Introduction to the 4 lectures. ①

What happens in a gauge th ?
strongly coupled : → becomes strong under RG
→ UV coupling tunable
⇒ make it strong by hand

With SUSY, quite tractable!

- $\mathcal{N}=4$ SYM $SU(N)$ ⟲ "S-duality"
- $\mathcal{N}=2$ $SU(2)+N_f=4$ ⟲ but the $SO(8)$ rep. of matter fields are exchanged
- $\mathcal{N}=2$ $SU(\underset{N}{3})+N_f=\underset{2N}{6}$ ↔ $\mathcal{N}=2$ $SU(2)+N_f=1$ + 'something'

Montonen-Olive '77

Seiberg-Witten '94

Argyres-Seiberg '07

Gaiotto '09 : general framework to understand S-duality with $\mathcal{N}=2$ SUSY.

- 'strange' matter content almost always appears.
- can be understood by compactifying 6d $\mathcal{N}=(2,0)$ theory on a Riemann surface.
- 6d $\mathcal{N}=(2,0)$ th is 'strange', so its cptf. is also 'strange'

'strange' = 'doesn't have useful Lag. desc.'

$Z_{\underset{S}{6d}}$ (X_4 × )

Z_{4d} ${}_{S[}$  ${}_{]}$ (X_4) $=$ $Z_{\underset{S[X_4]}{2d}}$ ()

if Z_{6d} is the susy part-func. indep. of the size of X_4 & 

S[] : class S theories S stands for six?

Introduction ②

$S[$  $_{S^4}] =$ 2d Liouville/Toda theory '09 Alday-Gaiotto-YT

$S[$  $_{S^3} \times O_{S^1}] =$ 2d q-deformed YM theory '11 Gadde-Rastelli-Razamat-Yan.

equation between QFT's !

- In the Trieste spring school '~~09~~ 10 I talked about the first equation.
- Aim this spring : introduce $\mathcal{N}=2$ S-dualities, focusing on the second equation.

Refs.

1312.2684 ← gen. review of Seiberg-Witten th & S-dualities

1412.7121 ← review of inst counting (relevant to the S^4 case)

http://member.ipmu.jp/yuji.tachikawa/tmp/

2d4d~~review~~.pdf ← light review on both $S^1\times S^3$ & S^4, only SU(2)

tn review.pdf ← review on the properties of the $T_N = S[$  $]$ th.

Contents.

- 2d qYM
- 4d $\mathcal{N}=2$ Lagrangian. — 1st lec.
- SU(2) $N_f=4$ and the SCI. — 2nd lec.
- 6d interpretation and $\mathcal{N}=4$ SYM.
- $SU(2) \to SU(3), SU(N)$? — 3rd lec.
- T_N theory.
- 'partial closure of punctures' & A-S duality. — 4th lec.

Sheet Thompson 9305120 Date

~~Blau-Thompson~~ ~~9310144~~

see, Ramgoolam & Moore.

~~Cordes~~ 9411210

2d YM.

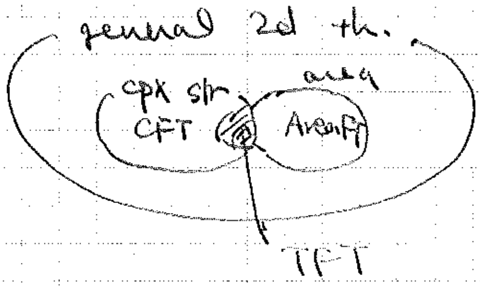
$$S \propto \underset{\uparrow}{\frac{1}{e^2}} \int d^2x \sqrt{\det g}\ \underbrace{\mathrm{tr} F_{\mu\nu}F^{\mu\nu}}_{\underbrace{(g^{00}g^{11} - g^{01}g^{10})\,\mathrm{tr}(F_{01})^2}_{\sqrt{\det g}^{-1}\ \mathrm{tr}(F_{01})^2}}$$

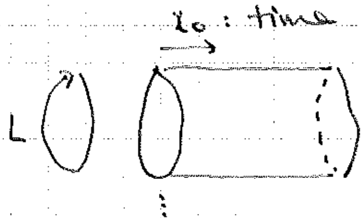
coupling const.

$F_{\mu\nu}$: G gauge field.

⇝ the metric only appears in the combination $\det g$.

⇝ doesn't depend on the precise form of the metric only the total area matters!





state space $\mathcal{H}_{S^1}$ ← func. of $U := P\exp\int_0^L A_1 dx' \in G$

residual G action at $x_1 = 0 = L$

$$U \to gUg^{-1}$$

⇝ $\psi(U) = \psi(gUg^{-1})$: 'class functions' on G.

spanned by $\chi_R(U)$ for irrep R.

$$\int_G \chi_R(U)\overline{\chi_{R'}(U)}\,[dU]_{\mathrm{Haar}} = \delta_{RR'} \quad \text{: ortho-normal.}$$

· What's the Hamiltonian? $H \propto \int dx_1 E_1^2$, $E_1^a(x) \sim \frac{\delta}{\delta A_1(x)}$.

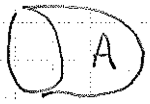
$$\propto \int dx_1 \frac{\delta}{\delta A_1(x)}\frac{\delta}{\delta A_1(x)}.$$

acting on $\chi_R(U) = \mathrm{tr}_R P\exp\int_0^L T^a A_1^a dx'$,

$$H\chi_R(U) \propto \mathrm{tr}_R \int_0^L T^a T^a dx'\ P\exp\int_0^L A_1 dx'$$

$$\propto L\,c_2(R)\chi_R(U), \quad c_2(R) = \mathrm{tr}_R T^a T^a.$$

on a torus $\ell\overset{T}{\boxed{}}$. $Z = \mathrm{tr}\, e^{-TH} = \sum_R e^{-\underbrace{TL}_{}\,c_2(R)}$ only the area matters!

How about ? this specifies a state $\in \mathcal{H}_{S^1}$.

Hartle-Hawking wave function.

$\psi_A(U)$.

$\psi_{A+A'}(U) = e^{-A'C_2}\psi_A(U)$

$\rightsquigarrow$ knowing $\psi_0(U)$ is enough.

$\propto \delta(U)$: delta f. on G.

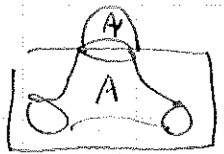
$$\text{to get } d_R.\ \text{do } \int \delta(U) \underset{\overset{\parallel}{\chi_{R'}(1) = \dim R'}}{\chi_{R'}(U)}\, dU = \int \Big(\overset{=\sum_R d_R \chi_R(U)}{\sum_R d_R \chi_R(U)}\Big) \chi_{R'}(U)\, dU$$

$\overset{\parallel}{d_{R'}}$ ← equal!

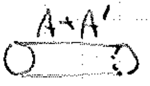
so $\psi_A(U) \overset{\propto}{=} \sum_R e^{-AC_2(R)} (\dim R)\, \chi_R(U)$.

How about ? this specifies a state in $\mathcal{H}_{S_1}\otimes\mathcal{H}_{S_1}\otimes\mathcal{H}_{S_1}$

$\sum_R \alpha \dim R\, \chi_R(U_1)$.



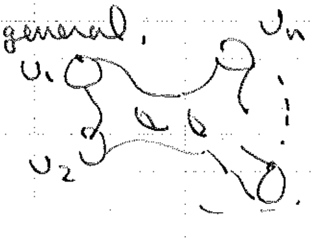
$=$



$\sum_R \chi_R(U_1)\chi_R(U_2)\, e^{-(A+A')C_2(R)}$.

$$\sum_R \frac{\chi_R(U_1)\chi_R(U_2)\chi_R(U_3)}{\alpha \dim R}\, e^{-AC_2(R)}$$

in general,



$$\rightsquigarrow \sum_R e^{-AC_2(R)} \frac{\prod_i \chi_R(U_i)}{(\boxed{\alpha}\cdot \dim R)^{2g-2+n}},$$

$$\delta S = \delta\beta \underbrace{\int d^2x \sqrt{g} R}_{} = \delta\beta(2-2g-n).$$

can be generated by { renormalization / regularization. of Planck mass.

$\rightsquigarrow \alpha \mapsto \alpha e^{-\beta}$.

q-deformation make the gauge gp. to be the quantum group.

Buffenoir-Roche, 9403066
Alekseev-Grosse-Schomerus
9405126

① matrix entries become noncommutative.
② Cartan torus is not deformed.

③ $\dim R \to \dim_q R = \chi_R(q^{\frac{N-1}{2}}, \cdots q^{\frac{1-N}{2}})$.

for explanation, see ✡

$$U_i{}^j = \begin{array}{c|cc} & \xrightarrow{j} & \\ \downarrow i & \alpha & \beta \\ & \gamma & \delta \end{array}$$

$$\begin{cases} \alpha\beta = q\beta\alpha \\ \alpha\gamma = q\gamma\alpha \\ \beta\gamma = \gamma\beta \\ \alpha\delta - q\beta\gamma = \delta\alpha - q^{-1}\gamma\beta = 1 \end{cases}$$

$$\beta\delta = q\delta\beta$$
$$\gamma\delta = q\delta\gamma$$

~~$U^{-1} = \begin{pmatrix} \delta & -q\gamma \\ q^{-1}\beta & \alpha \end{pmatrix}$~~

~~$\overset{\shortparallel}{U^\dagger} = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}$~~

~~$\bar{U}_{\bar{i}}{}^{\bar{J}} = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} = \begin{array}{c|cc} & \xrightarrow{\bar{J}} & \\ \downarrow \bar{i} & \delta & q^{-1}\beta \\ & -q\gamma & \alpha \end{array}$~~

$SU(2)_q$.

$U_i{}^j \, \bar{U}_{\bar{i}}{}^{\bar{J}}$

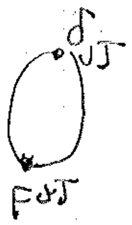
$$U^\dagger = U^{-1} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}$$

$$\bar{U}_{\bar{i}}{}^{\bar{J}} = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} = \begin{array}{c|cc} & \xrightarrow{\bar{J}} & \\ \downarrow \bar{i} & \delta & -q\gamma \\ & -q^{-1}\beta & \alpha \end{array}$$

$$\delta_{j\bar{J}} \, U_i{}^j \, \bar{U}_{\bar{i}}{}^{\bar{J}} = \delta_{i\bar{i}}$$

$$\delta^{i\bar{i}} \, U_i{}^J \, \bar{U}_i{}^{\bar{J}} \neq \delta^{j\bar{J}}$$

$$F^{i\bar{i}} \, U_i{}^j \, \bar{U}_{\bar{i}}{}^{\bar{J}} = F^{j\bar{J}} \qquad F = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$$



$= q + q^{-1}$.

4d $\mathcal{N}=2$ Lagrangians ①

Vector mult. $\mathcal{N}=2$

$$\begin{array}{ll|l|l}
W_\alpha : & & \lambda^\alpha & F_{\alpha\beta} \\
\Phi : & \phi & \psi^a & \\
\hline
& 2 & 1 & 0
\end{array} \quad \text{SU(2)}_R \text{ doublet}$$

hypermult. (full)

$$\begin{array}{ll|l|l}
Q : & & q & \psi_\alpha \\
\tilde{Q}^\dagger : & \tilde{\psi}^\dagger_\alpha & \tilde{q}^\dagger & \\
& 1 & 0 & -1
\end{array}$$

When Q is in a pseudo-real rep, can impose $\tilde{Q}_a = J_{ab} Q^b$ compatible with SUSY. "half-hyper"

Lagrangian in $\mathcal{N}=1$ notation:

$$\int d^2\theta\ \tau\ \mathrm{tr} W^\alpha W_\alpha + \frac{1}{g^2}\int d^4\theta\ \mathrm{tr}\,\Phi^\dagger\Phi$$

$$+ \int d^2\theta\ \mathrm{I}\ \tilde{Q}^i_a \Phi^a_b Q^b_i + \int d^4\theta \left(Q^\dagger e^V Q + \tilde{Q} e^V \tilde{Q}^\dagger\right).$$

where $\tau = \frac{4\pi}{g^2} i + \frac{\theta}{2\pi}$.

One-loop beta

$$\propto 2T(\text{adj}) - \sum_i \underbrace{T(R_i) \oplus T(\overline{R_i})}_{\text{full hyper in } R\oplus\overline{R}}$$

where $T(R)$ is $\mathrm{tr}_R T^a T^b = T(R)\delta^{ab}$

$T(\square) = \frac{1}{2}$ $T(\text{adj}) = N$ for $SU(N)$.

$= 0$ for
- $\mathcal{N}=4$ SYM where $R = \text{adj}$
- $SU(N)$ with $2N$ fund $\oplus$ $\overline{\text{fund}}$

one-loop beta = 0 ⇒ conformal

in holomorphic scheme, τ : one-loop exact
1 in front of $\tilde{Q}\Phi Q$ } not renormalized.

$\frac{1}{g^2}$ i.f.o. $\mathrm{tr}\,\phi^\dagger\phi$ ⟷ τ

1 in front of $\tilde{Q}\Phi Q$ ⟷ 1 i.f.o. $Q^\dagger e^V Q + \tilde{Q} e^V \tilde{Q}^\dagger$

locked by $\mathcal{N}=2$ SUSY.
↓
no wavef. renormal. either

⇝ the theory is conformal,
τ is a parameter one can choose in UV.

Basic S-duality. ①

What happens when we make g very big in these theories?

$G = SU(2)$ potential $= \mathrm{tr}[\phi, \phi^\dagger]^2$ + terms involving Q's

$\phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$, $Q = 0 = \tilde{Q}$ is a SUSY vac.

breaks $SU(2) \to U(1)$. ⇝ 't Hooft-Polyakov monopole.

massive Q-fields

massive W-boson

$m_W = |a|$

$m_M = |\tau a|$.

$m_Q = |a|$: $\mathcal{N}=4$ SYM

$= \frac{1}{2}|a|$: SU(2) with 4 flavors

$\underbrace{\text{perturb. excitations}}$ $\ll$ $\underbrace{\text{solitonic excitations.}}$

when g small

$\gg$

when g very big.

S-duality in $\mathcal{N}=4$ SYM : $\underline{\text{monopole}} \longleftrightarrow$ W-bosons $|a|$

SU(2) 4 fl. : $\underline{\text{monopole}} \longleftrightarrow$ Q-fields. $\frac{1}{2}|a|$

needs to be in $\mathcal{N}=4$ vect mult.

needs to be in $\mathcal{N}=2$ hyp. mult.

To see this, for $\mathcal{N}=4$,

$$\begin{array}{ccccccccc} & \Phi & & Q & & \tilde{Q} & & & \\ \text{give vev} \leftarrow & \boxed{\phi_1}\ \phi_2 & & \phi_3\ \phi_4 & & \phi_5\ \phi_6 & \leftarrow & SO(6)_R \simeq SU(4)_R & \mathbf{6} \\ & \lambda\ \lambda & & \lambda & & \lambda & \leftarrow & SU(4)_R. & \mathbf{4} \end{array}$$

$$\begin{array}{c} SO(6)_R \\ \downarrow \\ SO(5)_R. \end{array}$$

$\uparrow$ $USp(4)_R \simeq SO(5)_R$

monopole bkg : zero modes of $\lambda^{i=1,2,3,4}_{\alpha=1,2}$ ⇝ $b^{i=1,2,3,4}_{\alpha=1,2}$

become Q.M. operators with

$$\{b^i{}_\alpha, b^j{}_\beta\} = J^{ij}\varepsilon_{\alpha\beta} \quad , \quad (b^i{}_\alpha)^\dagger = J_{ij}\,\varepsilon^{\alpha\beta}\, b^j{}_\beta .$$

$\overrightarrow{8}$ gamma matrices in a funny basis.

$\nwarrow$ SO(8) dummy vector $= \mathbf{2} \otimes \mathbf{4}$

$b^i{}_\alpha \underbrace{|\text{monopole}\rangle}_{\text{SO(8) dummy spinor + spinor}}$

$\mathbf{3} \oplus \mathbf{5}$ $\quad$ $\mathbf{2} \otimes \mathbf{4}$

$\mathcal{N}=4$ vector multiplet! $\longrightarrow \left\{ \right.$ R-singlet vector $\longrightarrow \mathbf{3}$; $\mathbf{5}_R$ scalars $\uparrow$ ($\mathbf{5}$); fermions in $\mathbf{4}_R$ $\uparrow$ ($\mathbf{2} \otimes \mathbf{4}$)

Basic S-duality ②

For $\mathcal{N}=2$ $SU(2)$ $N_f=4$, first note

Q_a^i $\tilde{Q}^b_j$ $\quad a,b=1,2$ $\quad i,j=1,2,3,4$ $\qquad 2 \simeq \bar{2}$ for $SU(2)$. pseudoreal.

$\downarrow$

$Q_{a=1,2}^{I=1\cdots 8}$. $\qquad W=\tilde{Q}\Phi Q$ becomes

$W = Q^I_a \Phi^{(ab)} Q^J_b \delta_{IJ}$. $\leftarrow SO(8)_F$ invariant.

Φ
$\lambda\lambda$ ⤳ still gives zero modes $\lambda^{i=1,2}_{\alpha=1,2} \to b^{i=1,2}_{\alpha=1,2}$
$F_{\mu\nu}$

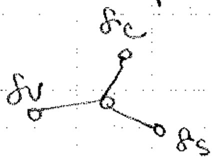
Q^I
ψ^I ⤳ gives zero modes $\qquad C^{I=1\cdots 8}$

$\underbrace{b^i_\alpha}_{2\otimes 2 \text{ of } SO(4) \text{ dummy}}$, C^I ⤳ $\underbrace{|\text{Monopole}\rangle}_{2\otimes 1 \oplus 1\otimes 2 \text{ of } SO(4)\text{ dummy}}$ $\leftarrow SU(2)_R$ doublet scalar + singlet fermion.

$\uparrow$

8_v of $SO(8)$ $\qquad$ in 8_s of $SO(8) \oplus 8_c$ of $SO(8)$.

so: $\quad$ Q-fields $\longleftrightarrow$ monopoles
hyper in 8_v $\qquad$ hyper in 8_s.



$SU(2)$ + 4 flavors in 8_v at τ $\longleftrightarrow$ $SU(2)$ + 4 flavors in 8_s at $2\tau' = -\frac{1}{2\tau}$.

Gaiotto's trick: 4 flavors = 2 flavors + 2 flavors.

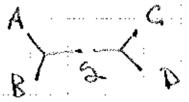
$SO(8) \supset SO(4)_1 \times SO(4)_2$

$SU(2)_g \quad \nearrow SU(2)_g \times SU(2)_{g'} \quad \searrow SU(2)_C \times SU(2)_D$.

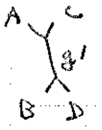
$SU(2)_A \times SU(2)_B$

$SO(8)_V \to$ Q^I_g ⤳ Q_{ABg} $\quad Q_{gCD}$

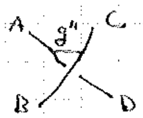
half-hyper in trifundamental.



$SO(8)_S \to$ Q' ⤳ $Q_{ACg'}$ $\quad Q_{g'BD}$



$SO(8)_C \to$ Q'' ⤳ $Q_{ADg''}$ $\quad Q_{g''BC}$



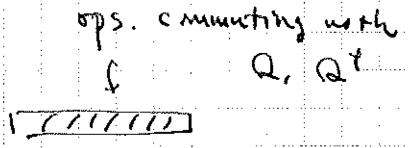
SCI ①

- 4d $\mathcal{N}=2$ SCFT $\mathcal{T}$: $SU(2,2|2) \times G_F$

Put it on $S^3 \times \mathbb{R}$ by conf. transf.

Pick a supercharge Q.

$$SCI_{\mathcal{T}} := \mathrm{Tr}_{\mathcal{H}(S^3)} (-1)^F e^{-\beta\{Q,Q^\dagger\}}$$



- Witten index → indep. of β of def. preserving Q.

Pick $Q_{1\dot{\alpha}}$ among $\underset{SU(2)_R}{Q_{i\alpha}}, \underset{\text{spinors of } SO(4)}{Q_{i\dot{\alpha}}}$

$$SCI_{\mathcal{T}}(p.q.t;x_i) = \mathrm{Tr}_{\mathcal{H}(S^3)} (-1)^F p^{j_2+j_1-\frac{r}{2}} q^{j_2-j_1-\frac{r}{2}} t^{I_3-\frac{r}{2}} \prod x_i^{F_i}$$

$$\{Q,Q^\dagger\} = \Delta - 2j_2 - 2I_3 + \frac{r}{2} = 0$$

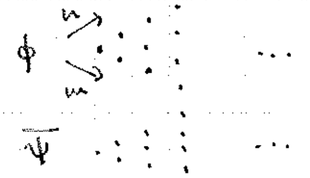
I_3 : Cartan of $SU(2)_R$ $\quad$ r : $U(1)_R$ charge
$j_{1,2}$: Cartan of spacetime spins
$F_{1\ldots n}$: charge under the Cartan of G_F.

- free hyper of $U(1)$ charge ± 1 … $\mathcal{H}(S^3)$ is a Fock sp.

cpx scalar w.f. on S^3 : $\underset{\updownarrow}{\overset{\phi \swarrow \text{top}}{\bigoplus_j V_j \otimes V_j}} \oplus \underset{\updownarrow}{\overset{\bar{\phi}}{\bigoplus_j V_j \otimes V_j}}$

Weyl f. w.f. on S^3 : $\underset{\psi_\alpha}{\bigoplus_j V_j \otimes V_{j-1/2}} \oplus \underset{\overline{\psi}_{\dot{\alpha}}}{\bigoplus_j V_j \otimes V_{j+1/2}} \Leftarrow \text{top}$

for each j,



and … remain.

$$\rightsquigarrow SCI_{\text{free hyper}}(p.q.t;x) = \prod_{n,m\geq 0} \frac{1-t^{-1/2}x^{-1}p^{m+1}q^{n+1}}{1-t^{1/2}\underset{\phi}{x}p^{m}q^{n}} \, \frac{1-t^{-1/2}xp^{m+1}q^{n+1}}{1-t^{1/2}\underset{\tilde{\phi}}{x^{-1}}p^{m}q^{m}}$$

(numerators: $\overline{\psi}$, $\overline{\tilde{\psi}}$)

$$= \Gamma_{p.q}(t^{1/2}x)\,\Gamma_{p.q}(t^{1/2}x^{-1}).$$

where $\Gamma_{p.q}(z) := \prod_{n,m\geq 0} \frac{1-z^{-1}p^{m+1}q^{n+1}}{1-z\,p^{m}q^{m}}$: elliptic Gamma.

SCI

For a gauge theory, say SU(N),
first regard it as a flavor sym. and proj. down. to gauge inv.

$\mathrm{diag}(z_1 \cdots z_N) \in SU(N)$, $\prod z_i = 1$

$$SCI^{\text{vect.mult.}}(p,q,t;z_1 \cdots z_N) = \left(\frac{1}{\Gamma_{p,q}(t)\,\Gamma_{p,q}}\right)^{N-1} \prod_{i\neq j} \frac{1}{\Gamma_{p,q}(t^{z_i}/_{z_j})\,\Gamma_{p,q}(^{z_i}/_{z_j})}$$

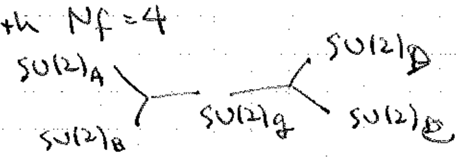
where $\Gamma_{p,q} := \lim (1-z)\,\Gamma_{p,q}(z)$

&

$$SCI_{\text{T gauged by } SU(N)} = \frac{1}{N!}\oint \frac{dz_i}{2\pi\sqrt{-1}\,z_i}\, SCI^{\text{vect. mult}}_{SU(N)}(z_1 \cdots z_N) \cdot SCI_{T}(z_1 \cdots z_N; x_1 \cdots x_a)$$

- Horrible but readily computable as formal p.s. in p & q.
- indep. of coupling τ ⤳ should be equal between S-dual pairs.

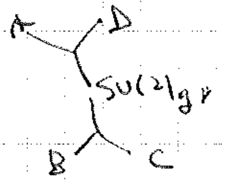
SU(2) with $N_f = 4$



$$I(a,b;c,d) := \frac{1}{2}\oint \frac{dz}{2\pi i z} \frac{1}{\Gamma_{p,q}(t)\Gamma_{p,q}} \prod_{\pm} \frac{1}{\Gamma_{p,q}(tz^{\pm 2})\,\Gamma_{p,q}(z^{\pm 2})} \cdot \prod_{\pm\pm\pm} \Gamma_{p,q}(t^{1/2} z^{\pm 1} a^{\pm 1} b^{\pm 1}) \prod_{\pm\pm\pm} \Gamma_{p,q}(t^{1/2} z^{\pm 1} c^{\pm 1} d^{\pm}) .$$

Should be invariant under $b \leftrightarrow c$

$$I(a,b;c,d) = I(a,c;b,d)$$



can be checked order by order but NOT obvious!

L. Rastelli noticed that this should be true. in ~~late~~ '09.
Asked a mathematician who's an expert ...
who told him that a PhD student just happened to have proved exactly this equality $\wedge$ as mero. func and writing it up in a thesis! Van de Bult.

— demonstration.

having p.q.t is too cumbersome. Let's set $q=t$

hyper: $$\Gamma_{p,q}(t^{1/2}x)\Gamma_{p,q}(t^{1/2}x^{-1}) = \frac{\prod_{m,n}}{\prod_{m,n}} \frac{1-t^{-1/2}x^{-1}p^{m+1}q^{n+1}}{1-t^{1/2}x p^{m}q^{n}} \frac{\cdots}{\cdots}$$

$$= \prod_{n\geq 0} \frac{1}{1-q^{1/2+n}x} \frac{1}{1-q^{1/2+n}x^{-1}} .$$

(It's a general prop. of SCI that when $q=t$ p drops out.]

SCI of trifund $= \prod_{\pm\pm\pm} \prod_{n\geq 0} \frac{1}{1-q^{\frac{1}{2}+n}z^{\pm}a^{\pm}b^{\pm}}$.

The vector contribution also simplifies : in the SU(2) case.

$$\frac{1}{2}\oint \frac{dz}{2\pi i z}(1-z^{2})(1-z^{-2})K(z)^{-2} \text{ (matter contribution)}$$

where $K(z)^{-1} = \prod_{n\geq 0}(1-q^{n+1}) \prod_{\pm}\prod_{n\geq 0}(1-q^{n+1}z^{\pm 2})$

In fact, SCI of trifund $= \frac{K(z)K(a)K(b)}{K_0} \sum_{n\geq 1} \frac{\chi_n(z)\chi_n(a)\chi_n(b)}{\chi_n(q^{1/2})}$

where $K_0^{-1} = \prod_{n\geq 0}(1-q^{2+n})$

$\chi_n(z) = z^{n-1}+z^{n-3}+\cdots+z^{-n+1}$ ⇐ characters in n-dim rep of SU(2).

Then: SCI of SU(2), $N_f=4$ at $q=t$

$$I_{q=t}(a,b;c\,d) = \underbrace{\frac{1}{2}\oint\frac{dz}{2\pi i z}(1-z^{2})(1-z^{-2})}\,\underline{\cancel{K(z)}^{-2}}$$

Haar measure of SU(2) restricted to U(2) $|z|=1$

$$\times \frac{\cancel{K(z)}K(a)K(b)}{K_0}\sum_{n\geq 1}\frac{\underline{\chi_n(z)}\chi_n(a)\chi_n(b)}{\chi_n(q^{1/2})}$$

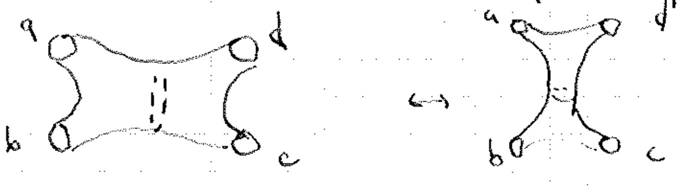
$$\times \frac{\cancel{K(z)}K(c)K(d)}{K_0}\sum_{m\geq 1}\frac{\underline{\chi_m(z)}\chi_m(c)\chi_m(d)}{\chi_m(q^{1/2})}$$

The issue was the sym under b↔c

$$= \frac{K(a)K(b)K(c)K(d)}{K_0^{2}}\sum_{n\geq 1}\frac{\chi_n(a)\chi_n(b)\chi_n(c)\chi_n(d)}{\chi_n(q^{1/2})^{2}} .$$

This is the part. f. of q-deformed YM !!

with $\alpha = K_0$ & funny inner product on SU(2).



SCI and the 6d theory ①

Why is the SCI of 4d $\mathcal{N}=2$ th. of this particular class given by the q-deformed YM ?

FACT. $\exists$ 6d $\mathcal{N}=(2,0)$ SCFT, labeled by $G = A_n, D_n, E_n$. s.t. its S^1 cptf in the IR is the 5d $\mathcal{N}=2$ SYM with G gauge.

existence proof : N M5 branes $\rightsquigarrow$ A_{N-1}. etc.

①

$$\begin{tikzcd}
\mathcal{N}=(2,0) \text{ in 6d} \arrow[r, "R_6"'] \arrow[dr, bend right, "R_5"'] & \mathcal{N}=2 \text{ in 5d} \arrow[r, "R_5"'] & \mathcal{N}=4 \text{ in 4d} \\
 & {} \arrow[r, "R_6"'] & {}
\end{tikzcd}$$

$\uparrow$ no scale

$\int d^5 x \underbrace{\frac{1}{g_{5d}^2}}_{\uparrow \text{ dimensionfull}} \text{tr}FF$

$\rightsquigarrow g_{5d}^2 \propto R_6$

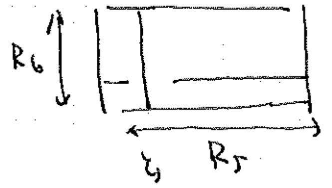
$\underbrace{\frac{1}{g_5^2}\int dx_5}\int d^4x\, FF$

$\frac{1}{g_4{}^2} \sim \frac{R_5}{R_6}$

S-dual!

$\frac{1}{g_4{}^2} \sim \frac{R_6}{R_5}$

· 4d S-duality of $\mathcal{N}=4$ SYM is just the Lorentz inv. in 6d.



$\rightsquigarrow m = R_5 \times \text{tension}$: monopole

$\swarrow$ ratio is $\frac{R_5}{R_6}$. cf $m_W \sim |a|$, $m_{\text{mono}} \sim |\tau a|$

$m' = R_6 \times \text{tension}$

W-boson.

$\Rightarrow$ If somebody finds a nice Lagrangian desc. of 6d $\mathcal{N}=(2,0)$ th, 4d $\mathcal{N}=4$ S-duality will be <u>manifestly</u> understood.

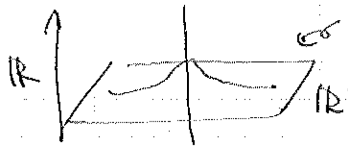
· Another peculiarity in 6d $\to$ 5d.

If we cptfy a D-dim scalar on S^1, of radius R

$$\phi(\vec{x}, x_0) = \sum \phi_{(n)}(\vec{x})\, e^{n i x_0/R}$$

$\overset{\uparrow}{\text{massless}}$ $\qquad$ $\overset{\uparrow}{\text{mass}} = \frac{|n|}{R}$: KK tower.

In the case of 6d $\to$ 5d from (2,0) theory, 5d gauge th. contains



$\rightsquigarrow$ instanton · particle of mass $\frac{8\pi^2}{g_{5d}^2} \propto \frac{1}{R_6}$

By equating we can fix the precise coeff : $g_{5d}^2 = \frac{8\pi^2 R_6}{\cancel{2\pi^2}} = 4\pi R_6$.

SCI and the 6d theory ③

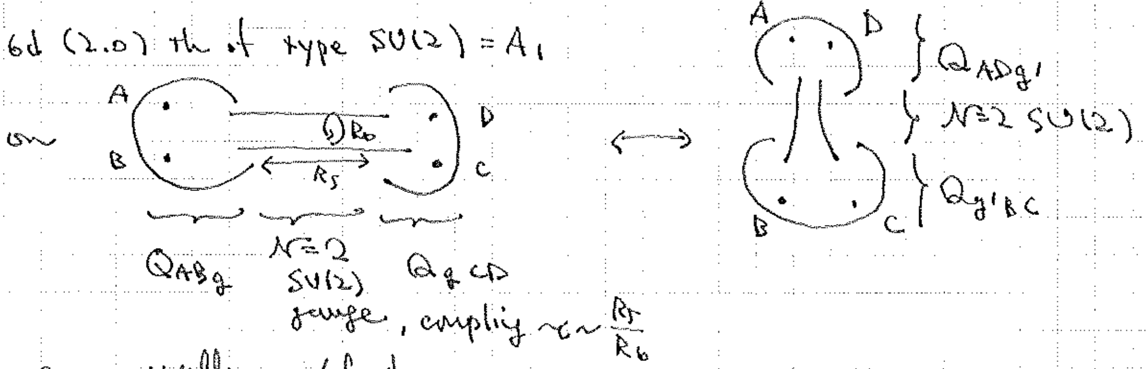
- Empirical fact

As far as the BPS quantities are concerned, 5d gauge th. computation including instanton effects reproduces 6d on S^1 including the KK towers. You should not add KK towers by hand.

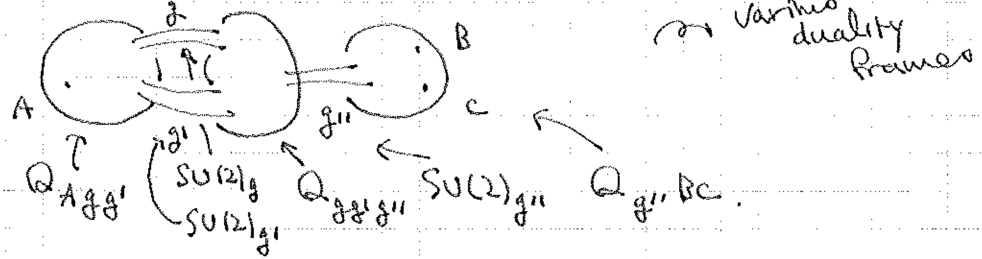
⟹ If there's a nice 6d $\mathcal{N}=(2,0)$ Lagrangian, its S^1 cptification should be such that all KK modes can be gauged away ...

I consider it unlikely, to be able to write down such a Lag.
That said, I thought the same for the M2-brane theory before ABJM...

② 6d (2.0) th of type $SU(2) = A_1$

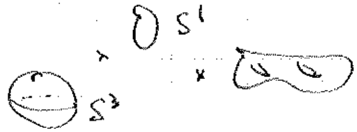


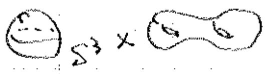
more generally, 6d th. on

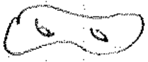


Now we can finally understand why $SCI = q\text{YM}$:

$SCI = \mathrm{tr}_{\mathcal{H}(S^3)}(-1)^F q^{\Delta - I_3} = Z_{4d}\Big($  $\Big)$

$= Z_{\substack{6d\\ \mathcal{N}=(2,0)}}\Big($  $\Big)$

$= Z_{\substack{5d\\ \mathcal{N}=2\text{ gauge}}}\Big($  $\Big)$

$= Z_{\substack{2d\\ \text{YM}}}\Big($  $\Big)$

YM, dressed by KK modes around S^3

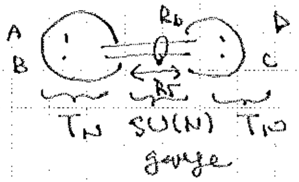
explicit computation shows it gives precisely the q-deformation!

~~Generalization to SU(N) & partial closure of punctures~~ ①

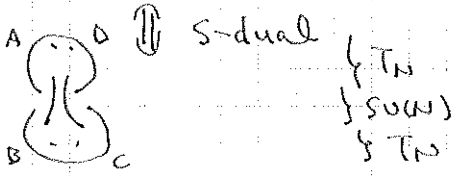
Now, take 6d $\mathcal{N}=(2,0)$ th. of type SU(N), $N>2$.

on 

: 4d $\mathcal{N}=2$ theory 'T_N' : $SU(N)^3$ flavor sym.

s.t. 

has tunable coupling $\tau \sim \frac{R_F}{R_6}$



This is a natural generaliz. of the S-duality of $\mathcal{N}=2$ 4d $N_f=4$ SU(2).

But what's this T_N theory?

$T_{N=2}$: Q_{ABC} : trifund half-hypermult.

$T_{N\geq3}$: ??? isolated strongly coupled SCFTs.
↳ doesn't have tunable coupling.

Its SCI can be found via qYM:

$$SCI_{T_N}(a.b.c) = \frac{K(a)K(b)K(c)}{K_0}\sum_\lambda \frac{\chi_\lambda(a)\chi_\lambda(b)\chi_\lambda(c)}{\chi_\lambda(\mathrm{diag}(q^{\frac{N-1}{2}}\cdots q^{\frac{1-N}{2}}))}$$

where $K(a)^{-1} = \prod_{n\geq1}\left[(1-q^n)^{N-1}\prod_{i\neq j}\left(1-q^n\frac{a_i}{a_j}\right)\right] = 1-q\chi_{adj}(a)+O(q^2)$

and $K_0^{-1} = \prod_{d=2}^{N}\prod_{n\geq0}(1-q^{d+n})$.

e.g. $SCI_{T_2} = 1 + \underbrace{q^{1/2}}\,\chi_\square(a)\chi_\square(b)\chi_\square(c) + \cdots$

↳ comes from $\frac{1}{q^{-1/2}+q^{1/2}} = q^{1/2}+O(q^{3/2})$

$$=\prod_{n\geq0}\prod_{\pm\pm\pm}\frac{1}{1+q^{1/2+n}a^\pm b^\pm c^\pm}$$

from $K(a)K(b)K(c)$

$$SCI_{T_3} = 1 + q\big(\chi_{adj}(a)+\chi_{adj}(b)+\chi_{adj}(c)$$

from $\lambda=\square$ $\rightarrow$ $+\chi_\square(a)\chi_\square(b)\chi_\square(c)$

$\lambda=\bar\square$; $\rightarrow$ $+\chi_{\bar\square}(a)\chi_{\bar\square}(b)\chi_{\bar\square}(c)\big) + \cdots$

prefactor $\sim \frac{1}{q^{-1}+1+q^{1}}$.

- Not a hyper (no $q^{1/2}$ term)

- note $8_A\oplus8_B\oplus8_C\oplus3_A\otimes3_B\otimes3_C\oplus\bar{3}_A\otimes\bar{3}_B\otimes\bar{3}_C = 78_{E_6}$

⇝ M-N's E_6 th. known from '96

T_4 and up: new theory, no enhancement...

Partial closure of punctures ①

Can we understand the S-duality of $\mathcal{N}=2$ $SU(N)$ with $N_f=2N$?
Yes, but we need to learn a new trick, called 'partial closure.'

In general, any 4d $\mathcal{N}=2$ SCFT with flavor sym G_F has

$$\underset{\substack{\uparrow\\ SU(2)_R\\ \text{triplet}}}{\mu_{(ij)}^{a}} \qquad \underset{\substack{\uparrow\\ SU(2)_R\\ \text{doublet}}}{\lambda_{\alpha i}^{a}} \qquad J_\mu^{a}$$

e.g. for free $\underset{\text{bifund}}{\hat{}}$hyper $Q^i{}_a$ $\tilde{Q}^b{}_j$ $\qquad U(N)_1 \times U(N)_2$

chiral μ^+ $\quad Q_a^i \tilde{Q}_i^b$
Kähler μ^0 $\quad Q^{\dagger b}_i Q^i{}_a - \tilde{Q}_i{}^b Q^{\dagger i}{}_a$ $\qquad$ for $U(N)_1$.
antichi. μ^- $\quad (Q^\dagger)^a_i (\tilde{Q}^\dagger)^i_b$

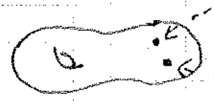
<u>partial closure</u> : giving a nilpotent vev $\mu^+ = \begin{pmatrix} 0\ 1 & & \\ \ \ 0\ 1 & & \\ \ \ \ \ 0 & & \\ & \begin{smallmatrix}0 & 1\\ & 0\end{smallmatrix} & \\ & & 0 \end{pmatrix}$

characterized by $N = n_1 + n_2 + \cdots$.
e.g. $6 = 3+2+1$

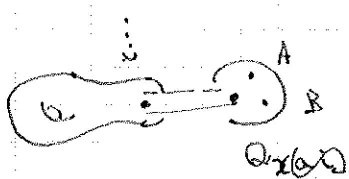
SCFT with $SU(N)$ sym.
$\downarrow$
SCFT with reduced sym + Nambu-Goldstone mult.
assoc. to broken $SU(N)$ sym.

complete closure $\sim$ biggest $\mu^+ = \begin{pmatrix} 0\ 1 & & \\ & \ddots & \ddots \\ & & 0 \end{pmatrix}$
$N = N$.

Start with the simplest case : $SU(2)$.



$\cdots SU(2)_A \rightsquigarrow \mu_A$: let's close this via
$\mu_A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
$\cdots SU(2)_B \rightsquigarrow \mu_B$
(there's my complete closure for $N=2$.)



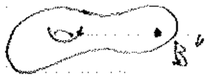
$\underset{\substack{\uparrow\\ SU(2)_g}}{Q_x}\underset{\substack{\uparrow\\ SU(2)_A}}{(a)}b_c$ — $SU(2)_B$.

$\mu^+_{(ab)} = Q_{xia} Q_{yjb}\, \varepsilon^{xy} \varepsilon^{ij}$

$\mu_A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ can be achieved via

$Q_{xia} = \varepsilon_{xi}\, \delta_{a1}$.

* $SU(2)_A$ action sends this to $\varepsilon_{xi}\, q_a$
$\rightsquigarrow$ one hyper is N-G mode.



puncture A gone!

no hypers remain. $\Leftarrow$ [* $SU(2)_{\text{gauge}} \times SU(2)_B \to \underset{\text{diag}}{SU(2)_{B'}}$
eats three hypers

Partial closure

Again, the leading contrib is easy to find:

$$1 + \boxed{q_b^{1/2}} \left(\chi_{\square}(a)\, \chi_{\bar{\square}}(b^{\text{}})\, \alpha + \chi_{\bar{\square}}(a)\, \chi_{\square}(b)\, \alpha^{-1} \right)$$

$\uparrow$ $SU(N)_1$ fund × $SU(N)_2$ anti fund × baryon charge 1.

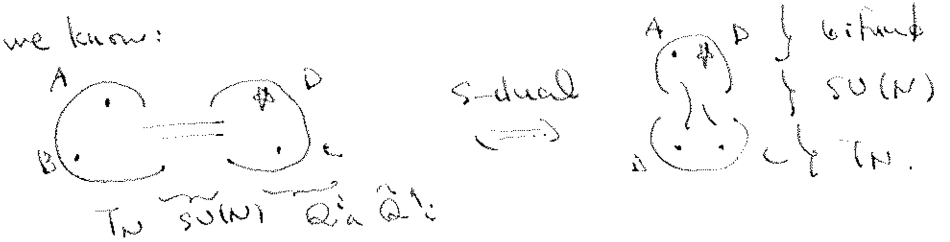
$\uparrow$ $SU(N)_1$ antifund × $SU(N)_2$ fund × bar.ch. -1.

$\downarrow$ comes from

$$\frac{q_b^{\frac{2-N}{2}}\alpha^{\pm 1}}{q_b^{\frac{1-N}{2}}(1+O(q_b))}$$

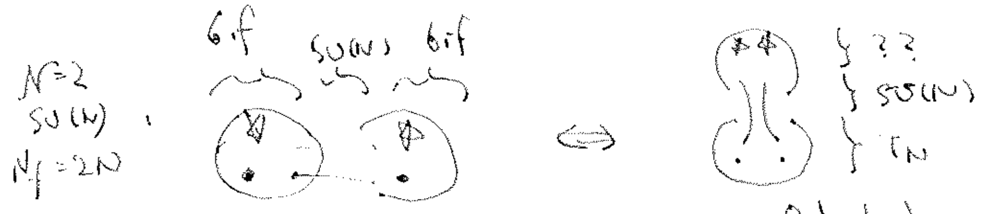
$\rightsquigarrow$ bifundamental $Q^a{}_i$, $\tilde{Q}^i{}_a$!

Now we know:



obtained by partially closing A.

Then we can consider



$Q_i{}^a$, $\tilde{Q}^i{}_a$

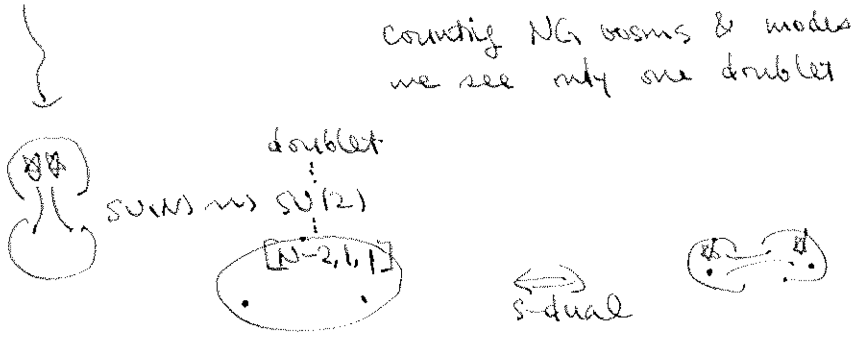
$$\mu^+_A = Q^a{}_i \tilde{Q}^i{}_b = \left(\begin{array}{c|c} \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{matrix} & \\ \hline & \begin{matrix} \ddots & \\ & 0 \end{matrix} \end{array}\right)$$

can be solved with $\overset{\shortparallel}{Q^i{}_b}$, $Q^a{}_i = \left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix} & \\ \hline & \begin{matrix} 0 & \\ & 0 \end{matrix} \end{array}\right)$.

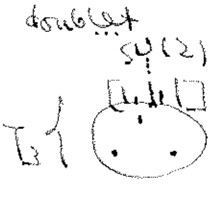
$$\rightsquigarrow \mu^+_{\text{gauge}} = \tilde{Q}^{\dot{a}}{}_a Q^a{}_j = \left(\begin{array}{c|c} \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 0 \end{matrix} & \\ \hline & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{array}\right)$$

breaks $SU(N)_g \rightsquigarrow SU(2)_g$.

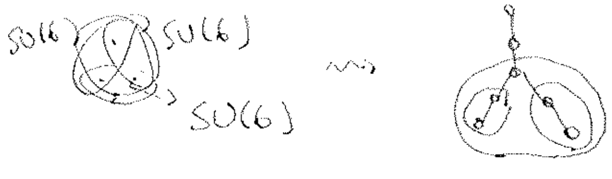
Counting NG bosons & modes eaten by Higgsing, we see only one doublet remaining.



When $N=3$,



$\Longleftrightarrow$ $\mathcal{N}=2$ SU(3) with $N_f=6$.

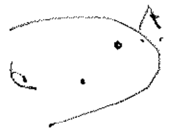


: E_6 !

Partial closure

What happens to the SCI under the closure?

For SU(2), we already know what happens: lose one puncture.
Can we restate abstractly? so that it can be applied generally?



$$\cdots SU(2)_A \cdots \underset{\sim\sim}{M_A}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

highest w. of $SU(2)_R$ & highest w. of $SU(2)_A$
at the same time.
$\Rightarrow$ $SU(2)_{R'}$: diag. combination remains unbroken.

a factor

$$\chi_\lambda(a) K(a) \rightsquigarrow \chi_\lambda((q^{1/2}, q^{-1/2}))\ \underbrace{K((q^{1/2}, q^{-1/2}))}_{\text{contains the contribution from NG bosons.}}$$

in SCI becomes

$$K(a) = \prod_{n\geq 1} \frac{1}{1-q^n a^{-2}} \cdot \frac{1}{1-q^n} \cdot \frac{1}{1-q^n a^2}$$

$$\xrightarrow[a \to q^{1/2}]{} \prod_{n\geq 1} \underbrace{\frac{1}{1-q^{n-1}} \cdot \frac{1}{1-q^n}}_{\text{N.G. modes}} \cdot \underbrace{\frac{1}{1-q^{n+1}}}_{\text{reminder}} \qquad \text{reminder} = \prod_{n\geq 2} \frac{1}{1-q^n} = K_0$$

∴ closure is done by replacing $\chi(a)K(a)$ by $\chi(q^{1/2})K_0$.

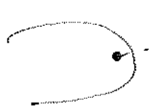
Indeed, with genus 0 with 4 punct. say,

$$\frac{\underset{\downarrow K_0}{\boxed{K(a)}}\, K(b) K(c) K(d)}{K_0^2} \sum_\lambda \frac{\underset{\downarrow \chi_\lambda(q^{1/2})}{\boxed{\chi_\lambda(a)}}\, \chi_\lambda(b) \chi_\lambda(c) \chi_\lambda(d)}{\chi_\lambda(q^{1/2})^2}$$

$$\Rightarrow \frac{K(b) K(c) K(d)}{K_0} \sum_\lambda \frac{\chi_\lambda(b) \chi_\lambda(c) \chi_\lambda(d)}{\chi_\lambda(q^{1/2})}$$, as it should be.

---

a generalization



$$\cdots SU(N)_A \cdots M_A^+ = \left(\underbrace{\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{matrix}}_{N-1} \;\middle|\; \underbrace{0}_{1} \right) \quad : \leftarrow U(1) \text{ sym } \begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ 0 & & & 1-N \end{pmatrix}$$

still unbroken

$$\chi(a) K(a) \xrightarrow{\text{replace}} \chi(q^{\frac{N-2}{2}}\alpha, q^{\frac{N-4}{2}}\alpha, \ldots, q^{\frac{2-N}{2}}\alpha, \alpha^{1-N}) \times K_{[N-1,1]}(\alpha)$$

where

$$K_{[N-1,1]}(\alpha) = \left(\prod_{d=1}^{N-1} \prod_{n\geq 0} \frac{1}{1-q^{d+n}}\right) \frac{1}{\prod_{\pm} \prod_{n\geq 0} 1-q^{\frac{N}{2}+n} \alpha^{\pm N}} .$$

Now we can compute the SCI of

$$: \frac{K_{[N-1,1]}(\alpha) K(a) K(b)}{K_0} \sum_\lambda \frac{\chi_\lambda(a) \chi_\lambda(b^{-1}) \chi_\lambda(q^{\frac{N-2}{2}}\alpha \cdots q^{\frac{2-N}{2}}\alpha, \alpha^{1-N})}{\chi_\lambda(q^{\frac{N-1}{2}}, \cdots, q^{\frac{1-N}{2}})} .$$