Percolation approach to many-body localization

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Brief outline

- The model
- Percolation approach to MBL
- Diagonalization of the Hamiltonian through displacement transformations

Many-body localization

- With strong enough disorder, all single-particle states are localized.
- Conductivity is then by hopping between localized states. Mott's variable range hopping.
- The standard driving nechanism for hopping is the phonon bath, but any extended, continuous bath could do the same role.
- Basko et al. (2006) proposed the electron-electron interaction as the driving mechanism above a certain temperature.
- The problem can be thought of as many-body delocalization in Fock space.
- Numerical simulations: mainly exact diagonalization (very small systems)

Model

- Single-particle Hamiltonian $H_0 = \sum_i \epsilon_i c_i^\dagger c_i + \sum_{\langle i,j \rangle} t c_j^\dagger c_i$, where $t = 1$ is our unit of energy and $\epsilon_i \in [-W/2, W/2]$
- Diagonalize H_0 and obtain a localized basis $|\alpha\rangle$
- The total Hamiltonian is in this basis

$$H = \sum_\alpha \phi_\alpha c_\alpha^\dagger c_\alpha + \frac{1}{2} \sum_{\alpha\beta\gamma\eta} V_{\alpha\beta\gamma\eta} c_\alpha^\dagger c_\beta c_\gamma^\dagger c_\eta$$

- Short range potential: nearest neighbours for fermions ($V = 1$)
- Periodic boundary conditions
- First compute $V_{\alpha\beta\gamma\eta}$

Matrix elements

Relevant information contained in the distribution of

$$V_{\alpha,\beta,\gamma,\eta}/(E_\alpha + E_\beta - E_\gamma - E_\eta)$$

Distribution

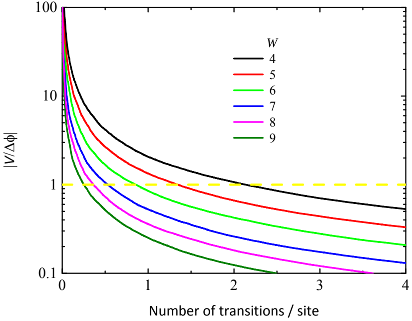

Figure: Distribution of the number of transitions of strength $|V/\Delta\phi|$ for several disorders W.

Matrix elements

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$$V_{\alpha,\beta,\gamma,\eta}/(E_\alpha + E_\beta - E_\gamma - E_\eta)$$

We simplify this to the number of resonances, i.e., configurations with

$$|V_{\alpha,\beta,\gamma,\eta}/(E_\alpha + E_\beta - E_\gamma - E_\eta)| > 1$$

Distribution

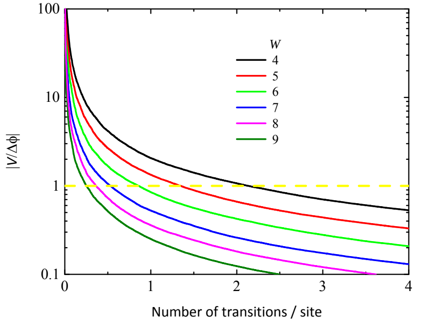

Figure: Distribution of the number of transitions of strength $|V/\Delta\phi|$ for several disorders W.

Resonances

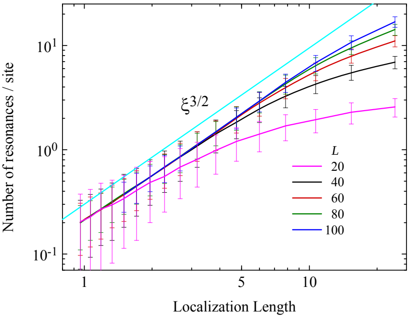

Figure: Number of resonances per site as a function of the localization length ξ for several L.

Spreading in configuration space

- Initial configuration: we occupied $L/2$ single-particle states $|\alpha\rangle$ at random
$$|0,1,1,0,0,0,1,0,1,1,0,\cdots\rangle$$

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- Iterate the procedure expanding layer by layer until there are no more resonating configurations or the number of configurations in a layer exceeds a maximum number (10^8).
- Only have to store three active layers

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- Calculate:
  - Size of the cluster Ω
  - Variance of the number of particles crossing a (virtual) boundary σ^2

Localized regime

We measure the accumulated probability $\mathcal{P}(\Omega) = \int_{\Omega}^{\infty} P(\Omega') d\Omega'$ as a function of the normalized cluster size Ω/Ω_L, where $\Omega_L = \binom{L}{L/2} \approx 2^L$

Disorder $W = 9$

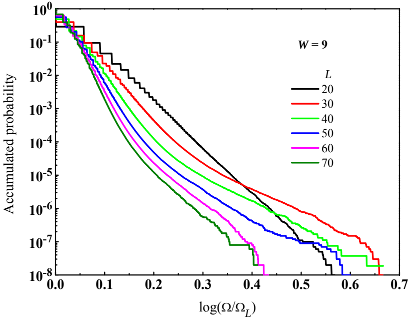


Figure: $\mathcal{P}(\Omega)$ **vs.** $\log(\Omega/\Omega_L)$ **for** $W = 9$.

Disorder $W = 8$

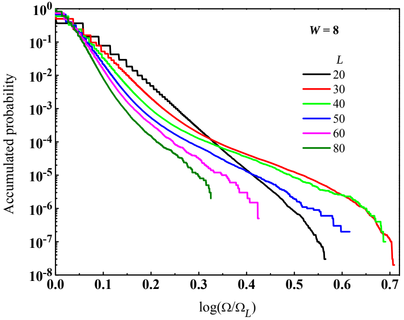
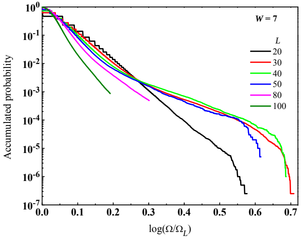


Figure: $\mathcal{P}(\Omega)$ **vs.** $\log(\Omega/\Omega_L)$ **for** $W = 8$.

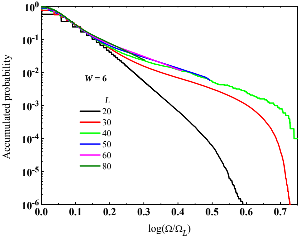
- Size $L = 20$ is quite anomalous
- Fluctuations in the number of resonances are important and cause long tails

Transition region

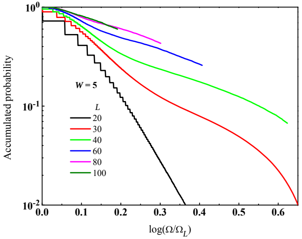
Disorder $W = 7$



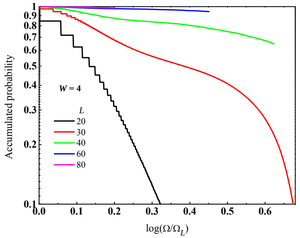
Disorder $W = 6$



Disorder $W = 5$



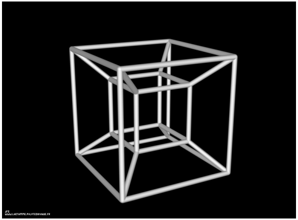
Disorder $W = 4$



Nature of the states

- **States are localized for** $W \gtrsim 6$
- **There is a transition around** $W_c \approx 6$
- **States are metallic for** $W < 4$
- Fluctuations are responsible for the complexity of the situation

Percolation in the hypercube



- Consider a hypercube of dimension L
- Occupy each edge with probability p
- Study the distribution of the size of the clusters

Hypercube

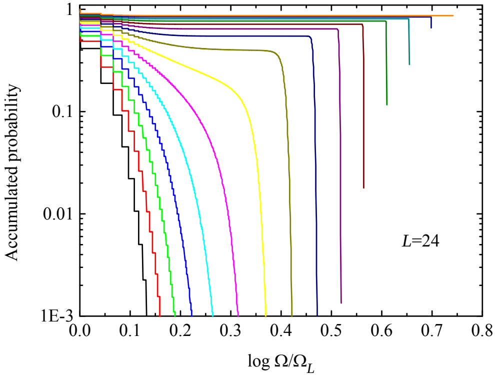


Figure: $\mathcal{P}(\Omega)$ **vs.** $\log(\Omega/\Omega_L)$ **for several values of the percolation probability and** $L = 24$.

Number fluctuations

Fluctuations

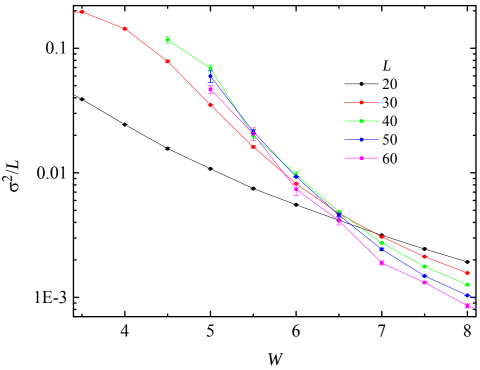


Figure: Variance of the number of particles crossing a surface divided by L as a function of disorder W.

- We study the variance σ^2 of the number of particles crossing a surface
- In the insulating regime $\sigma^2 \to$ constant
- In the metallic regime $\sigma^2 \propto L$

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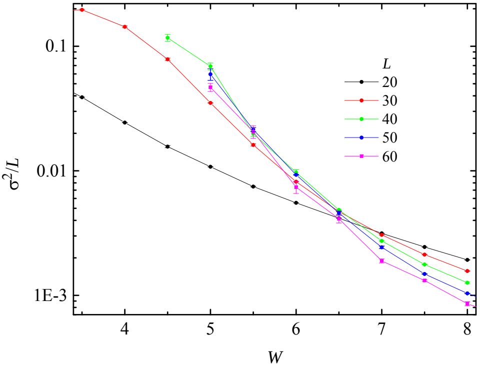

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- We study the variance σ^2 of the number of particles crossing a surface
- In the insulating regime $\sigma^2 \rightarrow$ constant
- In the metallic regime $\sigma^2 \propto L$
- There is a transition ($6 < W_c < 6.5$)
- Behavior in the extended phase is not clear

Diagonalization of the Hamiltonian

Local integrals of motion

We want to diagonalize the Hamiltonian

$$H = \sum_{\alpha} \phi_\alpha n_\alpha + \frac{1}{2} \sum_{\alpha\beta\gamma\eta} V_{\alpha\beta\gamma\eta} c_\alpha^\dagger c_\beta c_\gamma^\dagger c_\eta$$

The aim is to design an explicit procedure to find operators of the form

$$\tilde{n}_\alpha = U^\dagger n_\alpha U = n_\alpha + \sum_{\alpha\beta\gamma\eta} a_{\chi\beta\gamma\eta} c_\chi^\dagger c_\beta c_\gamma^\dagger c_\eta + \cdots$$

such that H can be written as

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Louk Rademaker (KITP) arXiv:1507.07276

Displacement transformation

Displacement transformation: $\mathcal{D}_X(\lambda) = \exp\{\lambda(X^\dagger - X)\}$

$$X = n_{\alpha 1} \cdots n_{\alpha k} c_{\beta}{}_1^\dagger c_{\beta}{}_2 c_{\beta}{}_3^\dagger \cdots c_{\beta}{}_l$$

(here we concentrate in the particular case $X = c_\alpha^\dagger c_\beta c_\gamma^\dagger c_\eta$)

$$\mathcal{D}_X(\lambda) = 1 + \sin\lambda(X^\dagger - X) + (\cos\lambda - 1)(X^\dagger X + XX^\dagger)$$

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$$\mathcal{D}_X(\lambda) = 1 + \sin\lambda(X^\dagger - X) + (\cos\lambda - 1)(X^\dagger X + XX^\dagger)$$

$$\tilde{n}_\delta = \mathcal{D}_X^\dagger(\lambda) n_\delta \mathcal{D}_X(\lambda) = n_\delta \pm \frac{1}{2}\sin 2\lambda(X^\dagger + X) \mp \sin^2\lambda(X^\dagger X - XX^\dagger)$$

upper sign if $\delta = \beta$ or η and lower sign if $\delta = \alpha$ or γ

$$\mathcal{D}_X^\dagger(\lambda)(X^\dagger + X)\mathcal{D}_X(\lambda) = \cos 2\lambda(X^\dagger + X) - \sin 2\lambda(X^\dagger X - XX^\dagger)$$

$$\mathcal{D}_X^\dagger(\lambda) H \mathcal{D}_X(\lambda) = \sum_\alpha \phi_\alpha n_\alpha + \left\{(-\phi_\alpha - \phi_\gamma + \phi_\beta + \phi_\eta)\frac{1}{2}\sin 2\lambda + \frac{1}{2} V_{\alpha\beta\gamma\eta}\cos 2\lambda\right\}$$

$$\times(X^\dagger + X) + \left\{(\phi_\alpha + \phi_\gamma - \phi_\beta - \phi_\eta)\sin^2\lambda - \frac{1}{2} V_{\alpha\beta\gamma\eta}\sin 2\lambda\right\}(X^\dagger X - XX^\dagger)$$

$$\tan 2\lambda = \frac{V_{\alpha\beta\gamma\eta}}{\phi_\alpha + \phi_\gamma - \phi_\beta - \phi_\eta}$$

Trivial example

One particle in two sites

- $H = \phi_1 n_1 + \phi_2 n_2 - t(c_1^\dagger c_2 + c_2^\dagger c_1)$ $\qquad \begin{pmatrix} \phi_1 & -t \\ -t & \phi_2 \end{pmatrix}$
- Define $X = c_1^\dagger c_2$
- $\tan 2\lambda = \dfrac{-2t}{\phi_1 - \phi_2}$
- $\tilde{n}_1 = n_1 - \sin 2\lambda (X^\dagger + X) + \sin^2 \lambda [(1-n_1)n_2 - n_1(1-n_2)]$
- $\tilde{n}_2 = n_2 + \sin 2\lambda (X^\dagger + X) - \sin^2 \lambda [(1-n_1)n_2 - n_1(1-n_2)]$
- $\langle 10|\tilde{n}_1|10\rangle = 1 - \sin^2 \lambda$
- $\langle 10|\tilde{n}_2|10\rangle = \sin^2 \lambda$
- $\sin^2 \lambda = \dfrac{4t^2}{(\phi_1 - \phi_2)^2 + 4t^2}$

Consecutive transformations

- Each transformation modifies the remaining quantum terms of H.
  For example, for $X = c_\alpha^\dagger c_\beta c_\gamma^\dagger c_\eta$, $Y = c_\alpha^\dagger c_i c_\gamma^\dagger c_j$, $Z = c_\eta^\dagger c_i c_\beta^\dagger c_j$

$$\mathcal{D}_X^\dagger(\lambda)(Y^\dagger + Y)\mathcal{D}_X(\lambda) = \cos\lambda(Y^\dagger + Y) - \sin\lambda(Z^\dagger + Z) + \cdots$$

- One start with the transformation corresponding to the higher $|\lambda|$, i.e. higher $|V_{\alpha\beta\gamma\eta}/(\phi_\alpha + \phi_\gamma - \phi_\beta - \phi_\eta)|$, and continues performing consecutive transformations with decreasing values of λ until all four operators terms in H have been cancelled (to a certain accuracy).
- Elliminating terms with a given number of operators de not generate terms with fewer operators.
- The final unitary transformation is

$$U = \prod_i \mathcal{D}_{X_i}(\lambda_i)$$

Occupation number

- As a proof of principle, we calculate $\langle\Phi_0|\tilde{n}_\alpha|\Phi_0\rangle$ where $|\Phi_0\rangle = c_\beta^\dagger \cdots c_\gamma^\dagger|0\rangle$ (we assume that $\beta, \cdots \gamma \neq \alpha$)
- There seems to be a localized regime for $W \gtrsim 6$ and probably a transition at $W_c \approx 6$.

Occupation number

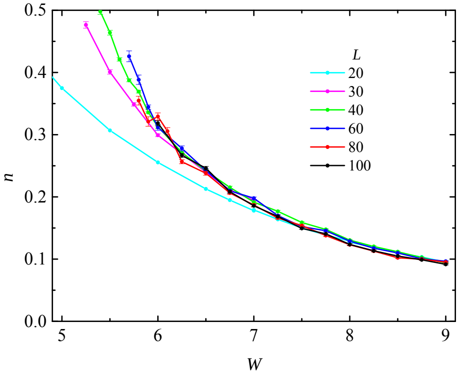

Figure: Average occupation number of a local integral of motion as a function of disorder.

Future possibilities

- Calculation of $\langle\Phi_0|\tilde{n}_\alpha \tilde{n}_\beta|\Phi_0\rangle$
- Quantum Coulomb gap
- Level statistics
- Higher dimensions
- Study convergence of the method