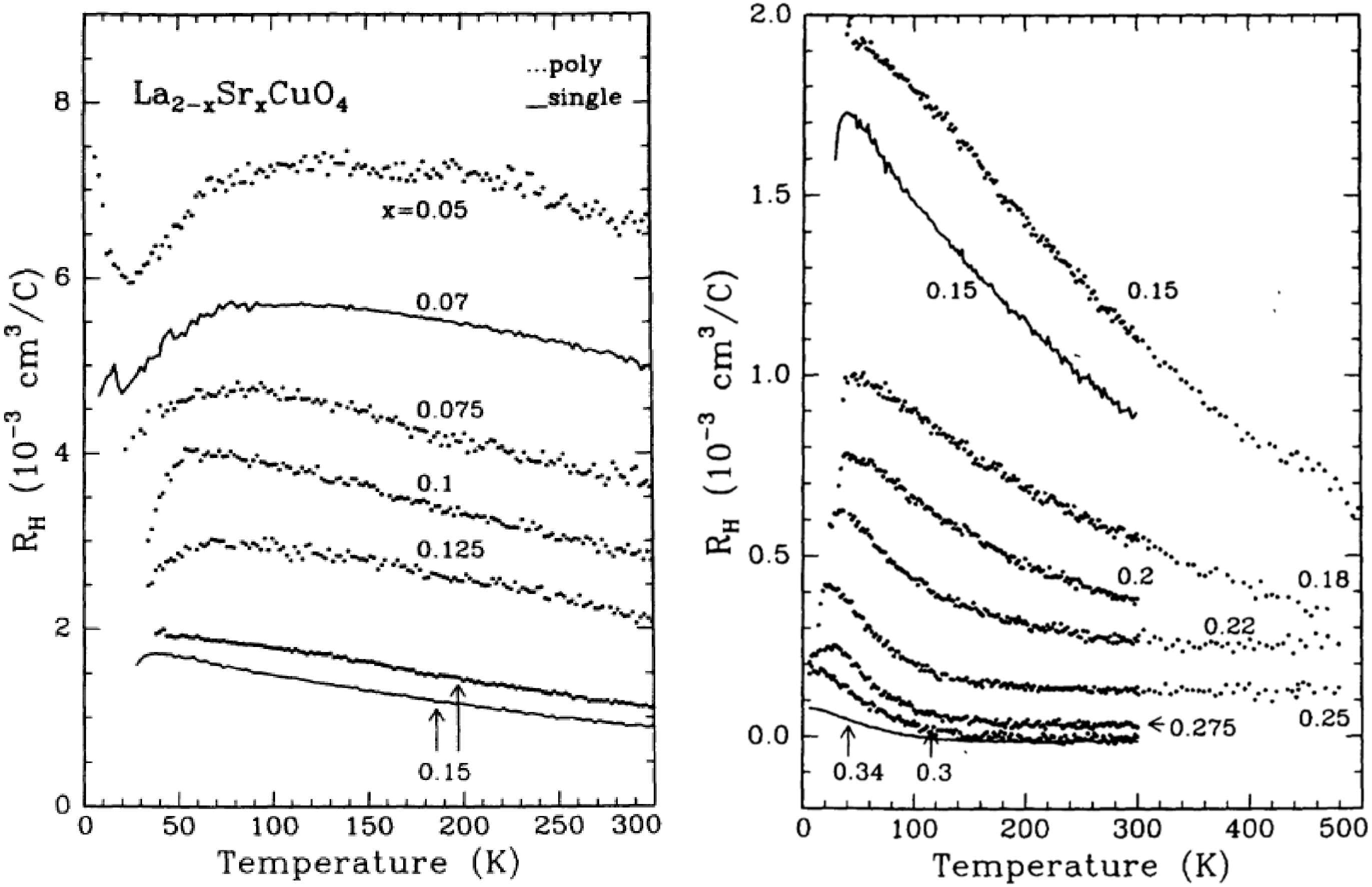

Hwang *et al.*, *PRL* **72** 2636 (94)

Strong *T*-dependence over a very wide temperature and doping range.
What does it signify?

Hall effect in hole-doped cuprates

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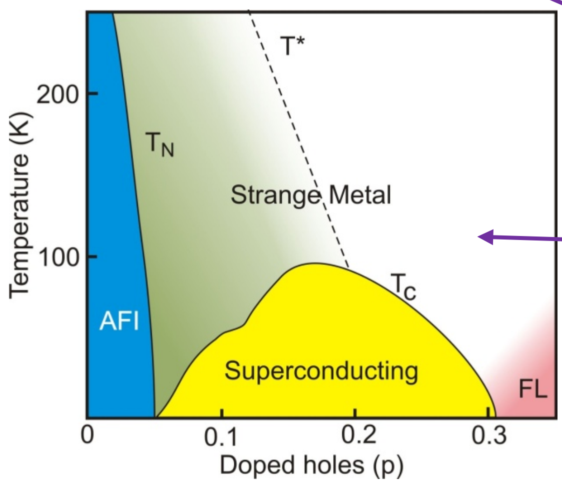
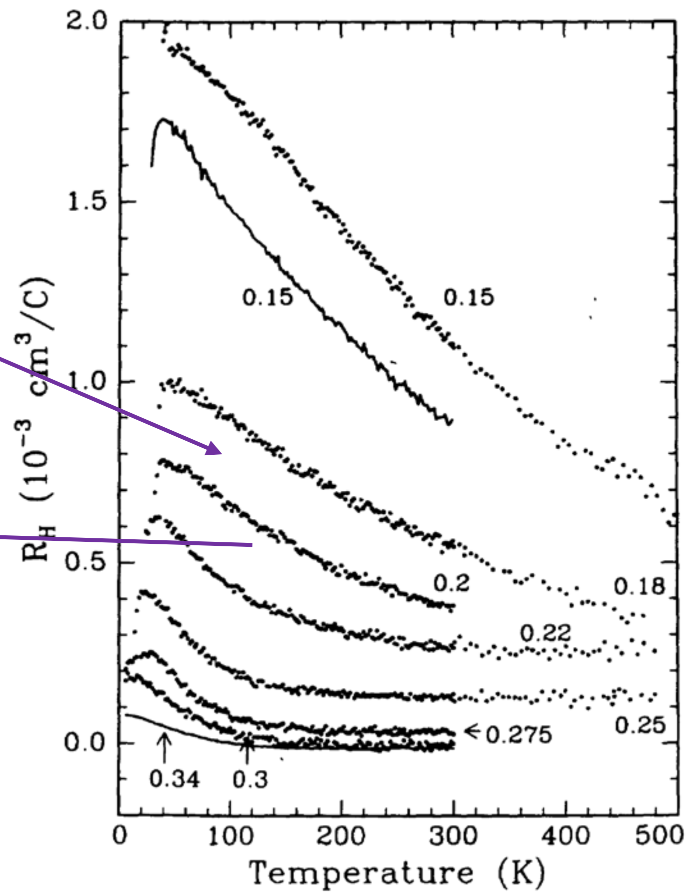
Marked, almost $1/T$ increase in $R_H(T)$ could indicate a reduction in carrier number with decreasing temperature (Recall that $R_H = 1/ne$)



However, this increase in $R_H(T)$ is occurring in range above pseudogap temperature.



Hwang *et al.*, *PRL* **72** 2636 (94)

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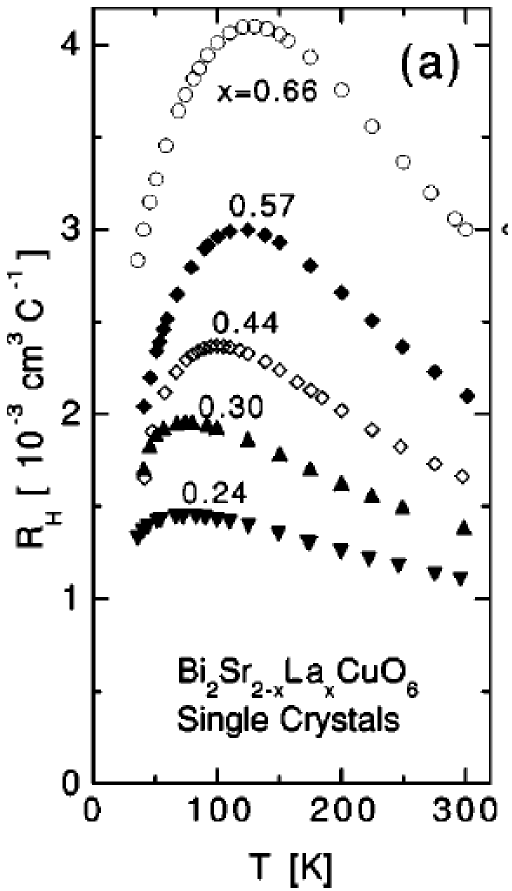
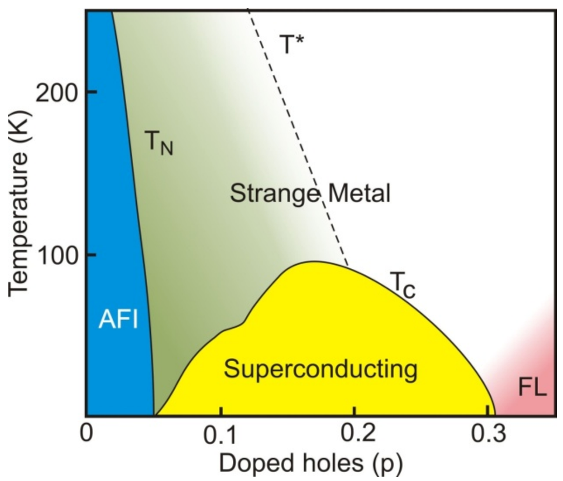


Ando + Murayama, *PRB* **60** R6991 (94)

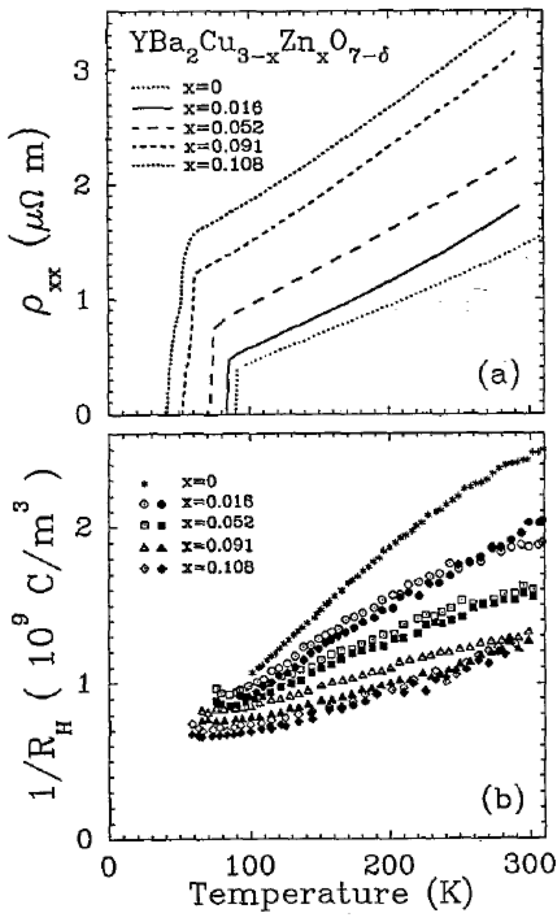
Moreover, in the underdoped regime, below $T = T^*$, $R_H(T)$ begins to decrease with T in the pseudogap regime.



This suggests that $R_H(T)$ does not in any way reflect the variation of $n(T)$.

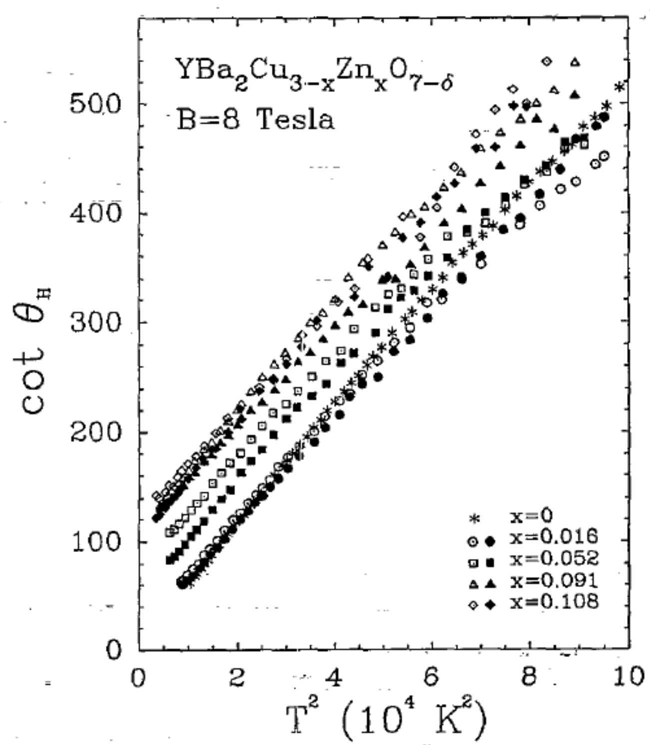
Cuprates are also largely single-band.

The inverse Hall angle



Chien *et al.*, *PRL* **67** 2088 (91)

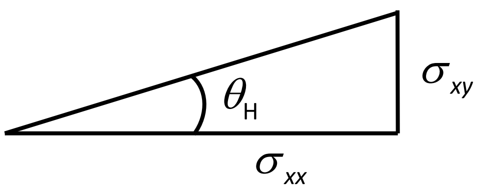
Variation in $\rho_{ab}(T)$ and $1/R_H(T)$ with Zn-doping in Y123 does not appear to be correlated with one another…



However when data plotted as $\cot\theta_H(T)$, it shows a very simple T^2 dependence.

The inverse Hall angle

$$R_H = \frac{\rho_{xy}}{B} = \frac{\sigma_{xy}}{B\sigma_{xx}^2}$$

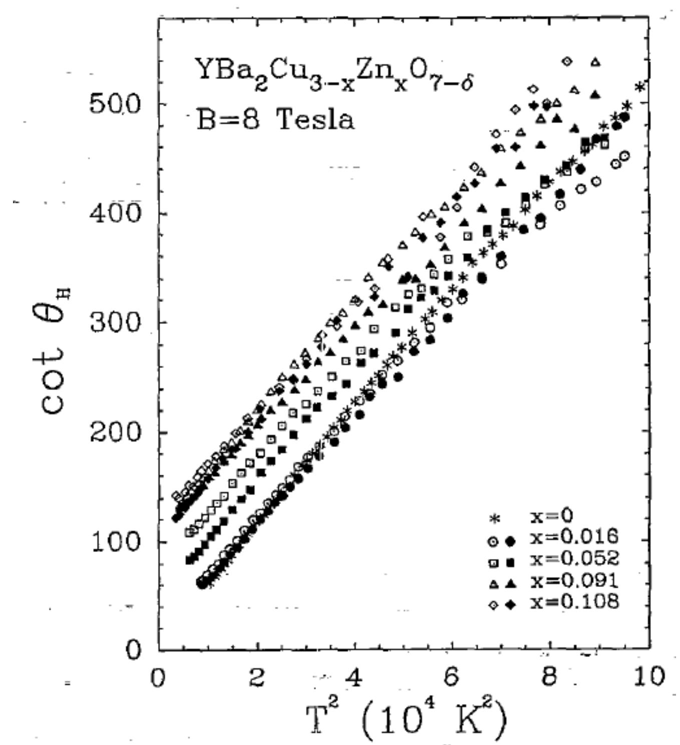


$$\tan\theta_H = \frac{\sigma_{xy}}{\sigma_{xx}}$$

$$\Rightarrow \cot\theta_H = \frac{\sigma_{xx}}{\sigma_{xy}} = \sigma_{xx} R_H B$$

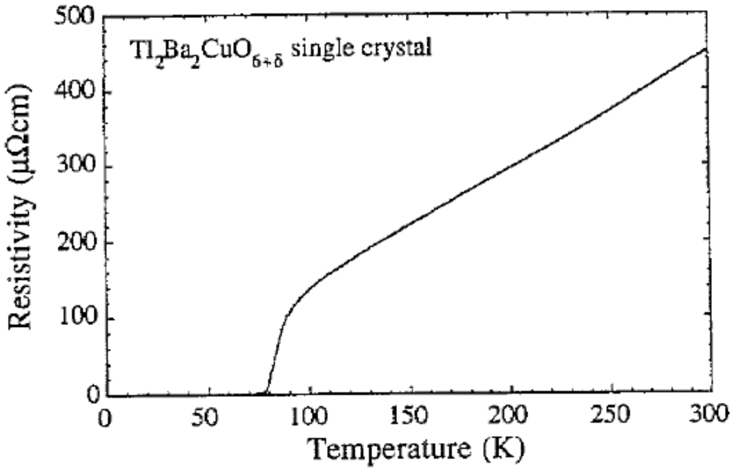
$$\Rightarrow \boxed{\cot\theta_H = \frac{R_H B}{\rho_{ab}}}$$

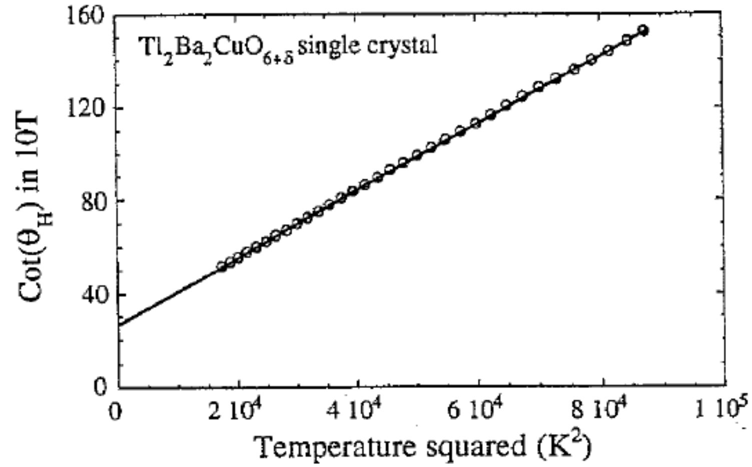
Variation in $\rho_{ab}(T)$ and $1/R_H(T)$ with Zn-doping in Y123 does not appear to be correlated with one another…



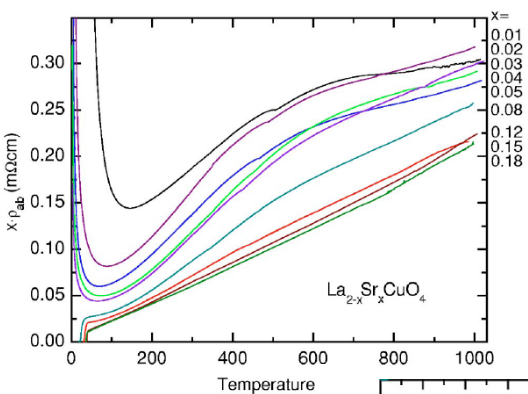
However when data plotted as $\cot\theta_H(T)$, it shows a very simple T^2 dependence.

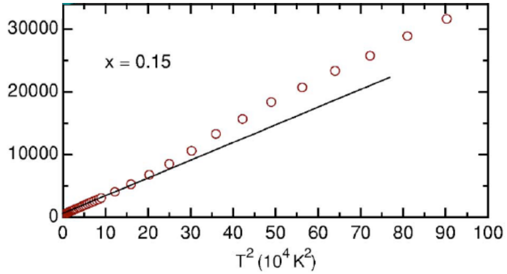
The separation of lifetimes





Tyler + Mackenzie, *Physica C* **282** 1185 (97)





Ono + Ando,
PRB **75** 024515 (07)

This so-called separation of lifetimes is most apparent at optimal doping, where $\rho_{ab}(T) \sim T$ and $\cot\theta_H(T) \sim T^2$.

Such remarkable transport behaviour is a hallmark of the cuprates that has engaged some of the greatest condensed matter thinkers.

We are not alone…

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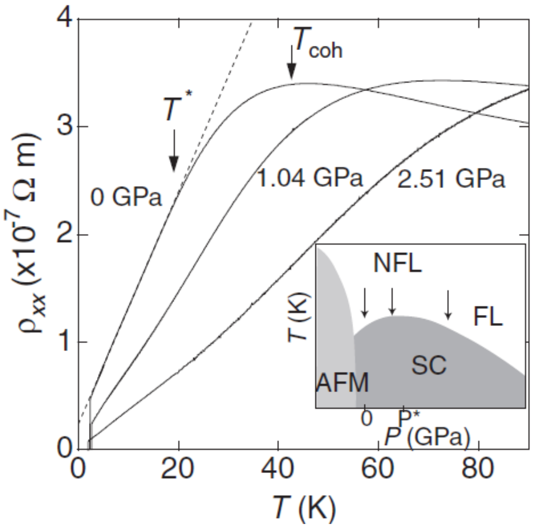
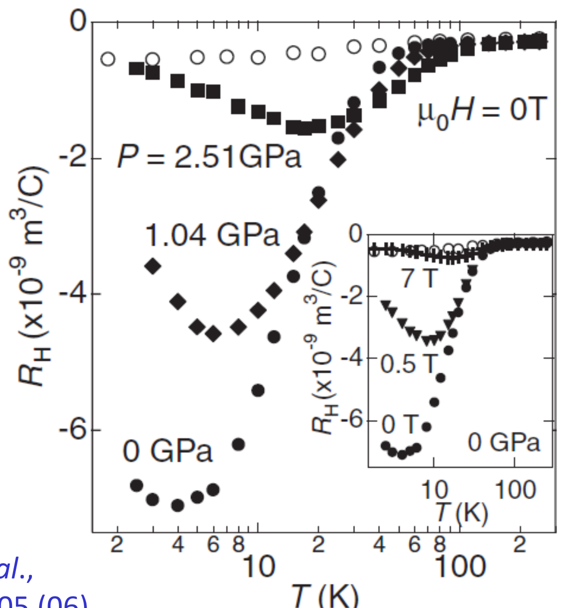
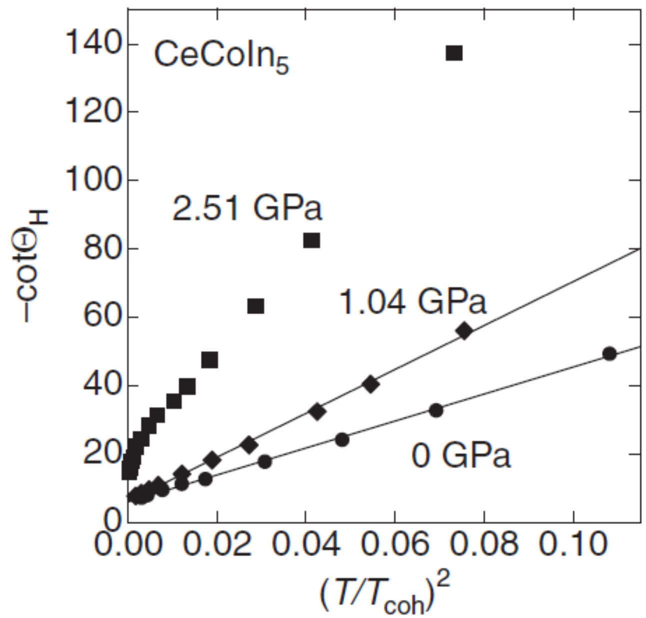






The quasi-2D heavy-fermion compound $CeCoIn_5$ shows a similar phenomenon of separation of lifetimes between T_c and T_{coh} ~ 20 K.

Nakajima *et al.*,
JPSJ **75** 023705 (06)

Transport theories – the "Big Three"

1) The two-lifetime model of Anderson and co-workers

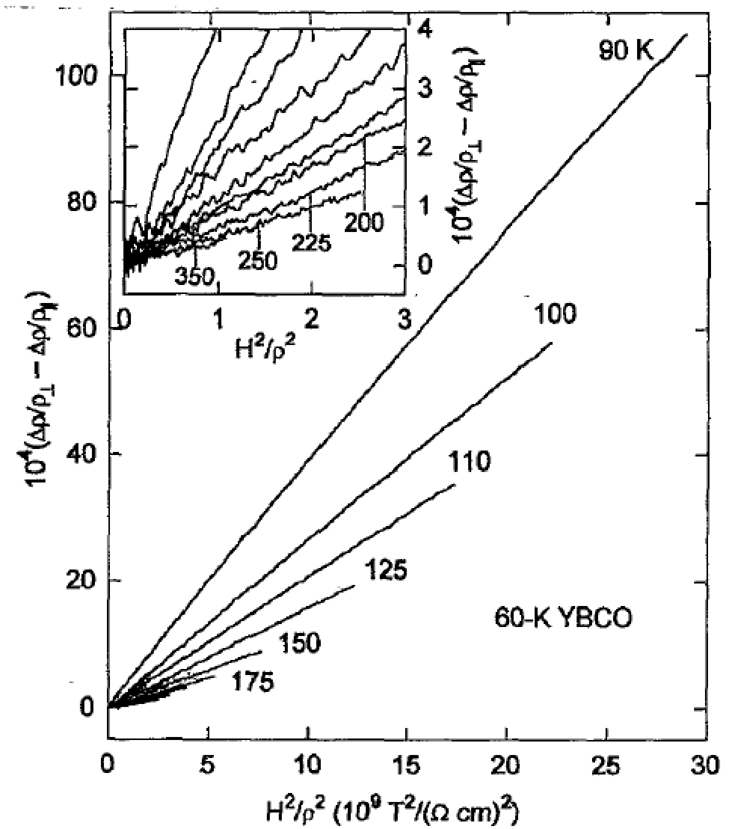
$$\boxed{\sigma_{ij}^{(n)} = \frac{e^3}{4\pi^3\hbar}\int v_i\left(\left(-\tau_{\mathrm{H}}\left[v_k \times B\right]\frac{\partial}{\partial k}\right)^n v_j \tau_{tr}\left(-\frac{\partial f_0}{\partial \varepsilon}\right)\right)\mathrm{d}^3k}$$

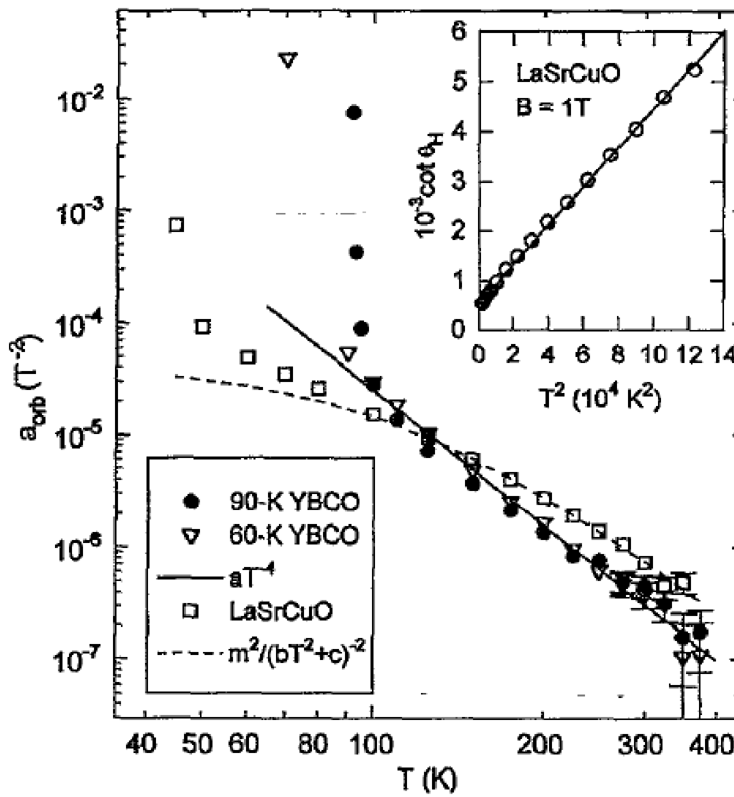
$$\frac{1}{\tau_{tr}} \propto T \qquad \Rightarrow \qquad \rho_{ab}(T) \propto \frac{1}{\tau_{tr}} \propto T$$

$$\frac{1}{\tau_{\mathrm{H}}} \propto T^2 \qquad \Rightarrow \qquad \cot\theta_{\mathrm{H}}(T) = \frac{\sigma_{xx}}{\sigma_{xy}^{(1)}} \propto \frac{\tau_{tr}}{\tau_{tr}\tau_{\mathrm{H}}} \propto \frac{1}{\tau_{\mathrm{H}}} \propto A + BT^2$$

$$\Rightarrow \qquad \frac{\Delta\rho_{ab}}{\rho_{ab}}(T) = -\frac{\sigma_{xx}^{(2)}}{\sigma_{xx}^{(0)}} - \left(\frac{\sigma_{xy}^{(1)}}{\sigma_{xx}^{(0)}}\right)^2 \propto \tau_{\mathrm{H}}^2 \propto \frac{1}{\left(A + BT^2\right)^2}$$

Violation of Kohler's rule





Harris *et al.*, *PRL* **75** 1391 (95)

Kohler's rule violated all the way up to room temperature

T-dependence of in-plane MR extremely well described by Anderson's two-lifetime model

The non-integer inverse Hall angle

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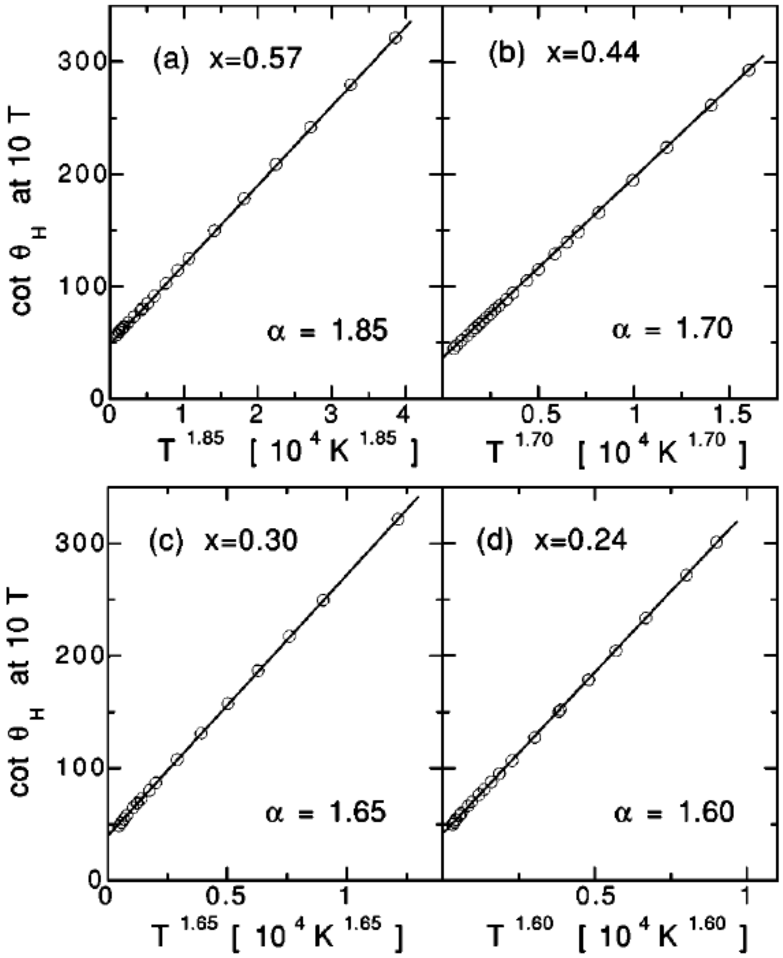
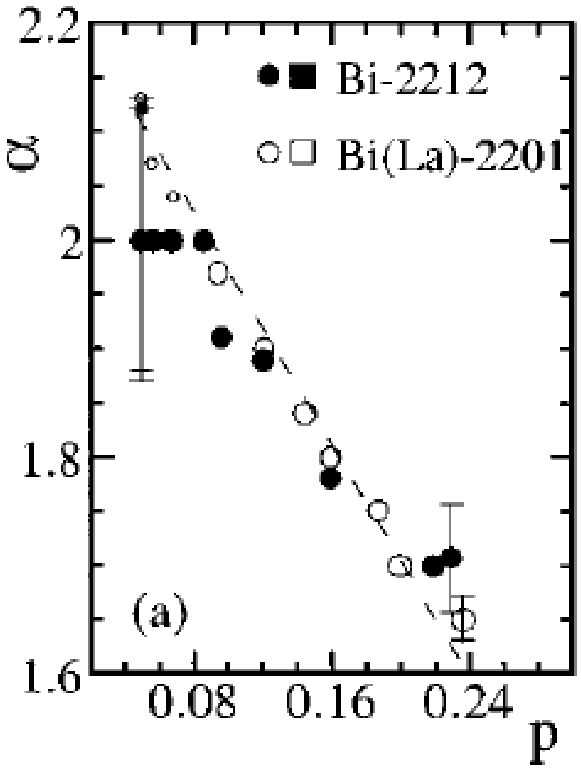


Ando + Murayama, *PRB* **60** R6991 (99)

This non-integer exponent in the inverse Hall angle and its doping-dependence are very hard to capture within this model

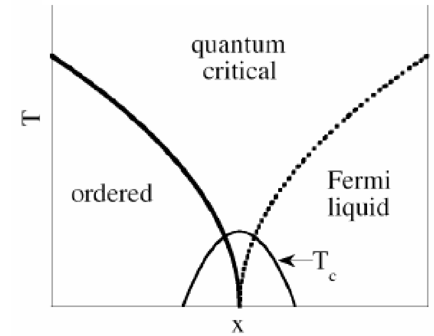


Konstantinovic *et al*., *PRB* **62** R11989 (00)

Transport theories – the "Big Three"

2) The marginal Fermi-liquid model of Varma and co-workers

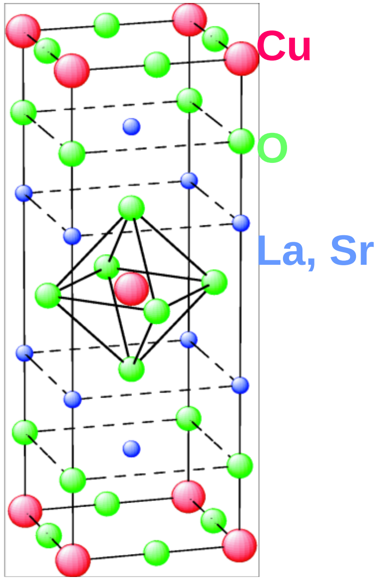
$$\boxed{\frac{1}{\tau_{tr}} \propto \lambda T + \gamma(\mathbf{k})}$$



The anisotropic T-independent elastic scattering term $\gamma(\mathrm{k})$ was argued to originate from small-angle scattering off impurities located out of the plane.

$\Rightarrow$ scattering rate anisotropy reflects that of DOS, i.e.

$$\boxed{\frac{1}{\tau}(\varphi) \propto \frac{1}{v_F}(\varphi)}$$



$\Rightarrow$ Violation of isotropic-l approximation and strong variation in $R_H(T)$ and

$$\boxed{\cot\theta_H(T) \propto \left(\frac{1}{\tau_{tr}}\right)^2}$$

Unfortunately, this model struggles to account for both the in-plane MR and the evolution of the exponents in OD cuprates.

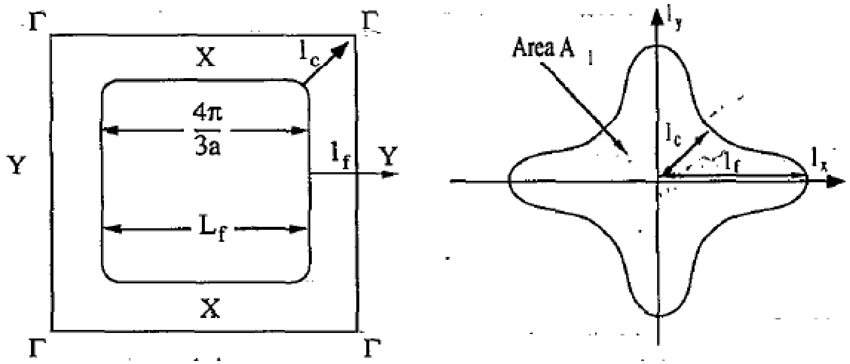
Carter+Schofield,
PRB **66** 241102(R) (02)

Transport theories – the "Big Three"

3) Anisotropic single lifetime models

3a) Nearly AFM FL model

Carrington *et al.*, *PRL* **69** 2855 (92)
Stojkovic+Pines, *PRL* **76** 811 (96)



$$\frac{1}{\ell_c} \propto T^2 \quad ; \quad \frac{1}{\ell_f} \propto T$$

3b) Cold spots model

Ioffe+Millis, *PRB* **58** 11631(98)

$$\Gamma(\varphi) \propto \Gamma_0 \sin^2(2\varphi) + \frac{1}{\tau_F}$$

3c) Anisotropic scattering rate saturation model

Hussey, *EPJB* **31** 495 (03)
Hussey, *JPCM* **20** 123201 (08)

$$\Gamma_{\text{ideal}}(\varphi) = \Gamma_0(\varphi) + \Gamma_1 \sin^2(2\varphi)T + \Gamma_2 T^2$$

$$\frac{1}{\Gamma_{\text{eff}}} = \frac{1}{\Gamma_{\text{ideal}}} + \frac{1}{\Gamma_{\max}}$$

Altogether now!

$$\frac{1}{\tau_{tr}} \propto \lambda T + \gamma(\mathbf{k})$$

MFL

Two-lifetime

$$\frac{1}{\tau_{tr}} \propto T \quad ; \quad \frac{1}{\tau_{\mathrm{H}}} \propto T^2$$

$$\Gamma_{\mathrm{ideal}}(\varphi) = \Gamma_0(\varphi) + \Gamma_1 \sin^2(2\varphi) T + \Gamma_2 T^2$$

$$\frac{1}{\Gamma_{\mathrm{eff}}} = \frac{1}{\Gamma_{\mathrm{ideal}}} + \frac{1}{\Gamma_{\max}}$$

$$\Gamma(\varphi) \propto \Gamma_0 \sin^2(2\varphi) + \frac{1}{\tau_r}$$

Cold Spots model

$$\frac{1}{\ell_f} \propto T \quad ; \quad \frac{1}{\ell_c} \propto T^2$$

NAFL