Aspects of Superstring Perturbation Theory

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Lecture 1: Overview

PLAN

1. Conventional approach and its shortcomings

2. Off-shell amplitudes

3. 1PI effective string field theory

Standard formulation of heterotic / type II string theory in a given background is based on a 2 dimensional (super-)conformal world-sheet field theory

– matter system + ghost system with total central charge zero.

On-shell states are described by BRST invariant vertex operators

– can be chosen to be dimension zero primary operators.

g-loop, n-point S-matrix elements

1. Compute certain CFT correlation functions on genus g Riemann surfaces with n punctures (marked points).

— insert vertex operators at the punctures

– insert a set of ghost operators following specific rules (lectures 2,3)

– in superstring theory also insert a certain set of ‘picture changing operators’ made of ghost fields and superstress tensor of the matter fields (lecture 3)

2. Integrate the result over the (6g-6+2n)-dimensional moduli space of the corresponding Riemann surface.

However this approach is insufficient for addressing many issues even within the perturbation theory.

1. Mass renormalization

2. Vacuum shift

LSZ formula for S-matrix elements in QFT

$$\lim_{k_i^2 \to -m_{i,p}^2} G^{(n)}_{a_1 \cdots a_n}(k_1, \cdots k_n) \prod_{i=1}^{n} \{ Z_i^{-1/2} \times (k_i^2 + m_{i,p}^2) \}$$

$G^{(n)}$: n-point Green's function

$a_1, \cdots a_n$: quantum numbers, $k_1, \ldots k_n$: momenta

$m_{i,p}$: physical mass of the i-th external state

– given by the locations of the poles of two point function in the $-k^2$ plane.

Z_i: wave-function renormalization factors, given by the residues at the poles.

In contrast, string amplitudes compute 'truncated Greens function on classical mass-shell'

$$\lim_{k_i^2 \to -m_i^2} G^{(n)}_{a_1 \cdots a_n}(k_1, \cdots k_n) \prod_{i=1}^{n} (k_i^2 + m_i^2) \,.$$

m_i: <u>tree level</u> mass of the i-th external state.

$k_i^2 \to -m_i^2$ condition is needed to make the vertex operators conformally invariant.

String amplitudes:

$$\lim_{k_i^2 \to -m_i^2} G^{(n)}_{a_1 \cdots a_n}(k_1, \cdots k_n) \prod_{i=1}^{n} (k_i^2 + m_i^2)\,,$$

The S-matrix elements:

$$\lim_{k_i^2 \to -m_{i,p}^2} G^{(n)}_{a_1 \cdots a_n}(k_1, \cdots k_n) \prod_{i=1}^{n} \{Z_i^{-1/2} \times (k_i^2 + m_{i,p}^2)\}$$

The effect of Z_i can be taken care of. Witten

The effect of mass renormalization is more subtle.

$\Rightarrow$ String amplitudes compute S-matrix elements directly if $m_{i,p}^2 = m_i^2$ but not otherwise.

– Includes BPS states, massless gauge particles and all amplitudes at tree level.

Problem with vacuum shift

Example: In many compactifications of SO(32) heterotic string theory on Calabi-Yau 3-folds, one loop correction generates a Fayet-Ilioupoulos term.

Effect: Generate a potential of a charged scalar ϕ of the form

$$c(\phi^*\phi - K\, g_s{}^2)^2$$

c, K: positive constants, g_s: string coupling

Dine, Seiberg, Witten; Atick, Dixon, A.S.; Dine, Ichinose, Seiberg
Atick, A.S.; Witten; D'Hoker, Phong; Berkovits, Witten

Correct vacuum: $|\phi| = g_s\sqrt{K}$

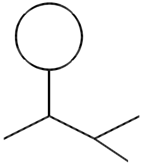
– not described by a world-sheet CFT

– conventional perturbation theory fails.

Even in absence of mass renormalization and vacuum shift we have to deal with infrared divergences in the integration over moduli space at intermediate stages.

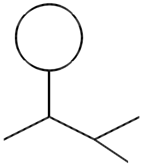
Witten

Consider a tadpole diagram in a QFT:



This diverges if a massless state propagates along the vertical propagator.

In the best possible scenario the result vanishes after loop integration (possibly due to SUSY).



In string theory, this translates to a specific regularization procedure for integration over moduli spaces of Riemann surfaces.

1. Put an upper cut-off L on certain moduli corresponding to the Schwinger parameter of the vertical propagator.

2. Do integration over the other moduli first.

3. Then let L go to infinity.

Witten

This works but requires an IR cut-off at the intermediate stages of calculation.

How do we circumvent these difficulties / need for IR cut-off?

Go off-shell.

Off-shell amplitudes

$\cdots$; Nelson; Zwiebach; Pius, Rudra, A.S.

1. Relax the constraint of conformal and BRST invariance on the vertex operators

– result will depend on the world-sheet metric around the punctures where the vertex operators are inserted.

2. Choose a local coordinate system w_i around the i-th puncture for each i and take the metric around the puncture $w_i = 0$ to be $|dw_i|^2$.

A different choice of the local coordinate system e.g. $y_i = f(w_i) \Rightarrow$ different metric $|dy_i|^2 = |f'(w_i)|^2\,|dw_i|^2$
$\Rightarrow$ different off-shell amplitudes for the same external states.

For superstring theories we need insertion of picture changing operators (PCO) on the Riemann surface.

Off-shell amplitudes depend not only on the choice of local coordinates at the punctures but also on the locations of the PCO's.

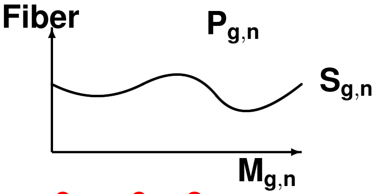
Are the physical quantities computed from off-shell amplitudes independent of the choice of local coordinates and PCO locations?

Some notations:

$M_{g,n}$: (6g-6+2n) dimensional moduli space of genus g Riemann surfaces with n punctures.

$P_{g,n}$: A fiber bundle with $M_{g,n}$ as the base and the choice of local coordinates at punctures and PCO locations as fibers (infinite dimensional).

A choice of local coordinate system and PCO locations corresponds to a section $S_{g,n}$ of this fiber bundle.



Dimension of $S_{g,n} = 6g - 6 + 2n$.

Procedure for constructing an off-shell amplitude

1. For a given set of external off-shell states collectively called ϕ, construct p-forms $\omega_{\mathbf{p}}(\phi)$ on $\mathbf{P_{g,n}}$ satisfying

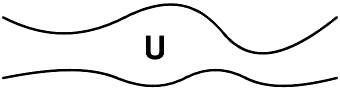
$$\omega_{\mathbf{p}}(\sum_{\mathbf{i}} \mathbf{Q_B^{(i)}} \phi) = (-1)^{\mathbf{p}} \mathbf{d}\omega_{\mathbf{p-1}}(|\phi\rangle)$$

$\mathbf{Q_B^{(i)}}$: BRST charge acting on i-th state

$\omega_{\mathbf{p}}$ is constructed from appropriate correlation functions of off-shell vertex operators and ghost insertions on the Riemann surface. (Lectures 2, 3)

2. Genus g, n-point amplitude

$$\int_{\mathbf{S_{g,n}}} \omega_{\mathbf{6g-6+2n}}(|\phi\rangle)$$



If U is the region bounded by the two sections then the difference in the integral over the two sections is

$$\int_U d\omega_{6g-6+2n}(|\phi\rangle) = -\int_U \omega_{6g-5+2n}\Big(\sum_i Q_B^{(i)}\phi\Big)$$

– vanishes for 'on-shell' states for which $Q_B^{(i)}|\phi\rangle = 0$.

However it does not vanish for off-shell states.

$\Rightarrow$ the off-shell amplitudes depend on the choice of the local coordinates at the punctures and PCO locations.

Goal: Prove that all physical quantities computed from the off-shell amplitudes are independent of the choice of local coordinates even though the amplitudes themselves are not.

For this we work within a specific class of local coordinates

– gluing compatible local coordinate system.

Gluing compatible sections

and

1PI amplitudes

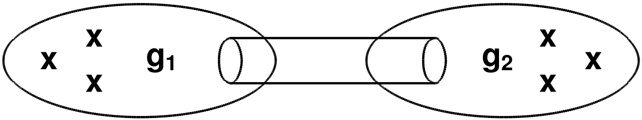
Consider a genus g_1, m-punctured Riemann surface and a genus g_2, n-punctured Riemann surface.

Take one puncture from each of them, and let w_1, w_2 be the local coordinates around the punctures at $w_1 = 0$ and $w_2 = 0$.

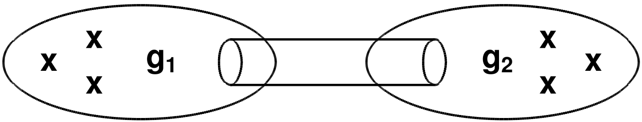
Glue them via the identification (plumbing fixture)

$$w_1 w_2 = e^{-s+i\theta}, \quad 0 \le s < \infty, \quad 0 \le \theta < 2\pi$$

– gives a family of new Riemann surfaces of genus $g_1 + g_2$ with (m+n-2) punctures.



Gluing compatibility: Choice of local coordinates at the punctures and the PCO locations on the genus $g_1 + g_2$ Riemann surface must agree with the one induced from the local coordinates at the punctures and PCO locations on the original Riemann surfaces.



Gluing compatibility allows us to divide the contributions to off-shell Green's functions into <u>1-particle reducible</u> (1PR) and <u>1-particle irreducible</u> (1PI) contributions.

Two Riemann surfaces joined by plumbing fixture

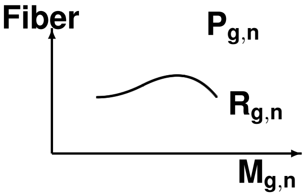
$\Updownarrow$

Two amplitudes joined by a propagator

Riemann surfaces which <u>cannot</u> be obtained by plumbing fixture of two or more Riemann surfaces contribute to <u>1PI</u> amplitudes.

Put another way, for a gluing compatible choice of sections, we can identify a subspace $R_{g,n}$ of the full section $S_{g,n}$ which we can call the 1PI subspace.

Lectures 2 and 3



All the Riemann surfaces corresponding to the full section $S_{g,n}$ are given by the Riemann surfaces in $R_{g,n}$ and their plumbing fixture in all possible ways.

Once this division has been made, we can define the 1PI amplitudes as

$$\int_{\mathbf{R_{g,n}}} \omega_{\mathbf{6g-6+2n}}$$

Generating function of these amplitudes is 1PI effective action.

Lecture 4

Tree amplitudes computed from 1PI action

= full off-shell string amplitude including loop corrections, given by integrals over the whole section $\mathbf{S_{g,n}}$

We can now apply standard field theory methods to compute renormalized masses and S-matrix from the 1PI action.

In this approach we

– first determine the vacuum by solving classical equations of motion derived from 1PI effective action,

– then do perturbation expansion around the vacuum.

As a result the perturbation expansion is free from any IR divergence associated with tadpoles.

– no need to regulate infrared divergences even at intermediate stages of the calculation.

– perfectly suitable for dealing with the vacuum shift.

The 1PI action depends on the choice of $S_{g,n}$.

However one finds that 1PI actions for different choices of $S_{g,n}$ are related by a <u>field redefinition</u>.

– all physical quantities remain unchanged under this.

– generalizes old result of Hata and Zwiebach in string field theory.

A bonus

One finds that this 1PI action automatically has infinite dimensional gauge invariance! (Lecture 4)

– includes general coordinate transformation, local supersymmetry etc.

Our task

1. Find a parametrization of the space $P_{g,n}$ and its tangent space.

2. For n given external states, collectively denoted as ϕ, construct p-forms ω_p on $P_{g,n}$ with the desired properties:

$$\omega_p(\sum_i Q_B^{(i)}\phi) = (-1)^p d\omega_{p-1}(|\phi\rangle)$$

$Q_B^{(i)}$: BRST charge acting on i-th state

3. Construct gluing compatible sections $S_{g,n}$ and 1PI subspaces $R_{g,n}$ of $P_{g,n}$.

Lecture 2: Warm-up with bosonic string theory

Our task

1. Find a parametrization of the space $P_{g,n}$ and its tangent space.

2. For n given external states, collectively denoted as ϕ, construct p-forms ω_p on $P_{g,n}$ with the desired properties:

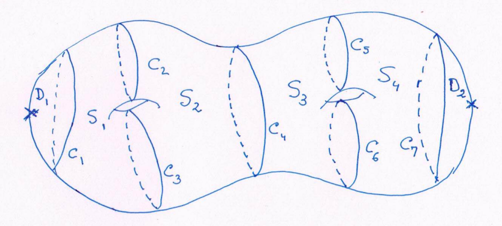
$$\omega_p(\sum_i Q_B^{(i)}\phi) = (-1)^p d\omega_{p-1}(|\phi\rangle)$$

$Q_B^{(i)}$: BRST charge acting on i-th state

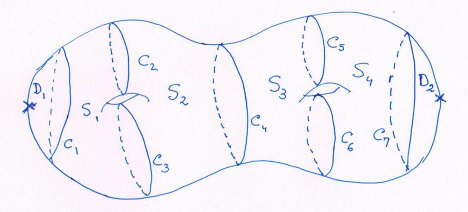
3. Construct gluing compatible sections $S_{g,n}$ and 1PI subspaces $R_{g,n}$ of $P_{g,n}$.

Parametrization of $P_{g,n}$

Given a genus g Riemann surface with n-punctures, consider this as a union of n disks $\{D_a\}$ and 2g-2+n spheres $\{S_i\}$ each with three holes.



3g-3+2n circles $\{C_s\}$ form boundaries between these different regions.



w_a: complex coordinate on D_a with a-th puncture at $w_a = 0$

z_i: complex coordinate on S_i.

If S_i and D_a share a common boundary then on the boundary $z_i = f_{ia}(w_a)$

If S_i and S_j share a common boundary then on the boundary $z_i = F_{ij}(z_j)$

$$z_i = f_{ia}(w_a), \qquad z_i = F_{ij}(z_j)$$

The transition functions f_{ia} and F_{ij} contain complete information on the Riemann surface and the choice of local coordinates.

– can be used as coordinates of $P_{g,n}$.

1. Changes induced by arbitrary reparametrization of z_i's give the same point in $P_{g,n}$.

2. In general reparametrizations of w_a give different points in $P_{g,n}$, but $w_a \to e^{i\alpha_a} w_a$ for constant α_a's are defined to give the same point in $P_{g,n}$.

$\Rightarrow$ $P_{g,n}$ contains information on local coordinates up to phases.

Tangent vectors of $P_{g,n}$ correspond to infinitesimal changes in f_{ia} and f_{ij}.

e.g. under $f_{ia} \to f_{ia} - \delta f_{ia}$ we have

$$z_i = f_{ia}(w_a) \quad \to \quad z_i = f_{ia}(w_a) - \delta f_{ia}(w_a)$$

Introduce infinitesimal vector field on the Riemann surface around the common boundary of S_i and D_a

$$v(z_i) = \delta f_{ia}({f_{ia}}^{-1}(z_i))$$

– labels an infinitesimal tangent vector of $P_{g,n}$.

Reparametrization of z_k for any k does not induce a change in the Riemann surface or the local coordinates at the punctures

– should give a zero tangent vector.

However it is represented by a collection of vector fields on the three circles forming the boundary of S_k.

These must be declared as zero tangent vectors of $P_{g,n}$.

This finishes our brief introduction to parametrization of $P_{g,n}$ and its tangent space.

Construction of $\omega_p(|\phi\rangle)$ on $P_{g,n}$

Nelson; Zwiebach

A bosnic string theory is based on

1. A matter CFT of c=26

2. A ghost CFT of $c = -26$ containing b, c, $\bar{b}$, $\bar{c}$ fields.

H: Hilbert space of the combined CFT annihilated by

$$b_0 - \bar{b}_0, \qquad L_0 - \bar{L}_0$$

$H^{\otimes n}$: n-fold tensor product of H.

$\langle A|B\rangle$: BPZ inner product between CFT states

Consider a point in $P_{g,n}$

– a Riemann surface S of genus g and n punctures with choice of local coordinates at each puncture.

Given any $|\phi\rangle \in H^{\otimes n}$, $\langle\phi\rangle_S$ is defined to be the correlation function of the n vertex operators of $|\phi\rangle$ on the Riemann surface, inserted with the chosen local coordinate system.

$$\omega_0(|\phi\rangle) \equiv (2\pi i)^{-3g+3-n}\langle\phi\rangle_S$$

$\omega_p(|\phi\rangle)$ is a p-form on $P_{g,n}$.

Given p tangent vectors $(V_1, \cdots V_p)$ of $P_{g,n}$, contraction of ω_p with these tangent vectors is a number.

We shall now prescribe this number $\omega_p(|\phi\rangle)[V_1, \cdots V_p]$

Recall that a tangent vector V_i of $P_{g,n}$ can be labelled by a vector field v_i on S around some circle C_{s_i} seperating two regions.

Define

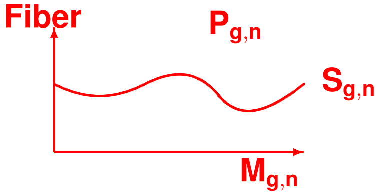
$$B[V_i] \equiv \oint_{C_{s_i}} dz\, b(z) v_i(z) + \oint_{C_{s_i}} d\bar{z}\, \bar{b}(\bar{z}) \bar{v}_i(\bar{z})$$

$$\omega_p(|\phi\rangle)[V_1, \cdots V_n] \equiv (2\pi i)^{-3g+3-n} \langle B[V_1] \cdots B[V_p]\, \phi\rangle_S$$

Using the standard manipulation involving CFT on Riemann surfaces one can prove that $\omega_{\mathbf{p}}(|\phi\rangle)$ defined this way satisfies the desired properties.

1. If any of the tangent vectors is zero tangent vector induced by reparametrization of one or more $\mathbf{z_i}$ then $\omega_{\mathbf{p}}(|\phi\rangle)[\mathbf{V_1}, \cdots \mathbf{V_n}]$ vanishes.

2.
$$\omega_{\mathbf{p}}(\sum_{\mathbf{i}} \mathbf{Q_B^{(i)}}\phi) = (-\mathbf{1})^{\mathbf{p}} \mathbf{d}\omega_{\mathbf{p-1}}(|\phi\rangle)$$



3. If $|\phi\rangle \in H^{\otimes n}$ is a tensor product of BRST invariant states of ghost number 2, then

$$\int_{S_{g,n}} \omega_{6g-6+2n}(|\phi\rangle)$$

for any section $S_{g,n}$ of $P_{g,n}$ gives the usual on-shell amplitudes in string theory.

Choosing gluing compatible sections of $P_{g,n}$

1. Begin with 3-punctured sphere and one punctured torus.

The first one has 0-dimensional moduli space and the second one has two dimensional moduli space.

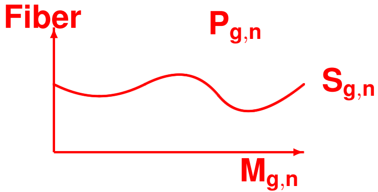
Choose local coordiates at the punctures arbitrarily consistent with symmetries

– exchange of punctures on the 3-punctured sphere

– modular transformation for the 1-punctured torus.

Choosing specific local coordinate system

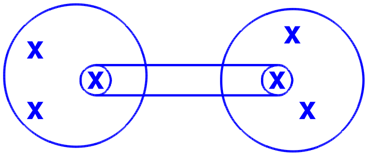
$\Rightarrow$ choosing sections of $P_{0,3}$ and $P_{1,1}$ respectively.



Declare these to be 1PI subspaces of $P_{0,3}$ and $P_{1,1}$ and the corresponding Riemann surfaces with local coordinates as 1PI Riemann surfaces.

$$R_{0,3} = S_{0,3}, \qquad R_{1,1} = S_{1,1}$$

Now take two 3-punctured spheres and glue them using plumbing fixture.



$$\mathbf{w_1 w_2 = q}, \quad \mathbf{q} \equiv \mathbf{e}^{-\mathbf{s}+\mathbf{i}\theta}, \quad \mathbf{0} \leq \mathbf{s} < \infty, \quad \mathbf{0} \leq \theta < \mathbf{2}\pi$$

Declare these to be 1PR 4-punctured spheres and choose the local coordinates to be those induced from 3-punctured spheres.

Repeat this for inequivalent permutations of the four punctures i.e. ‘sum over s, t and u-channel diagrams’.

Explicit form

On each of the two 3-punctured spheres labelled by z_1, z_2, choose the punctures to be at 0, 1, ∞.

Choose local coordinates around $z_i = 0$ on each to be $w_1 = f(z_1)$ and $w_2 = f(z_2)$, with $f(0) = 0$.

Plumbing fixture:

$$w_1 w_2 = q \quad \Rightarrow \quad z_1 = f^{-1}(q/f(z_2))$$

This gives a four punctured sphere with punctures at

$$z_1 = 1, \quad \infty, \quad f^{-1}(q/f(1)), \quad f^{-1}(q/f(\infty))$$

Locations of the punctures in z_1 plane

$$x_1 = 1, \quad x_2 = \infty, \quad x_3 = f^{-1}(q/f(1)), \quad x_4 = f^{-1}(q/f(\infty))$$

The cross ratio

$$\begin{aligned} x &\equiv (x_1 - x_2)(x_3 - x_4)/(x_1 - x_3)(x_2 - x_4) \\ &= -(f^{-1}(q/f(1)) - f^{-1}(q/f(\infty))/(1 - f^{-1}(q/f(1))) \end{aligned}$$

labels moduli space of 4-punctured sphere.

t and u-channel contributions are obtained by the exchange $x_2 \Leftrightarrow x_3$ and $x_2 \Leftrightarrow x_4$

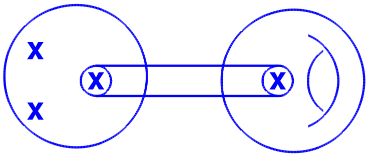
For $|q| \leq 1$, these cover some subspace of the moduli space of 4-punctured sphere.

Declare these to be 1PR 4-punctured spheres.

On these 1PR 4-punctured spheres choose the local coordinates at the punctures to be what is induced from the 3-punctured spheres.

On the rest of the 4-punctured spheres choose local coordinates arbitrarily consistent with symmetries and continuity and declare them to be 1PI 4-punctured spheres.

Similarly gluing 3-puntured spheres with 1-punctured tori we get a set of 2-punctured tori.



Declare them to be 1PR and choose local coordinates on them to be those induced from the constituents

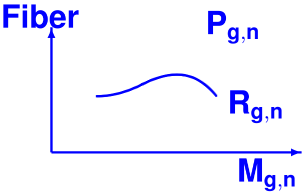
– covers part of the moduli space of 2-punctured tori.

Declare the rest of the 2-punctured tori to be 1PI and choose local coordinates on them arbitrarily maintaining symmetries and continuity.

Proceeding this way, for all $P_{g,n}$ we can choose

1. Gluing compatible sections $S_{g,n}$

2. Identify part of the section $S_{g,n}$ as 1PI subspace $R_{g,n}$ of $P_{g,n}$



Off-shell 1PI ampitudes are then defined as

$$\int_{R_{g,n}} \omega_{6g-6+2n}(|\phi\rangle)$$

Full amplitudes are given by $\int_{S_{g,n}} \omega_{6g-6+2n}(|\phi\rangle)$

One subtle point

Choosing local coordinates consistent with all symmetries is a strong constraint.

e.g. for a three punctured sphere, if we make an SL(2,C) transformation that permutes the three punctures, the local coordinates will have to be permuted.

It may not be convenient to make a single choice of local coordinates compatible with all symmetries.

Remedy: We can allow $S_{g,n}$ to be weighed average of several subspaces

$$S_{g,n} = \sum_i a_i \, S^{(i)}_{g,n}, \quad \sum_i a_i = 1$$

$S^{(i)}_{g,n}$ correspond to different sections of $P_{g,n}$.

$$\int_{S_{g,n}} \omega_{6g-6+2n}(|\phi\rangle) \equiv \sum_i a_i \int_{S^{(i)}_{g,n}} \omega_{6g-6+2n}(|\phi\rangle)$$

The notion of gluing compatibility and construction of 1PI subspaces can be generalized to these cases as well.

Lecture 3: Heterotic and Type II Strings

World-sheet theory

We shall focus on heterotic string theory but the generalization to type II string theories is straightforward.

The world-sheet theory contains a matter super-conformal field theory (SCFT) with central charge $(26, 15)$.

Ghost system contains:

Left-moving $\bar{b}$, $\bar{c}$ ghosts of central charge -26.

Right-moving b, c ghosts of central charge -26 and β, γ ghosts of total central charge 11.

‘Bosonization’ of β, γ system:

$$\gamma = \eta\, \mathbf{e}^{\phi}, \quad \beta = \partial\xi\, \mathbf{e}^{-\phi}, \quad \delta(\gamma) = \mathbf{e}^{-\phi}, \quad \delta(\beta) = \mathbf{e}^{\phi}$$

ξ, η are fermions and ϕ is a scalar with background charge.

(ghost number, picture number, GSO) assignments of various fields are:

$$\mathbf{c}, \bar{\mathbf{c}}: (\mathbf{1}, \mathbf{0}, +), \quad \mathbf{b}, \bar{\mathbf{b}}: (-\mathbf{1}, \mathbf{0}, +), \quad \gamma: (\mathbf{1}, \mathbf{0}, -), \quad \beta: (-\mathbf{1}, \mathbf{0}, -)$$

$$\xi: (-\mathbf{1}, \mathbf{1}, +), \quad \eta: (\mathbf{1}, -\mathbf{1}, +), \quad \mathbf{e}^{\mathbf{q}\phi}: (\mathbf{0}, \mathbf{q}, (-\mathbf{1})^{\mathbf{q}})\,.$$

Picture changing operator (PCO)

$$\mathcal{X}(\mathbf{z}) = \{\mathbf{Q_B}, \xi(\mathbf{z})\} = \mathbf{c}\partial\xi + \mathbf{e}^{\phi}\mathbf{T_F} - \frac{\mathbf{1}}{\mathbf{4}}\partial\eta\mathbf{e}^{\mathbf{2}\phi}\mathbf{b} - \frac{\mathbf{1}}{\mathbf{4}}\partial\left(\eta\mathbf{e}^{\mathbf{2}\phi}\mathbf{b}\right)\,.$$

$\mathbf{T_F}$: superpartner of matter stress tensor

Recall: H is defined to be the Hilbert space of states annihilated by

$$b_0 - \bar{b}_0, \qquad L_0 - \bar{L}_0$$

Off-shell NS sector states are states in H of picture number -1.

Off-shell R-sector states are states in H of picture number $-1/2$.

Getting non-zero amplitude on a genus g Riemann surface requires total picture number $2g-2$ carried by all the operators

– requires insertion of PCO's, e.g. 2g-2+n PCO's on genus g with n NS-punctures.

Our task now is the same as in bosonic string theory

1. Find a parametrization of the space $P_{g,m,n}$ and its tangent space.

(m,n): number of (NS,R) punctures

2. For given external states, collectively denoted as ϕ, construct p-forms ω_p on $P_{g,m,n}$ with the desired properties:

$$\omega_p(\sum_i Q_B^{(i)}\phi) = (-1)^p \, d\omega_{p-1}(|\phi\rangle)$$

$Q_B^{(i)}$: BRST charge acting on i-th state

3. Construct gluing compatible sections and 1PI subspaces of $P_{g,m,n}$.

Parametrization of $P_{g,m,n}$

This involves:

1. Parametrization of moduli space of (m+n) punctured Riemann surfaces and choice of local coordinates at the punctures.

– same as in the case of bosonic string theory.

2. Locations of the PCO's

2g-2+m + n/2 complex coordinates $\{y_j\}$ parametrizing the PCO locations.

Construction of $\omega_{\mathbf{p}}(|\phi\rangle)$

We now have two kinds of tangent vector

1. Tangent vectors associated with deformations of the moduli of the punctured Riemann surface or local coordinates at the punctures.

– parametrized in the same way as in bosonic string theory.

2. $\partial/\partial y_j$ describing deformation of PCO locations.

1. Contraction of $\omega_p(|\phi\rangle)$ with the tangent vectors of the first kind has the same expresssion as in bosonic string theory except for insertions of PCO's

$$\omega_p(|\phi\rangle)[V_1, \cdots V_p]$$

$$\equiv \quad (2\pi i)^{-3g+3-n} \langle B[V_1] \cdots B[V_p] \prod_{j=1}^{2g-2+m+n/2} \mathcal{X}(y_j)\, \phi\rangle_S$$

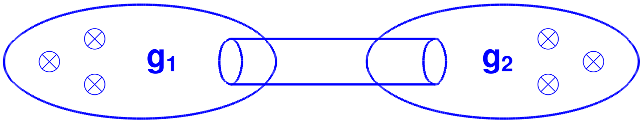
2. Contraction of $\omega_p(|\phi\rangle)$ with $\partial/\partial y_k$ has the effect of replacing the $\mathcal{X}(y_k)$ factor by $-\partial\xi(y_k)$.

The p-form $\omega_p(|\phi\rangle)$ on $P_{g,m,n}$ defined this way satisfies all the required identities.

Construction of gluing compatible sections

– proceeds in the same way as in the case of bosonic string theory.

Extra condition: on a 1PR Riemann surface obtained by plumbing fixture of one or more Riemann surfaces, the PCO locations should agree with those on the 1PI Riemann surfaces that are glued.



An exercise in counting:

Consider a genus g_1 surface with m_1 NS and n_1 R-punctures, glued to a genus g_2 surface with m_2 NS and n_2 R-punctures.

Total number of PCO's on the two surfaces is

$$(2g_1 - 2 + m_1 + n_1/2) + (2g_2 - 2 + m_2 + n_2/2)$$

If the gluing is at NS puncture then the glued surface has genus $g_1 + g_2$, $(m_1 + m_2 - 2)$ NS punctures and $(n_1 + n_2)$ R-punctures.

– required number of PCO's

$$2(g_1 + g_2) - 2 + (m_1 + m_2 - 2) + (n_1 + n_2)/2 \qquad \surd$$

If the gluing is at R puncture then the glued surface has genus $g_1 + g_2$, $(m_1 + m_2)$ NS punctures and $(n_1 + n_2 - 2)$ R-punctures.

– required number of PCO's

$$2(g_1 + g_2) - 2 + (m_1 + m_2) + (n_1 + n_2 - 2)/2 \qquad \times$$

– one more than what is induced from the component Riemann surfaces.

Question: Where should we insert the extra PCO when we glue two Riemann surfaces at R-punctures via

$$z\,w = e^{-s+i\theta} \qquad 0 \le s < \infty, \quad 0 \le \theta < 2\pi$$

A consistent prescription: Insert

$$\mathcal{X}_0 \equiv \oint \frac{dz}{z}\mathcal{X}(z) = \oint \frac{dw}{w}\mathcal{X}(w)$$

around either puncture.

$\mathcal{X}_0$ **has been used earlier for other purposes.**

Berkovits, Zwiebach; Erler, Konopka, Sachs

A technical issue: Spurious poles

The correlation function used for defining ω_p diverges at points where no vertex operators or PCO's coincide.

Verlinde, Verlinde

$$f(\{y_i\}, \{w_j\}, \{m_k\}) = 0$$

y_i: location of PCO's

$\{m_k\}$: moduli

w_j: locations of vertex operators

– a real codimension two subspace on the section

– appears even for on-shell amplitudes

– related to the fact that the gauge choice for the world-sheet gravitino breaks down at these points.

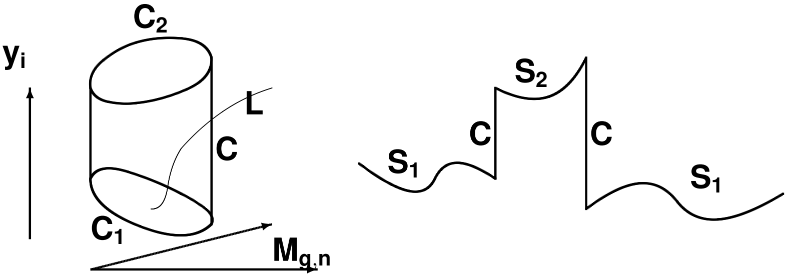
We could try to avoid it by judicious choice of section

– impossible for high enough genus when the supermoduli space is not holomorphically projected.

Donagi, Witten

How to integrate through these poles?

Resolution: Use 'vertical segment'



Integrate over a section S_1 outside C_1, then along C and then along a section S_2 inside C_2.

L: Path of the spurious pole.

We intercept the spurious pole along the vertical segment.

Along the vertical segment we have to contract $\omega_{6g-6+2n}$ with $\partial/\partial y_i$

$\Rightarrow$ **PCO at y_i is replaced by $-\partial\xi(y_i)$.**

Thus the vertical integration inserts

$$-\int_u^v \partial\xi(y_i)dy_i = \xi(u) - \xi(v)$$

– depends only on the initial and final points and has no singularity or ambiguity.

This is only the tip of the iceberg!

In general we have to add corrections not only on codimension 1 subspaces of $M_{g,n}$ but also on codimension 2 and higher subspaces all the way up to codimension K subspaces. A.S., Witten

K: number of PCO's

This finishes our description of construction of ω_p on $P_{g,m,n}$, gluing compatible 'sections' $S_{g,m,n}$ of $P_{g,m,n}$ and hence also the 1PI subspaces $R_{g,m,n}$ of $P_{g,m,n}$.

Lecture 4: 1PI effective action and its applications

Definitions and identities

NS-sector vertex operators are grassman even for even ghost number and grassmann odd for odd ghost number

For R-sector it is opposite.

Given N states $|A_1\rangle, \cdots |A_N\rangle$ in the CFT Hilbert space H, of which m are NS states and $n = N - m$ are R-states we define a multi-linear function

$$\{A_1 \cdots A_N\} = \sum_{g=0}^{\infty} {g_s}^{2g} \int_{R_{g,m,n}} \omega_{6g-6+2(m+n)}(|A_1\rangle, \cdots |A_N\rangle)$$

We also define $[A_1 \cdots A_N] \in H$ via

$$\langle A_0|c_0^-|[A_1 \cdots A_N]\rangle = \{A_0 A_1 \cdots A_N\}, \quad c_0^- \equiv (c_0 - \bar{c}_0)/2$$

1. Under exchange of A_i and A_j, $\{A_1 \cdots A_N\}$ pick up a sign

$$(-1)^{\gamma_i \gamma_j}$$

γ_i: grassmannality of A_i i.e. 0 if A_i is grassmann even and 1 if A_i is grassmann odd.

2.
$$\sum_{i=1}^{N} (-1)^{\gamma_1 + \cdots \gamma_{i-1}} \{A_1 \cdots A_{i-1} (Q_B A_i) A_{i+1} \cdots A_N\}$$
$$= -\frac{1}{2} \sum_{\substack{\ell, k \geq 0 \\ \ell + k = N}} \sum_{\substack{\{i_a; a=1,\cdots \ell\}, \{j_b; b=1, \cdots k\} \\ \{i_a\} \cup \{j_b\} = \{1, \cdots N\}}} \sigma(\{i_a\}, \{j_b\})$$
$$\{A_{i_1} \cdots A_{i_\ell} (G[A_{j_1} \cdots A_{j_k}])\}$$

$\sigma(\{i_a\}, \{j_b\})$: the sign that one picks up while rearranging $b_0^-, A_1, \cdots A_N$ to $A_{i_1}, \cdots A_{i_\ell}, b_0^-, A_{j_1}, \cdots A_{j_k}$

$$G|s\rangle \equiv \begin{cases} |s\rangle & \text{if } |s\rangle \in H_{NS} \\ \mathcal{X}_0 |s\rangle & \text{if } |s\rangle \in H_R \end{cases},$$

3. $\{A_1 \cdots A_k (G[\widetilde{A}_1 \cdots \widetilde{A}_\ell])\} = (-1)^{\gamma + \widetilde{\gamma} + \gamma\widetilde{\gamma}} \{\widetilde{A}_1 \cdots \widetilde{A}_\ell (G[A_1 \cdots A_k])\}$.

$\gamma, \widetilde{\gamma}$: total grassmannality of the A_i's, $\widetilde{A}_i$'s.

NS sector string field: An arbitrary state $|\psi_{NS}\rangle \in H$ carrying ghost number 2 and picture number -1.

R sector string field: An arbitrary state $|\psi_{R}\rangle \in H$ carrying ghost number 2 and picture number $-1/2$.

$$|\psi\rangle \equiv |\psi_{NS}\rangle + |\psi_{R}\rangle$$

If $|\phi_r\rangle$ is a basis in $H_{-1} + H_{-1/2}$, then we can expand

$$|\psi\rangle = \sum_r a_r |\phi_r\rangle$$

The coefficients a_r are the dynamical variables labelling the string field (in momentum space).

Coefficients of NS sector basis states are grassmann even and the coefficients of R-sector basis states are grassmann odd.

We shall first describe the 1PI effective action for NS sector fields.

$$\mathbf{S}(|\psi_{\mathbf{NS}}\rangle) = \mathbf{g_s}^{-2}\left[\frac{\mathbf{1}}{\mathbf{2}}\langle\psi_{\mathbf{NS}}|\mathbf{c_0^-}\mathbf{Q_B}|\psi_{\mathbf{NS}}\rangle + \sum_{\mathbf{n=1}}^{\infty}\frac{\mathbf{1}}{\mathbf{n!}}\{\psi_{\mathbf{NS}}{}^{\mathbf{n}}\}\right]$$

$\{\psi_{\mathbf{NS}}{}^{\mathbf{n}}\}$: $\{\psi_{\mathbf{NS}}\psi_{\mathbf{NS}}\cdots\psi_{\mathbf{NS}}\}$ **with n copies of** $\psi_{\mathbf{NS}}$ **inside** $\{\ \}$.

Infinitesimal gauge transformation law

$$\delta|\psi_{\mathbf{NS}}\rangle = \mathbf{Q_B}|\lambda_{\mathbf{NS}}\rangle + \sum_{\mathbf{n=0}}^{\infty}\frac{\mathbf{1}}{\mathbf{n!}}[\psi_{\mathbf{NS}}{}^{\mathbf{n}}\lambda_{\mathbf{NS}}]$$

$|\lambda_{\mathbf{NS}}\rangle$: **is an element of H with ghost number 1, picture number** -1.

Gauge invariance of $\mathbf{S}(|\psi_{\mathbf{NS}}\rangle)$ **can be proved using the identities involving** $\{\cdots\}$ **and** $[\cdots]$.

Equations of motion:

$$\mathbf{Q_B}|\psi_{\mathbf{NS}}\rangle + \sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{(\mathbf{n}-\mathbf{1})!}[\psi_{\mathbf{NS}}{}^{\mathbf{n-1}}] = \mathbf{0}$$

Note: $\{\psi_{\mathbf{NS}}\}$ and [ ] are non-zero from genus 1 onwards

$|\psi_{\mathbf{NS}}\rangle = 0$ is not a solution to equations of motions.

We have to first solve the equations of motion and then expand the 1PI action around the solution.

Special importance: Vacuum solution carrying zero momentum

Iterative construction of the vacuum solution:

Suppose $|\psi_k\rangle$ is the solution to order ${g_s}^k$. $\quad (|\psi_0\rangle = 0)$

P: projection operator to $L_0^+ \equiv L_0 + \bar{L}_0 = 0$ states.

Then

$$|\psi_{k+1}\rangle = -\frac{b_0^+}{L_0^+} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (1-P)[\psi_k^{n-1}] + |\phi_{k+1}\rangle \,,$$

$|\phi_{k+1}\rangle$ is an $L_0^+ = 0$ state satisfying

$$Q_B |\phi_{k+1}\rangle = -\sum_{n=1}^{\infty} \frac{1}{(n-1)!} P[\psi_k^{n-1}] + \mathcal{O}({g_s}^{k+2}) \,.$$

$$|\psi_{k+1}\rangle = -\frac{b_0^+}{L_0^+}\sum_{n=1}^{\infty}\frac{1}{(n-1)!}(1-P)[\psi_k^{n-1}] + |\phi_{k+1}\rangle\,,$$

$$Q_B|\phi_{k+1}\rangle = -\sum_{n=1}^{\infty}\frac{1}{(n-1)!}P[\psi_k^{n-1}] + \mathcal{O}({g_s}^{k+2})\,.$$

Possible obstruction to solving these arise from the last equation.

rhs could contain a component along a non-trivial element of BRST cohomology.

– reflects the existence of zero momentum massless tadpoles in perturbation theory.

Unless this equation can be solved we have to declare the vacuum inconsistent.

$$|\psi_{k+1}\rangle = -\frac{b_0^+}{L_0^+}\sum_{n=1}^{\infty}\frac{1}{(n-1)!}(1-P)[\psi_k^{n-1}] + |\phi_{k+1}\rangle\,,$$

$$Q_B|\phi_{k+1}\rangle = -\sum_{n=1}^{\infty}\frac{1}{(n-1)!}P[\psi_k^{n-1}] + \mathcal{O}({g_s}^{k+2})\,.$$

Once these equations have been solved, we do not encounter any further tadpole divergence in perturbation theory.

Note: The full solution $|\psi_v\rangle$ is $|\psi_\infty\rangle$, but in practice we shall stop at some fixed order in g_s.

This also allows us to deal with the cases involving vacuum shift, e.g. when a scalar field χ in low energy theory has potential

$$\mathbf{c}(\chi^2 - \mathbf{K}\,\mathbf{g_s}^2)^2 \,.$$

At order $\mathbf{g_s}$ we have three solutions $\chi = \mathbf{0}, \pm \mathbf{g_s}\sqrt{\mathbf{K}}$.

In 1PI effective feld theory this will be reflected in the existence of multiple solutions for $|\phi_1\rangle$.

The solution corresponding to $\chi = \mathbf{0}$ will have non-zero dilaton one point function at higher order

$\Rightarrow$ an obstruction to extending the corresponding 1PI effective field theory solution to higher order.

The solutions corresponding to $\chi = \pm \mathbf{g_s}\sqrt{\mathbf{K}}$ will not encounter such obstructions.

For Ramond sector states it is not possible to write down an action with local kinetic term.

We can only write down the equation of motion.

– related to the fact that Ramond sector states carry picture number $-1/2$ and the inner product between two such states vanish by picture number conservation.

For a string field theory this would be problematic since we would not know how to quantize the theory.

However for 1PI theory this is not a problem since we only need to work at the tree level.

General structure (including NS and R-sector):

A general string field configuration corresponds to a state $|\psi\rangle \in \mathbf{H}$ of ghost number 2 and picture number $(-1, -1/2)$ in (NS,R) sector.

1PI equation of motion:

$$\mathbf{Q_B}|\psi\rangle + \sum_{\mathbf{n}=1}^{\infty} \frac{1}{(\mathbf{n}-1)!} \mathbf{G}[\psi^{\mathbf{n}-1}] = 0$$

$\mathbf{Q_B}$: BRST operator

G: identity in NS sector $\quad \mathcal{X}_0 \equiv \oint \mathbf{z}^{-1} \mathbf{dz}\, \mathcal{X}(\mathbf{z})$ in R sector

Gauge transformations

The infinitesimal gauge transformation parameters correspond to states $|\lambda\rangle$ of ghost number 1 and picture number $(-1,-1/2)$ in (NS,R) sector.

Gauge transformation law

$$\delta|\psi\rangle = \mathbf{Q_B}|\lambda\rangle + \sum_{\mathbf{n}=0}^{\infty} \frac{\mathbf{1}}{\mathbf{n}!}\mathbf{G}[\psi^{\mathbf{n}}\,\lambda]$$

Once we have a vacuum solution $|\psi_v\rangle$ we can expand the equations of motion around $|\psi_v\rangle$.

Define: $|\chi\rangle \equiv |\psi\rangle - |\psi_v\rangle$

$$\{A_1 \cdots A_k\}' \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \{\psi_v^n A_1 \cdots A_k\}, \qquad \text{for } k \geq 3,$$

$$[A_1 \cdots A_k]' \equiv \sum_{n=0}^{\infty} \frac{1}{n!} [\psi_v^n A_1 \cdots A_k], \qquad \text{for } k \geq 2,$$

$$\{A_1\}' \equiv 0, \qquad [\,]' \equiv 0, \qquad \{A_1 A_2\}' \equiv 0, \quad [A_1]' \equiv 0,$$

$$\widehat{Q}_B|A\rangle \equiv Q_B|A\rangle + \sum_{k=0}^{\infty} \frac{1}{k!} G[\psi_v{}^k A].$$

New 'shifted' equations of motion

$$\widehat{Q}_B|\chi\rangle + \sum_{n=2}^{\infty} \frac{1}{n!} G[\chi^n]' = 0$$

1. Under exchange of A_i and A_j, $\{A_1 \cdots A_N\}'$ pick up a sign

$$(-1)^{\gamma_i \gamma_j}$$

γ_i: grassmannality of A_i i.e. 0 if A_i is grassmann even and 1 if A_i is grassmann odd.

2.
$$\sum_{i=1}^{N} (-1)^{\gamma_1 + \cdots \gamma_{i-1}} \{A_1 \cdots A_{i-1} (\widehat{Q}_B A_i) A_{i+1} \cdots A_N\}'$$
$$= -\frac{1}{2} \sum_{\substack{\ell, k \geq 0 \\ \ell + k = N}} \sum_{\substack{\{i_a; a=1,\cdots \ell\}, \{j_b; b=1,\cdots k\} \\ \{i_a\} \cup \{j_b\} = \{1,\cdots N\}}} \sigma(\{i_a\}, \{j_b\})$$
$$\{A_{i_1} \cdots A_{i_\ell} (G[A_{j_1} \cdots A_{j_k}]')\}'$$

$\sigma(\{i_a\}, \{j_b\})$: the sign that one picks up while rearranging $b_0^-, A_1, \cdots A_N$ to $A_{i_1}, \cdots A_{i_\ell}, b_0^-, A_{j_1}, \cdots A_{j_k}$

$$G|s\rangle \equiv \begin{cases} |s\rangle & \text{if } |s\rangle \in H_{NS} \\ \mathcal{X}_0 \, |s\rangle & \text{if } |s\rangle \in H_R \end{cases},$$

3. $\{A_1 \cdots A_k (G[\widetilde{A}_1 \cdots \widetilde{A}_\ell]')\}' = (-1)^{\gamma + \widetilde{\gamma} + \gamma \widetilde{\gamma}} \{\widetilde{A}_1 \cdots \widetilde{A}_\ell (G[A_1 \cdots A_k]')\}'$.

$\gamma, \widetilde{\gamma}$: total grassmannality of the A_i's, $\widetilde{A}_i$'s.

Linearized equations of motion around $|\psi_{\mathbf{v}}\rangle$:

$$\widehat{\mathbf{Q}}_{\mathbf{B}}|\chi\rangle = 0$$

– has two kinds of solution:

1. Solutions which exist for all momentum k of $|\chi\rangle$

– have the form $\widehat{\mathbf{Q}}_{\mathbf{B}}|\lambda\rangle$ for some $|\lambda\rangle$ and are pure gauge.

2. Solutions which exist for special values of k^2

– represent physical states with the corresponding values of $-k^2$ giving <u>renormalized mass^2</u>.

This abstract definition can be developed into a fully systematic perturbative scheme.

A similar procedure can be given for the S-matrix elements starting from the LSZ formalism.

Pius, Rudra, A.S.

Section dependence

The definition of $\{A_1 \cdots A_N\}$ and all subsequent analysis depends on the choice of 1PI subspace $R_{g,m,n}$.

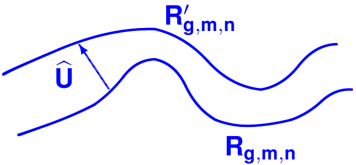
A different choice of gluing compatible ‘sections’

$\Rightarrow$ a different choice of $R_{g,m,n}$

$\Rightarrow$ a different set of equations of motion.

Do the renormalized masses and S-matrix elements depend on this choice?

We shall consider the case of infinitesimal deformations from $\mathbf{R_{g,m,n}}$ to $\mathbf{R'_{g,m,n}}$ labelled by some tangent vector $\widehat{\mathbf{U}}$ of $\mathbf{P_{g,m,n}}$ at every point of $\mathbf{R_{g,m,n}}$.



Result: The change in the equation of motion can be compensated by a field redefinition $|\psi\rangle \to |\psi\rangle + \delta|\psi\rangle$ where

$$\langle\phi|\mathbf{c_0^-}|\delta\psi\rangle = -\sum_{\mathbf{g=0}}^{\infty} \mathbf{g_s}^{\mathbf{2g}} \sum_{\mathbf{m,n=0}}^{\infty} \frac{\mathbf{1}}{\mathbf{m!n!}}$$

$$\Big[\int_{\mathbf{R_{g,m+1,n}}} \omega_{\mathbf{6g-5+2m+2n+2}}[\widehat{\mathbf{U}}](\mathbf{G}|\phi_{\mathbf{NS}}\rangle\rangle, |\psi_{\mathbf{NS}}\rangle^{\otimes\mathbf{m}}, |\psi_{\mathbf{R}}\rangle^{\otimes\mathbf{n}})$$

$$\int_{\mathbf{R_{g,m,n+1}}} \omega_{\mathbf{6g-5+2m+2n+2}}[\widehat{\mathbf{U}}](|\psi_{\mathbf{NS}}\rangle^{\otimes\mathbf{m}}, \mathbf{G}|\phi_{\mathbf{R}}\rangle, |\psi_{\mathbf{R}}\rangle^{\otimes\mathbf{n}})\Big]$$

Thus renormalized masses and S-matrix elements remain unchanged.

Future prospects

1. Demand of infinite dimensional gauge invariance more or less fixes the perturbative scattering amplitude

– integral over the full integration cycle $S_{g,m,n}$.

Could it constrain the structure of non-perturbative corrections to the 1PI effective action?

2. Can we use the off-shell action to study string theory in weak RR background field perturbatively?

References

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Pius, Rudra, A.S., , arXiv:1311.1257, 1401.7014, 1404.6254

A.S., arXiv:1408.0571, 1411.7478, 1501.00988

A.S., Witten, to appear

(Other relevant references can be found in these)

Translation of convention from slides $\Rightarrow$ the papers
(used for simplifying notation in the slides)

$\mathbf{M_{g,n}} \Rightarrow \mathcal{M}_{\mathbf{g,n}}$
$\mathbf{P_{g,n}} \Rightarrow \widehat{\mathcal{P}}_{\mathbf{g,n}}$ for bosonic string theory
$\mathbf{P_{g,n}} \Rightarrow \widetilde{\mathcal{P}}_{\mathbf{g,n}}$ (also $\widetilde{\mathcal{P}}_{\mathbf{g,m,n}}$) for heterotic / type II string theory
$\mathbf{R_{g,n}} \Rightarrow \mathcal{R}_{\mathbf{g,n}}$ (also $\mathcal{R}_{\mathbf{g,m,n}}$) for heterotic / type II string theory
$\{\cdots\}' \Rightarrow \{\cdots\}'', \quad [\cdots]' \Rightarrow [\cdots]''$