Lecture 2

ICTP 2015 Summer school

Neutrinos
selected topics

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Propagation of wave packets

What happens?

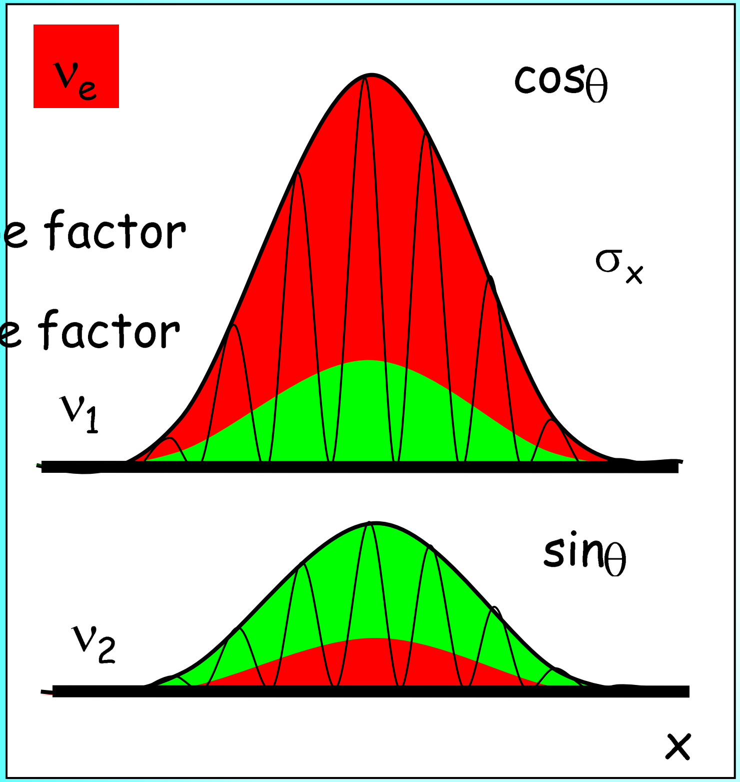
Phase difference change	Due to different masses (dispersion relations) → phase velocities	**Oscillations**
Separation of wave packets	Due to different group velocities	**Loss of coherence**
Spread of individual wave packets	Due to presence of waves with different momenta and energy in the packet	**Loss of coherence within within WP**

Wave packet picture

2ν– example

Shape factor

Phase factor



$$\nu_e = \cos\theta\ \nu_1 + \sin\theta\ \nu_2$$
$$\nu_\mu = -\sin\theta\ \nu_1 + \cos\theta\ \nu_2$$

opposite phase

$$\nu_1 = \cos\theta\ \nu_e - \sin\theta\ \nu_\mu$$
$$\nu_2 = \cos\theta\ \nu_\mu + \sin\theta\ \nu_\varepsilon$$

Interference of the same flavor parts

$\phi = 0$

effective, frequency

$$|\nu(x,t)\rangle = \cos\theta\ g_1(x - v_1 t) e^{i\phi_1} |\nu_1\rangle + \sin\theta\ g_2(x - v_2 t) e^{i\phi_2} |\nu_2\rangle$$

$\phi = \phi_2 - \phi_1$

Oscillation phase

Changes with (x,t), for $\phi \neq 0$
components ν_μ will not cancel
→ appearance of ν_μ

Oscillation phase

$\phi = \phi_2 - \phi_1$ $\phi_i = - E_i t + p_i x$ $p_i = \sqrt{E_i^2 - m_i^2}$ Dispersion relation

$\phi = \Delta E t - \Delta p x$

These are averaged characteristics of WP

where

$\Delta p = (dp/dE)\Delta E + (dp/dm^2)\Delta m^2 = 1/v_g \, \Delta E + (1/2p)\, \Delta m^2$

group velocity

insert

$$\phi = \Delta E/v_g \, (v_g t - x) + \frac{\Delta m^2}{2E} x$$

standard oscillation phase

$\Delta E \sim \Delta m^2/2E$ $< \sigma_x$

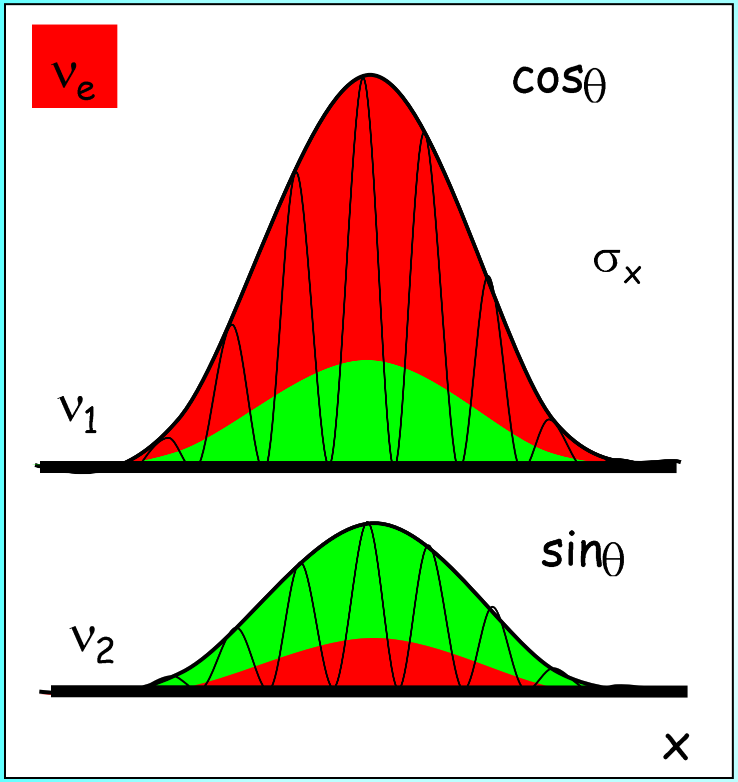
Averaged energies

$\sigma_x \Delta m^2/2E$

Oscillation effect over the size of WP usually- small

Phase difference along the wave packets is the same

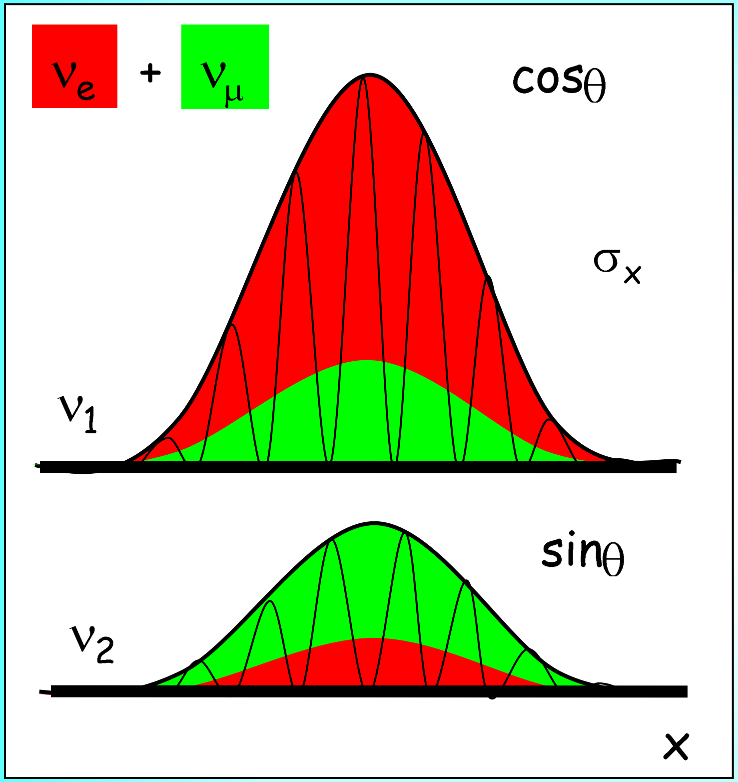
Oscillations



$\phi = 0$

- Destructive interference of the muon parts
- Constructive interference of electron parts

Oscillations



$\phi = \pi$

- Destructive interference of the electron parts
- Constructive interference of muon parts

Detection:

As important as production
should be considered symmetrically with production

Detection effect can be included in
the generalized shape factors

$$g_k(x - v_k t) \rightarrow G_k(L - v_k t)$$

$x \rightarrow L$ - distance between central points of the production and detection regions

HOMEWORK…

Oscillation probability

Amplitude of (survival) probability

$$A(\nu_e) = \langle\nu_e|\nu(x,t)\rangle = \cos^2\theta\, g_1(x - v_1 t) + \sin^2\theta\, g_2(x - v_2 t)\, e^{i\phi}$$

Probability in the moment of time t

interference

$$P(\nu_e) = \int dx\, |\langle\nu_e|\nu(x,t)\rangle|^2 =$$

$$= \cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta\cos\phi \int dx\, g_1(x - v_1 t)\, g_2(x - v_2 t)$$

if $\int dx\, |g|^2 = 1$

If $g_1 = g_2$

$$P(\nu_e) = 1 - 2\sin^2\theta\cos^2\theta\,(1 - \cos\phi) = 1 - \sin^2 2\theta \sin^2 \tfrac{1}{2}\phi$$

depth of oscillations

$$\phi = \frac{\Delta m^2\, x}{2E} = \frac{2\pi x}{l_\nu}$$

$$l_\nu = \frac{4\pi E}{\Delta m^2}$$

Oscillation length

Physics summary

Appearance probability

$$P(\nu_\mu) = \sin^2 2\theta \sin^2 \frac{\pi x}{l_\nu}$$

All complications are "absorbed" in normalization
or reduced to partial averaging of oscillations
or lead to negligible corrections of order $m/E \ll 1$

Oscillations - effect of the phase difference increase between mass eigenstates

Admixtures of the mass eigenstates ν_i in a given neutrino state do not change during propagation

Flavors (flavor composition) of the eigenstates are fixed by the vacuum mixing angle

Coherence in propagation

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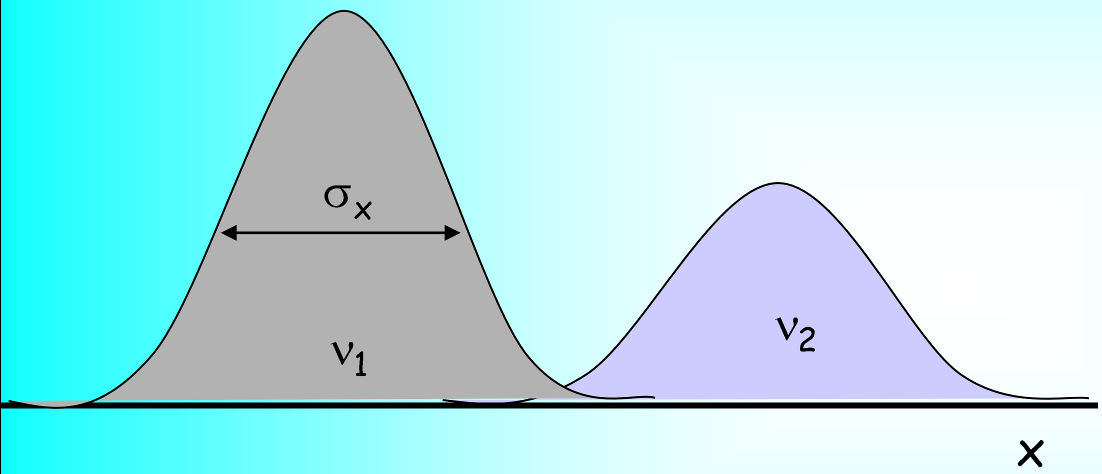
In the configuration space: separation of the wave packets due to difference of group velocities



$\Delta v_{gr} = \Delta m^2 / 2E^2$

separation: $\Delta v_{gr} L = \Delta m^2 L/2E^2$

no overlap: $\Delta v_{gr} L > \sigma_x$

coherence length:

$L_{coh} = \sigma_x E^2/\Delta m^2$

In the energy space: averaging over oscillations

Oscillatory period in the energy space $E^T = 4\pi E^2/(\Delta m^2 L)$

Averaging (loss of coherence) if energy resolution σ_E is $E^T < \sigma_E$

→ leads to the same coherence length

If $E^T > \sigma_E$ - restoration of coherence even if the wave packets separated

Equivalence of considerations in p- and x- spaces

**

on blackboard…

L Stodolsky theorem

or in stationary state approximation

if there is no time tagging

Wave packets are unnecessary for computation of observable effects

The only what is needed to make correct integration over

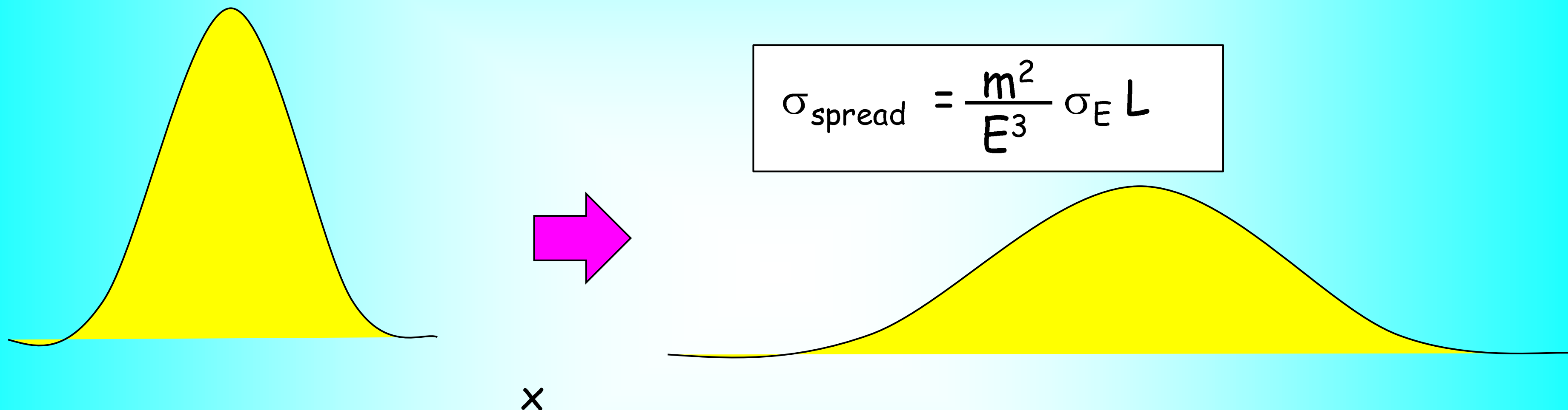
Production energy spectrum

Energy resolution of a detector

Spread of wave packets **

on blackboard…

$$\sigma_{spread} = \frac{m^2}{E^3}\,\sigma_E\,L$$



Loss of coherence between different parts of the WP

Becomes classical describing that the highest energy neutrinos arrive first

No effect if considered in the p-space

Master equation

If loss of coherence and other complications related to WP picture are irrelevant – `` point-like" picture

$$i \frac{d\Psi}{dt} = H\Psi$$

$$\Psi = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix}$$

$$H = \frac{M M^+}{2E} + V(t)$$

generalization of

$$E \sim p + \frac{m^2}{2E}$$

V - matter effects

M is the mass matrix

$V = \mathrm{diag}\,(V_e, 0, 0)$ – effective potential

Mixing matrix in vacuum

$$M M^+ = U M_{diag}^2 U^+$$

$$M_{diag}^2 = \mathrm{diag}\,(m_1^2, m_2^2, m_3^2)$$

Neutrino polarization vectors

$\psi = \begin{pmatrix} \nu_e \\ \nu_\tau, \end{pmatrix}$ ➡ Polarization vector:

$$\mathbf{P} = \psi^+ \sigma/2\, \psi \qquad (*)$$

$$\mathbf{P} = \begin{pmatrix} \mathrm{Re}\, \nu_e^+ \nu_\tau, \\ \mathrm{Im}\, \nu_e^+ \nu_\tau, \\ \nu_e^+ \nu_e - 1/2 \end{pmatrix}$$

Evolution equation:

$$i \frac{d\Psi}{dt} = H \Psi \quad \Rightarrow \quad i \frac{d\Psi}{dt} = (\mathbf{B}\, \sigma)\, \Psi$$

where $\mathbf{B} = \frac{2\pi}{l_m} (\sin 2\theta_m,\ 0,\ \cos 2\theta_m)$

Differentiating **P** from (*) and using equation of motion for Ψ

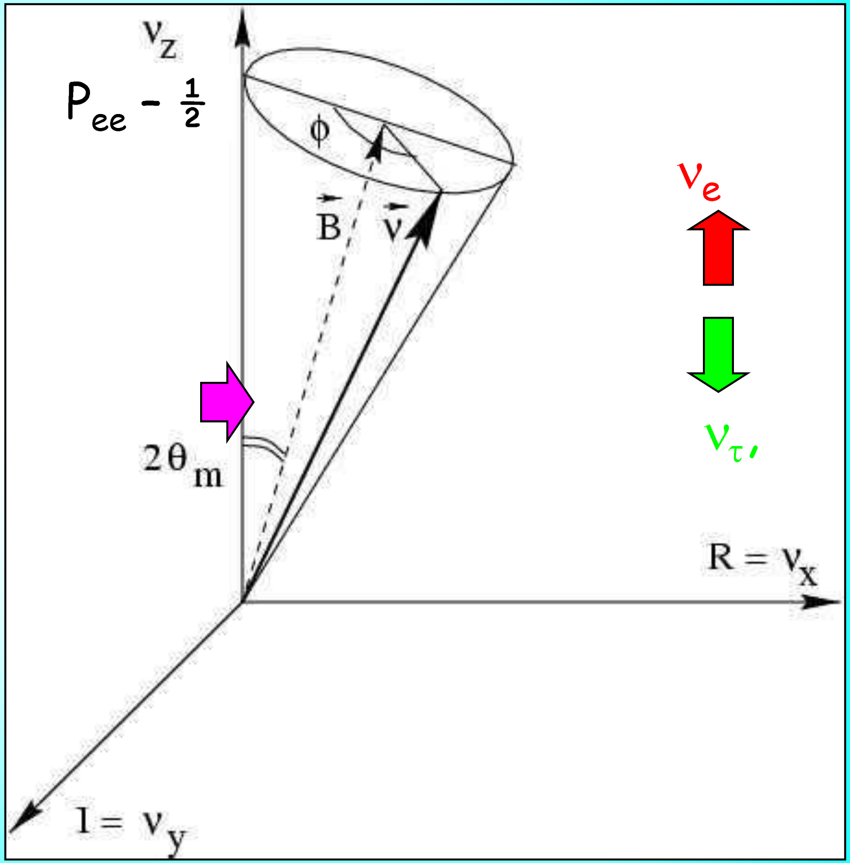
$$\frac{d\mathbf{P}}{dt} = (\mathbf{B} \times \mathbf{P})$$

Coincides with equation for the electron
spin precession in the magnetic field

Graphical representation

$\vec{\nu} = \mathbf{P} =$
$(\mathrm{Re}\ \nu_e^+ \nu_\tau, \quad \mathrm{Im}\ \nu_e^+ \nu_\tau, \quad \nu_e^+ \nu_e - 1/2)$

$$\mathbf{B} = \frac{2\pi}{l_m}(\sin 2\theta_m,\ 0,\ \cos 2\theta_m)$$



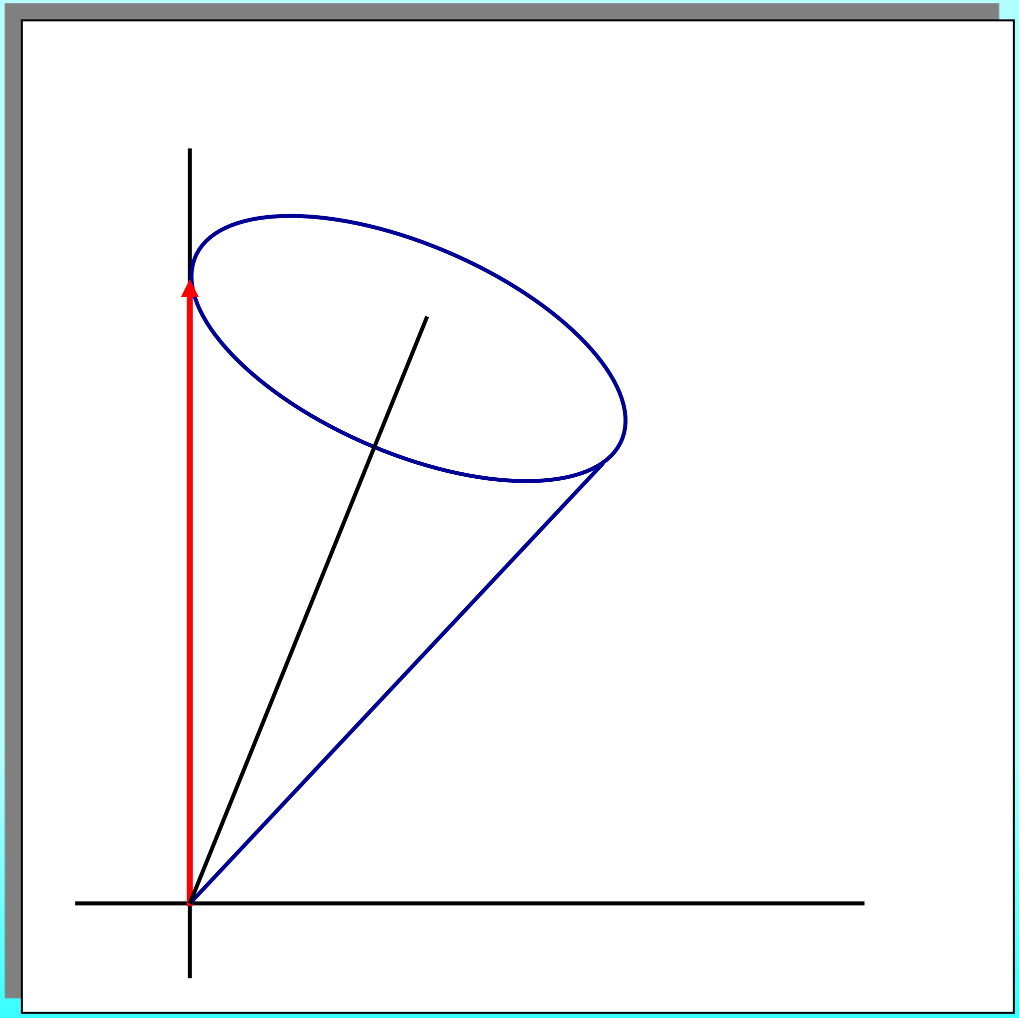
Evolution equation

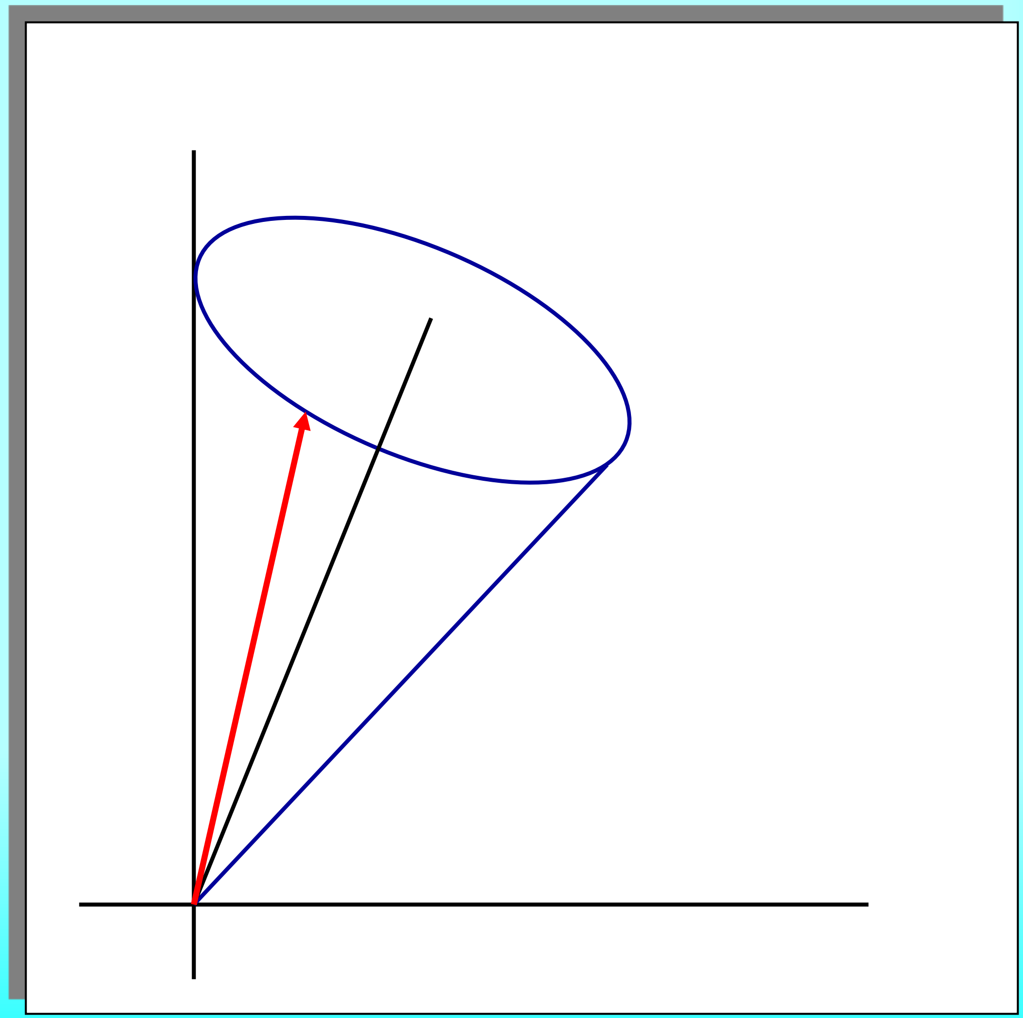
$$\frac{d\vec{\nu}}{dt} = (\vec{B} \times \vec{\nu})$$

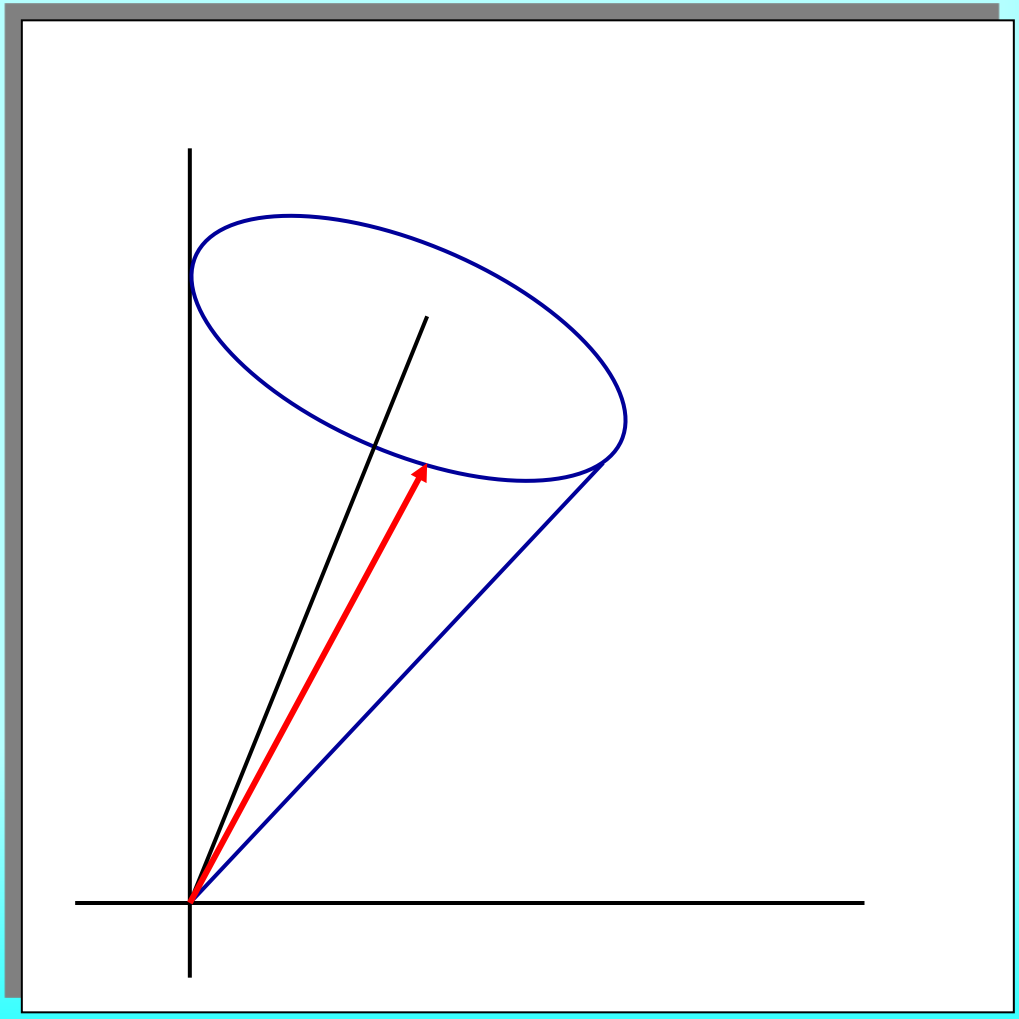
$\phi = 2\pi t / l_m$ - phase of oscillations

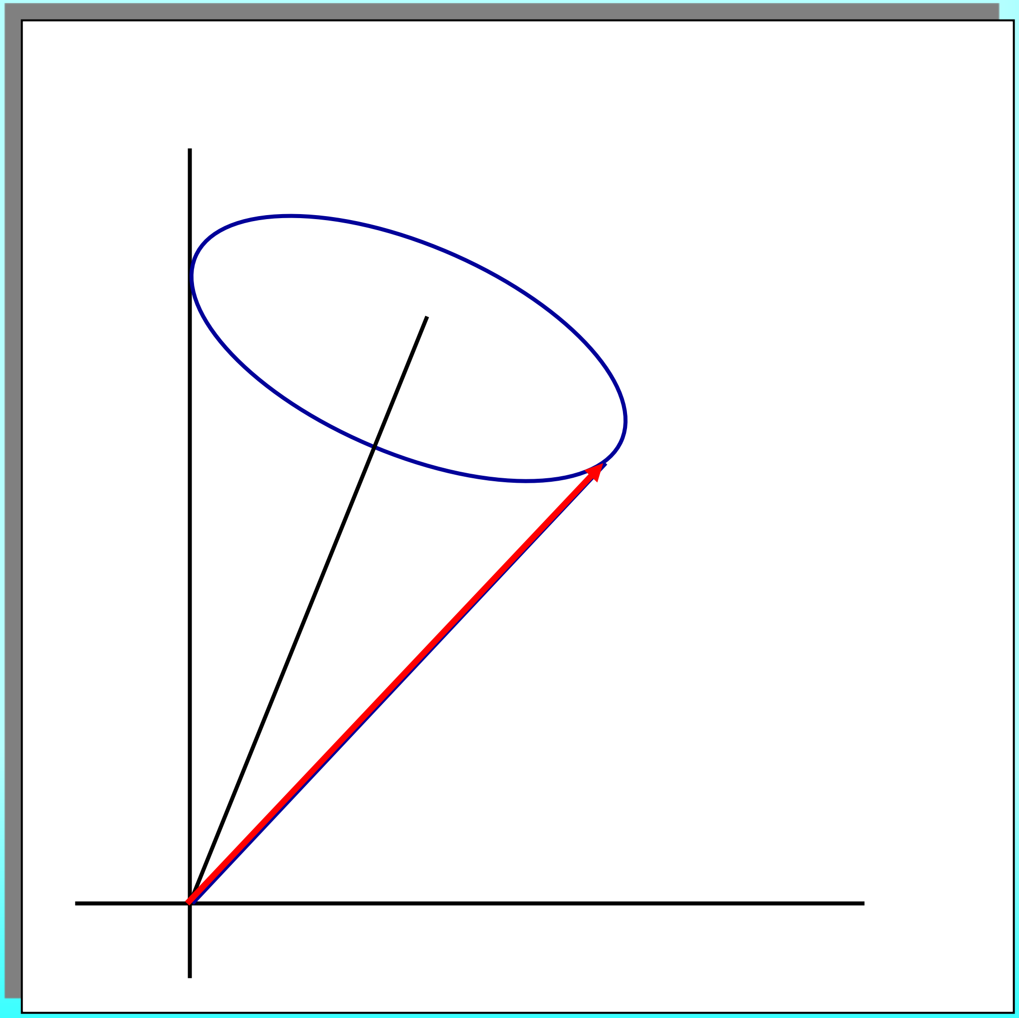
$P_{ee} = \nu_e^+ \nu_e = \nu_Z + 1/2 = \cos^2\theta_Z/2$ probability to find ν_e

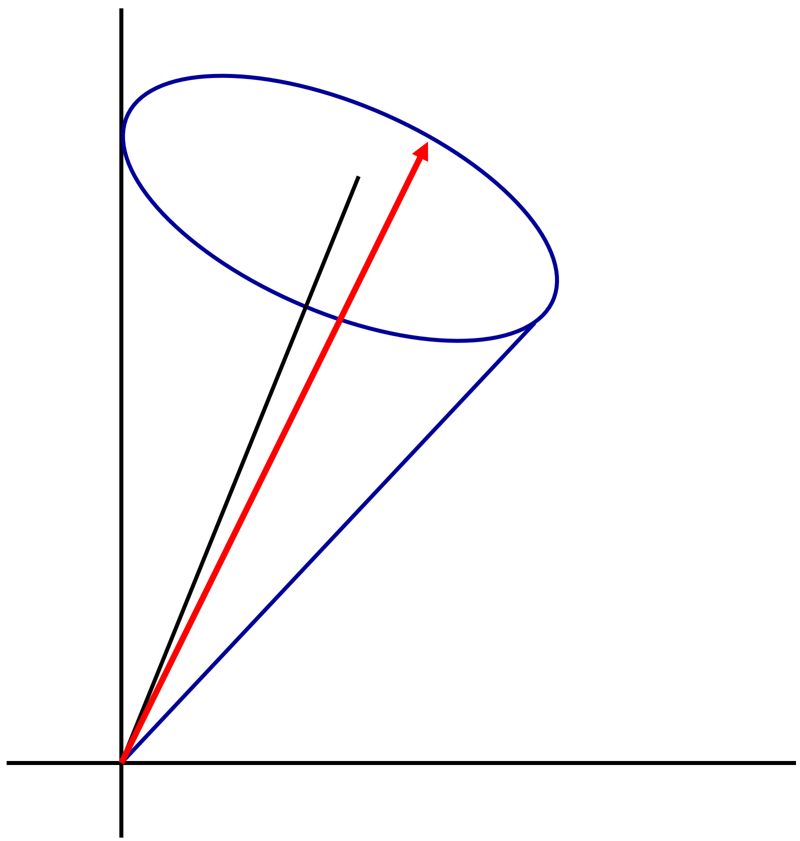
Oscillations

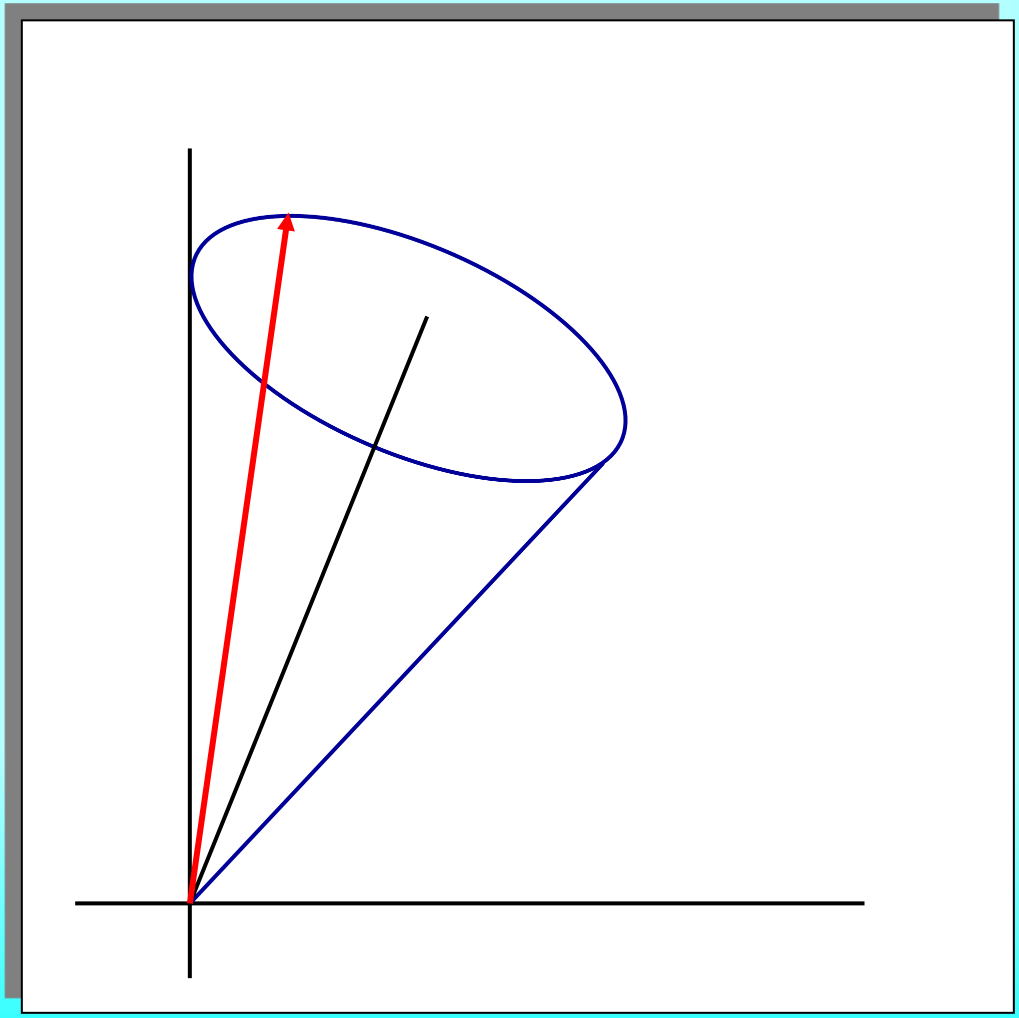


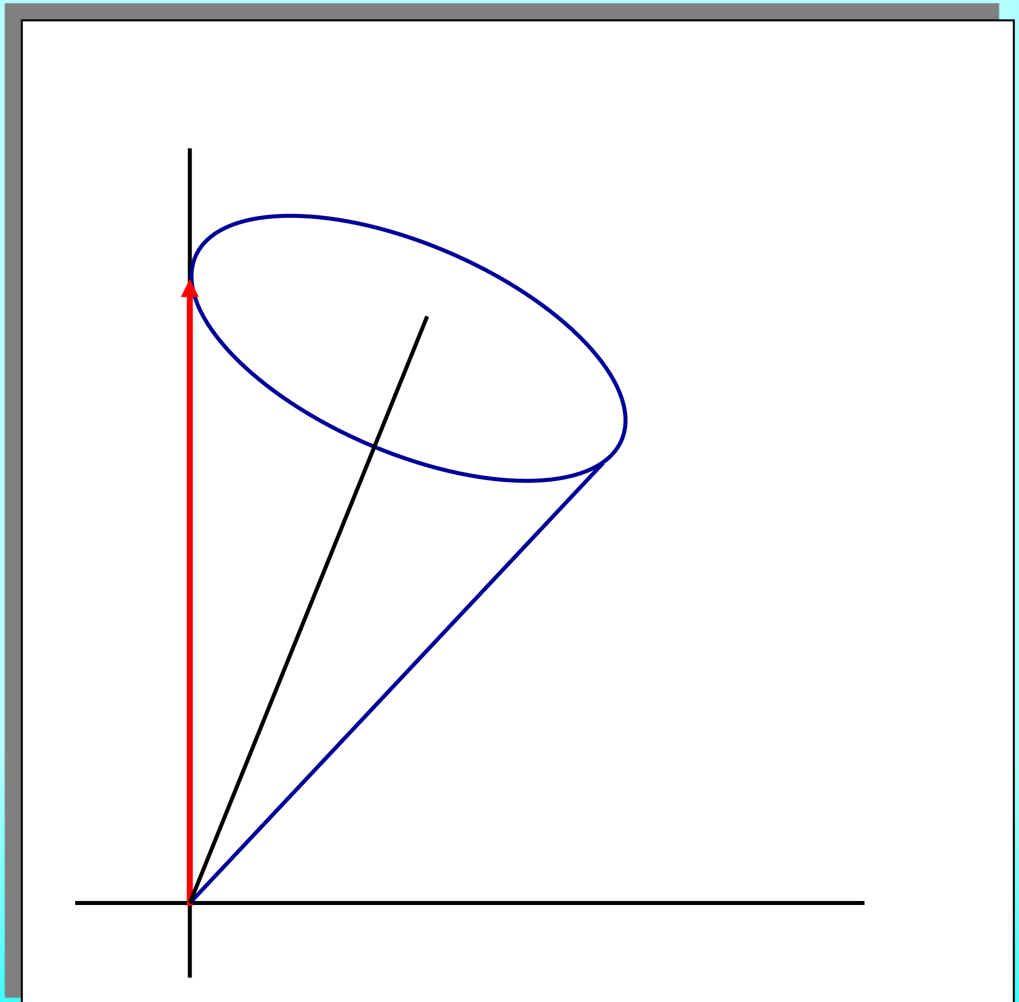




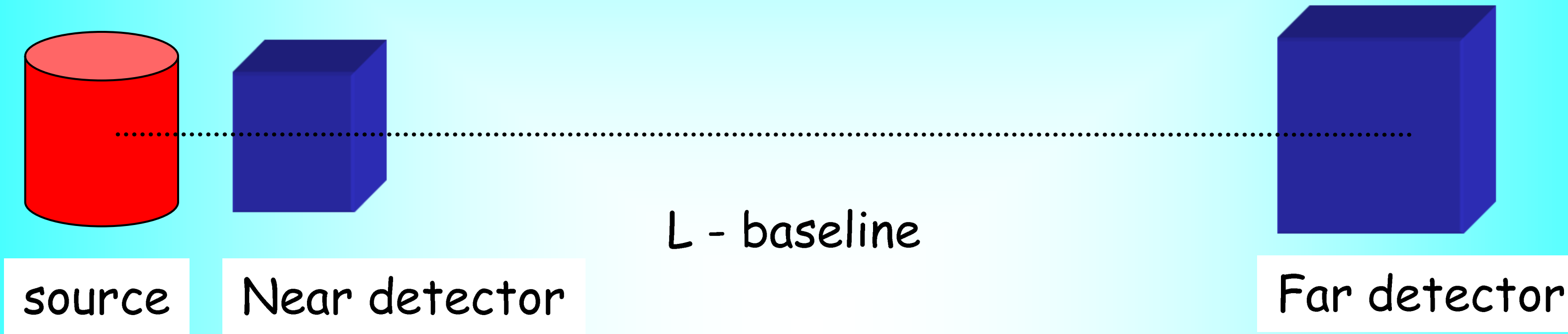








Experimental set-up



ν_α

ν_α disappearance

ν_β appearance

$$\phi = \frac{\Delta m^2 x}{2E}$$

Oscillation probability – periodic function of

- distance L and
- inverse energy 1/E

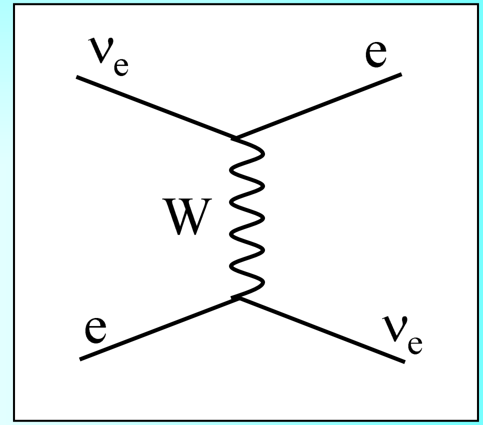
Matter effects:
Oscillations & flavor conversion

Matter potential

L. Wolfenstein, 1978

for ν_e ν_μ

at low energies Re A >> Im A
inelastic interactions can be neglected



Elastic forward scattering ⇒ V_e, V_μ

potentials

Refraction index:

$$n - 1 = V / p$$

for E = 10 MeV

$$n - 1 = \begin{cases} \sim 10^{-20} & \text{inside the Earth} \\ < 10^{-18} & \text{inside the Sun} \end{cases}$$

difference of potentials

$$V = V_e - V_\mu = \sqrt{2}\, G_F\, n_e$$

$V \sim 10^{-13}$ eV inside the Earth

Refraction length:

$$l_0 = \frac{2\pi}{V}$$

Matter potential

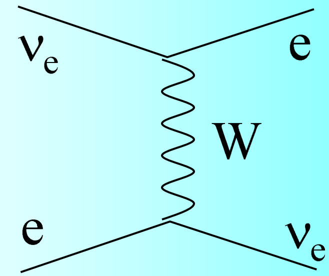
derivation

At low energies: neglect the inelastic scattering and absorption effect is reduced to the elastic forward scattering (refraction) described by the potential V:

$$H_{int}(\nu) = \langle \psi \,|\, H_{int} \,|\, \psi \rangle = V\, \bar{\nu}\nu$$

ψ is the wave function of the medium

CC interactions with electrons



$$H_{int} = \frac{G_F}{\sqrt{2}}\, \bar{\nu}\gamma^{\mu}(1-\gamma_5)\nu\; \bar{e}\gamma_{\mu}(1-\gamma_5)e$$

$\langle \bar{e}\gamma_0(1-\gamma_5)e\rangle = n_e$ - the electron number density

$\langle \bar{e}\vec{\gamma}e\rangle = n_e\vec{v}$

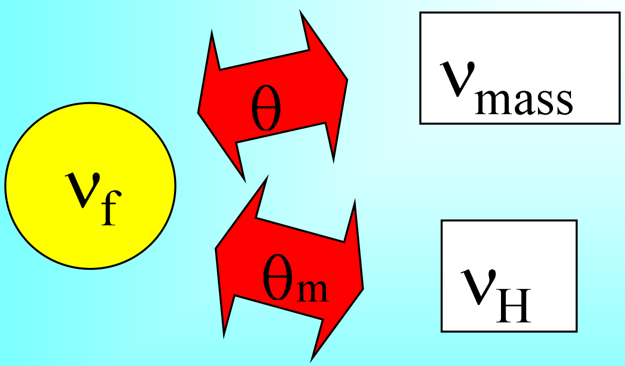
$\langle \bar{e}\vec{\gamma}\gamma_5 e\rangle = n_e\vec{\lambda}_e$ - averaged polarization vector of e

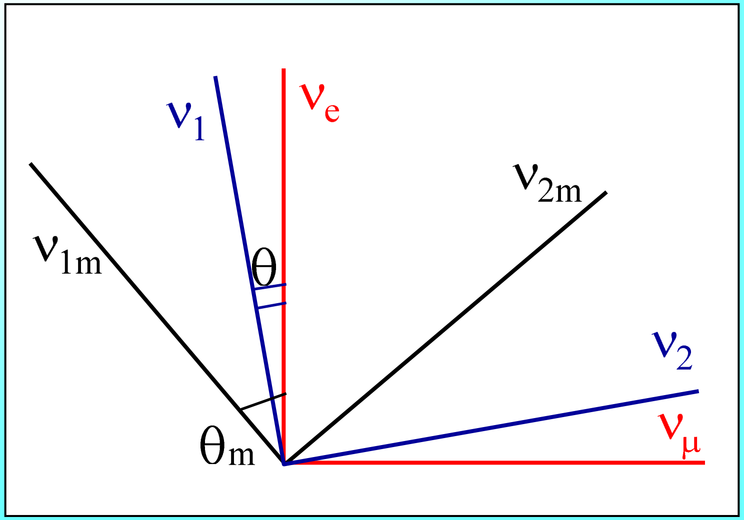
For unpolarized medium at rest:

$$V = \sqrt{2}\, G_F n_e$$

Mixing in matter

	in vacuum:		in matter:	
Effective Hamiltonian	H_0	⇒	$H = H_0 + V$	
Eigenstates	ν_1, ν_2	⇒	ν_{1m}, ν_{2m}	depend on n_e, E
Eigenvalues	$m_1^2/2E,\ m_2^2/2E$	⇒	H_{1m}, H_{2m}	instantaneous





Mixing angle determines flavors (flavor composition) of eigenstates of propagation

Evolution equation

$$i \frac{d\nu_f}{dt} = H_{tot}\, \nu_f \qquad \nu_f = \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}$$

$H_{tot} = H_{vac} + V$ is the total Hamiltonian

$H_{vac} = \frac{M^2}{2E}$ is the vacuum (kinetic) part

$V = \begin{pmatrix} V_e & 0 \\ 0 & 0 \end{pmatrix}$ matter part $\qquad V_e = \sqrt{2}\, G_F n_e$

H_{tot}

$$i \frac{d}{dt} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} -\frac{\Delta m^2}{2E}\cos 2\theta + V_e & \frac{\Delta m^2}{4E}\sin 2\theta \\ \frac{\Delta m^2}{4E}\sin 2\theta & 0 \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}$$

Mixing in matter

Diagonalization of the Hamiltonian:

$$\sin^2 2\theta_m = \frac{\sin^2 2\theta}{(\cos 2\theta - 2EV/\Delta m^2)^2 + \sin^2 2\theta}$$

$$V = \sqrt{2}\, G_F\, n_e$$

Mixing is maximal if

$$V = \frac{\Delta m^2}{2E} \cos 2\theta$$

Resonance condition

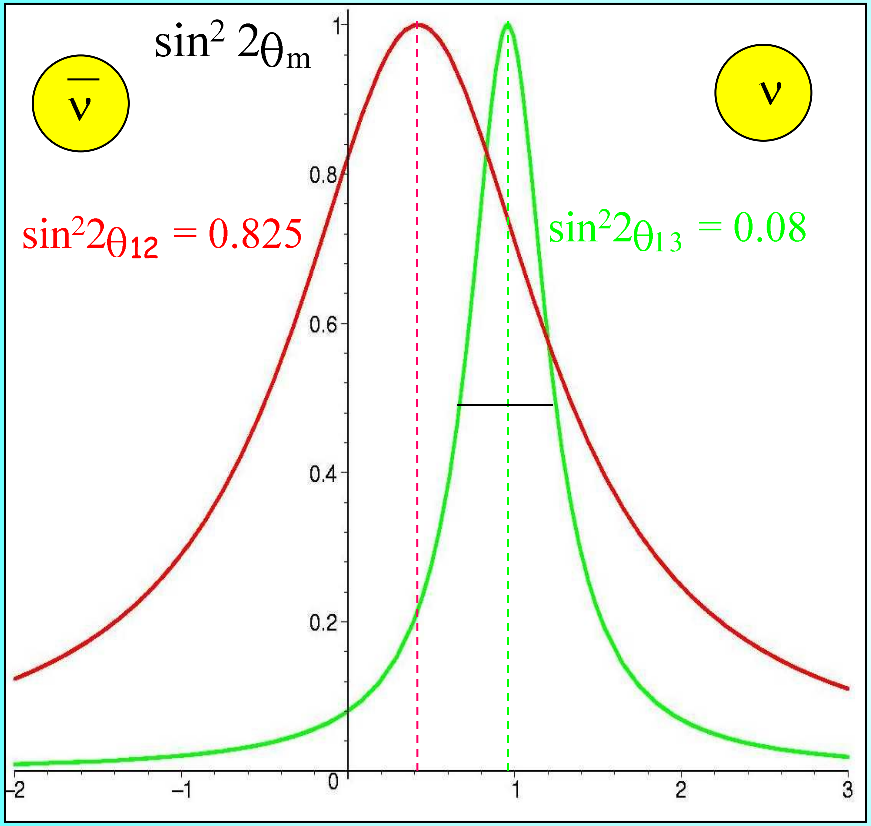
$$H_e = H_\mu$$

$$\sin^2 2\theta_m = 1$$

Difference of the eigenvalues

$$H_{2m} - H_{1m} = \frac{\Delta m^2}{2E} \sqrt{(\cos 2\theta - 2EV/\Delta m^2)^2 + \sin^2 2\theta}$$

Resonance



$l_\nu / l_0 \sim n E$

In resonance:

$\sin^2 2\theta_m = 1$

Flavor mixing is maximal

$l_\nu = l_0 \cos 2\theta$

Vacuum oscillation length $\approx$ Refraction length

Resonance width: $\Delta n_R = 2 n_R \tan 2\theta$

Resonance layer: $n = n_R +/- \Delta n_R$

Level crossing

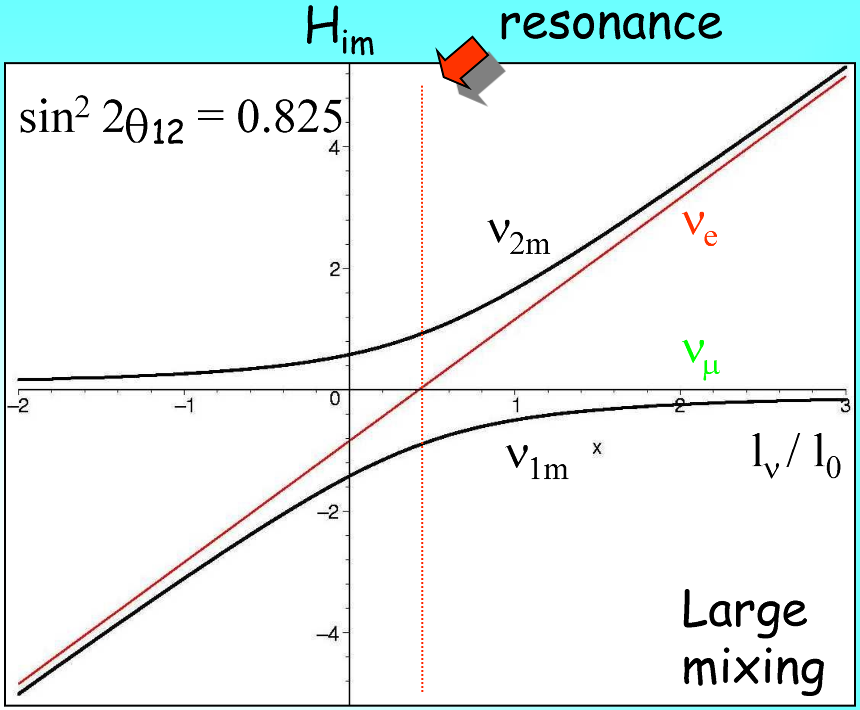
V. Rubakov, private comm.
N. Cabibbo, Savonlinna 1985
H. Bethe, PRL 57 (1986) 1271

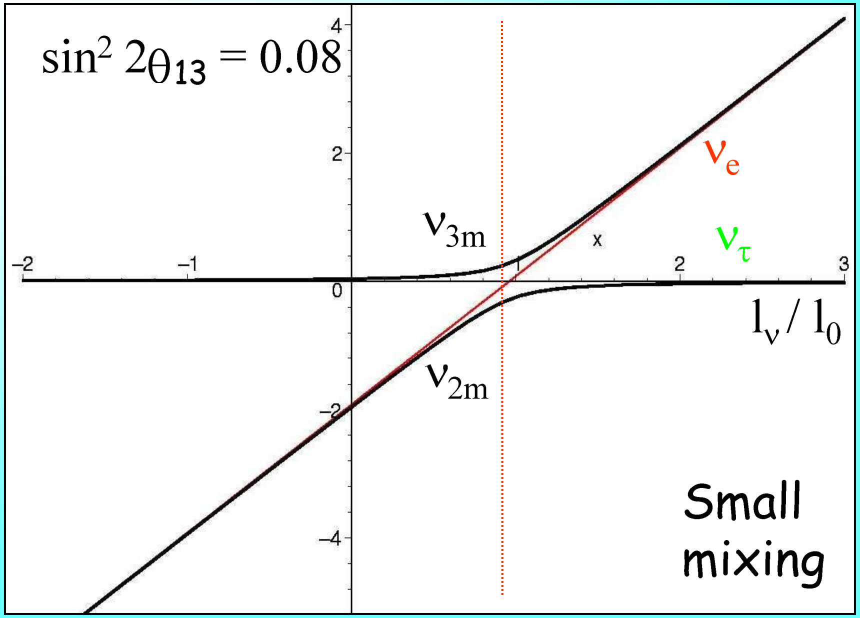
Dependence of the neutrino eigenvalues on the matter potential (density):

$$\frac{l_\nu}{l_0} = \frac{2E\,V}{\Delta m^2}$$

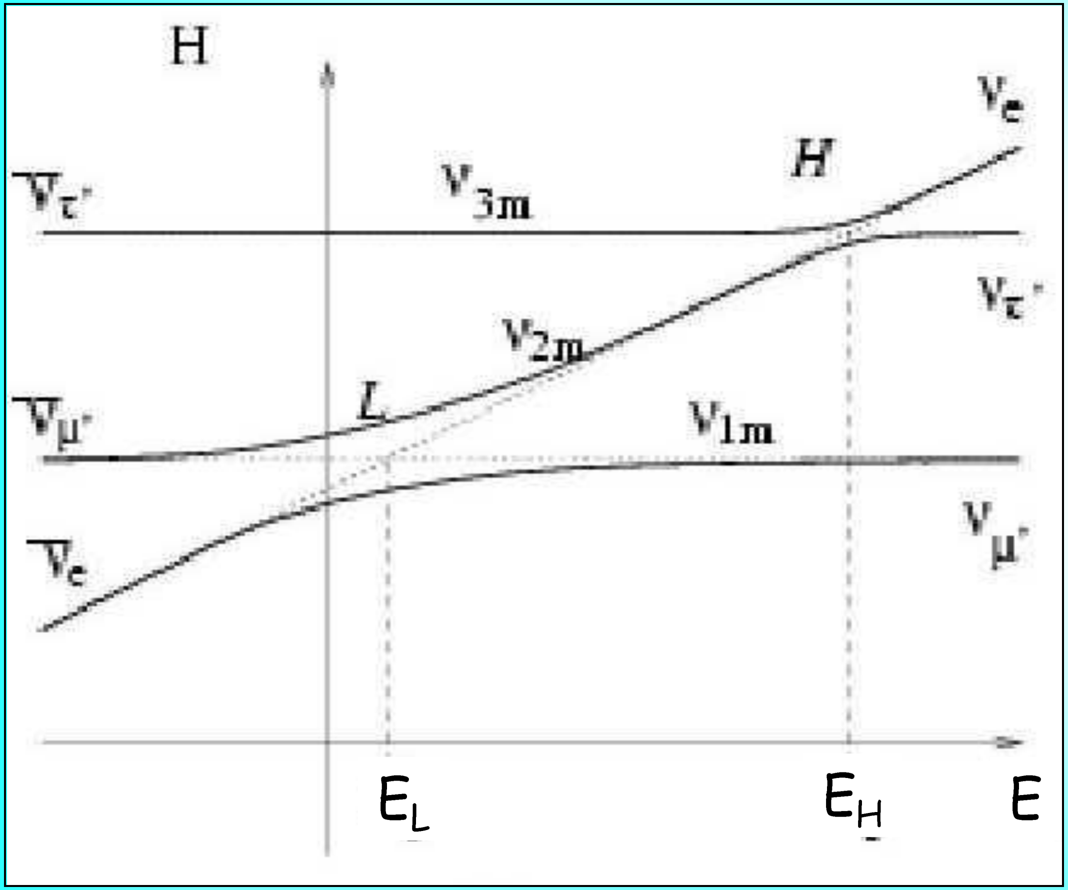
$$\frac{l_\nu}{l_0} = \cos 2\theta$$

Crossing point - resonance
- the level split is minimal
- the oscillation length is maximal





Level crossings



Normal mass hierarchy

0.1 GeV

6 GeV

Resonance region

High energy range

Oscillations in matter

Oscillation probability constant density

$$P(\nu_e \rightarrow \nu_a) = \sin^2 2\theta_m \sin^2\left(\frac{\pi L}{l_m}\right)$$

half-phase ϕ

Amplitude of oscillations

oscillatory factor

$\theta_m(E, n)$ - mixing angle in matter

$l_m(E, n)$ - oscillation length in matter

$$l_m = 2\pi/(H_{2m} - H_{1m})$$

In vacuum: $\theta_m \rightarrow \theta$, $l_m \rightarrow l_\nu$

Maximal effect:

$\sin^2 2\theta_m = 1$ ⇒ MSW resonance condition

$\phi = \pi/2 + \pi k$

Oscillation length in matter

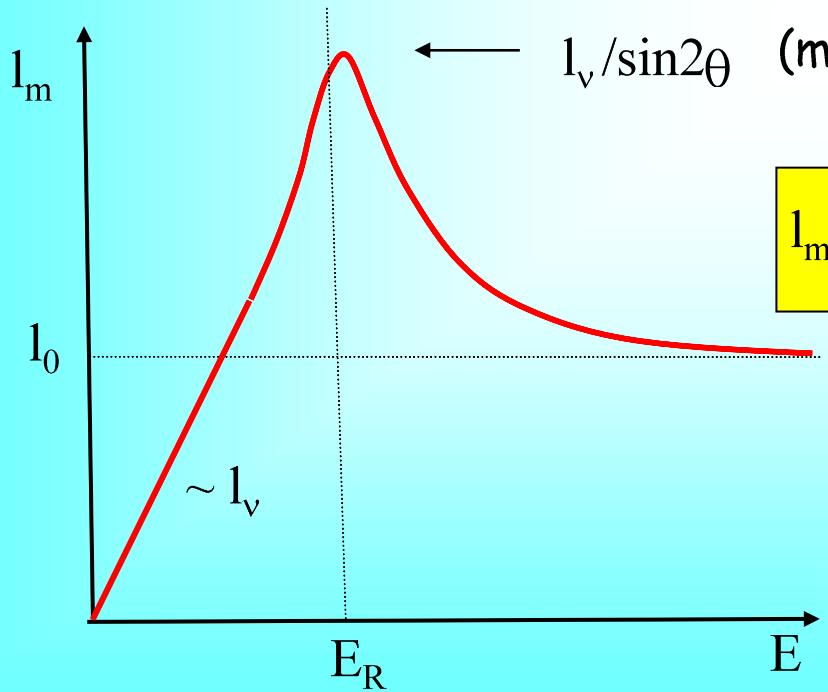
Oscillation length in vacuum

$$l_\nu = \frac{4\pi E}{\Delta m^2}$$

Refraction length

$$l_0 = \frac{2\pi}{\sqrt{2}\, G_F n_e}$$

- determines the phase produced by interaction with matter



$l_\nu / \sin 2\theta$ (maximum at $l_\nu = l_0 / \cos 2\theta$)

$$l_m = \frac{2\pi}{H_{2m} - H_{1m}}$$

shifts with respect resonance energy:

$$l_\nu (E_R) = l_0 \cos 2\theta$$

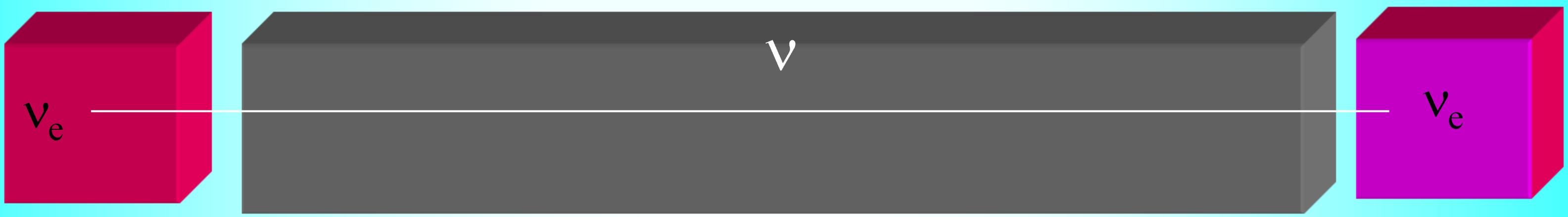
converges to the refraction length

Resonance enhancement of oscillations

Constant density

Resonance enhancement

Constant density



Source

$F_0(E)$

layer of length L

Detector

F(E)

Depth of oscillations determined by $\sin^2 2\theta_m$
the oscillation length, l_m
depends on neutrino energy

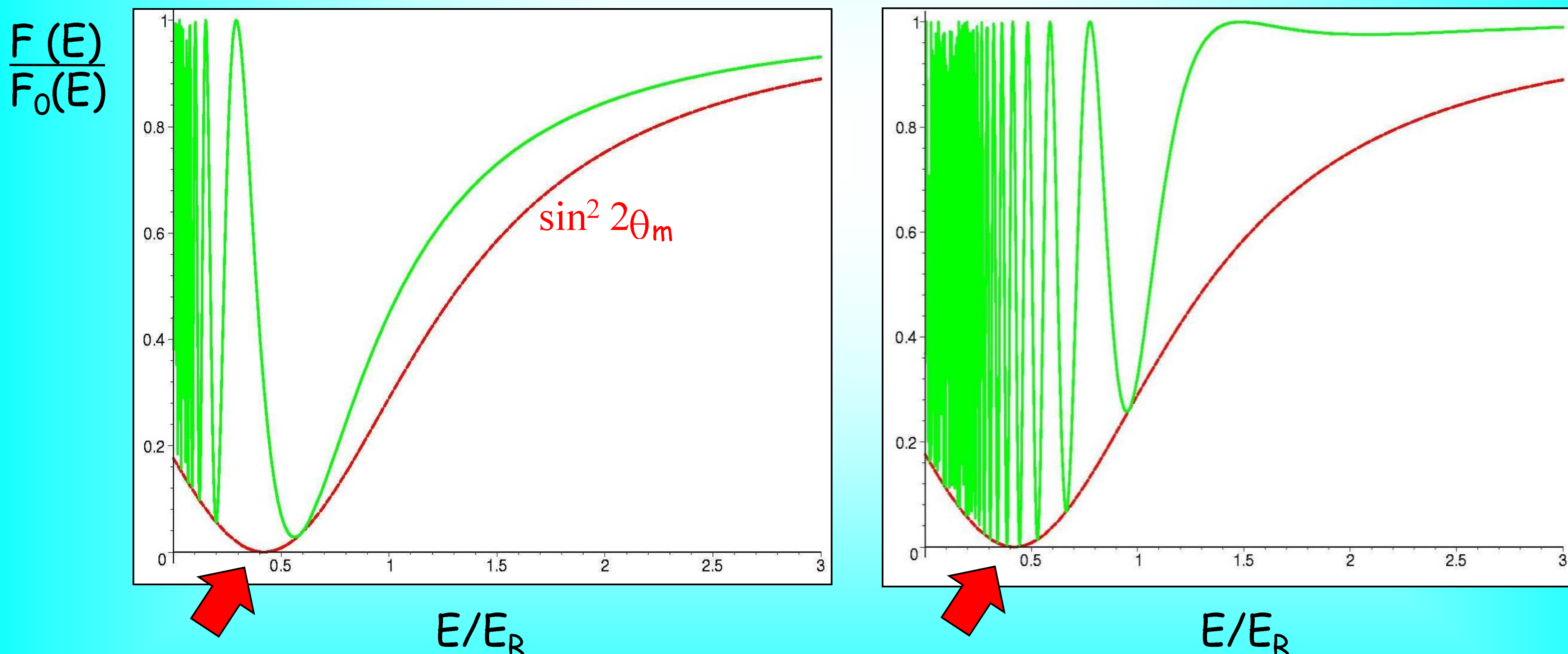
For neutrinos propagating
in the mantle of the Earth

Large mixing $\sin^2 2\theta = 0.824$

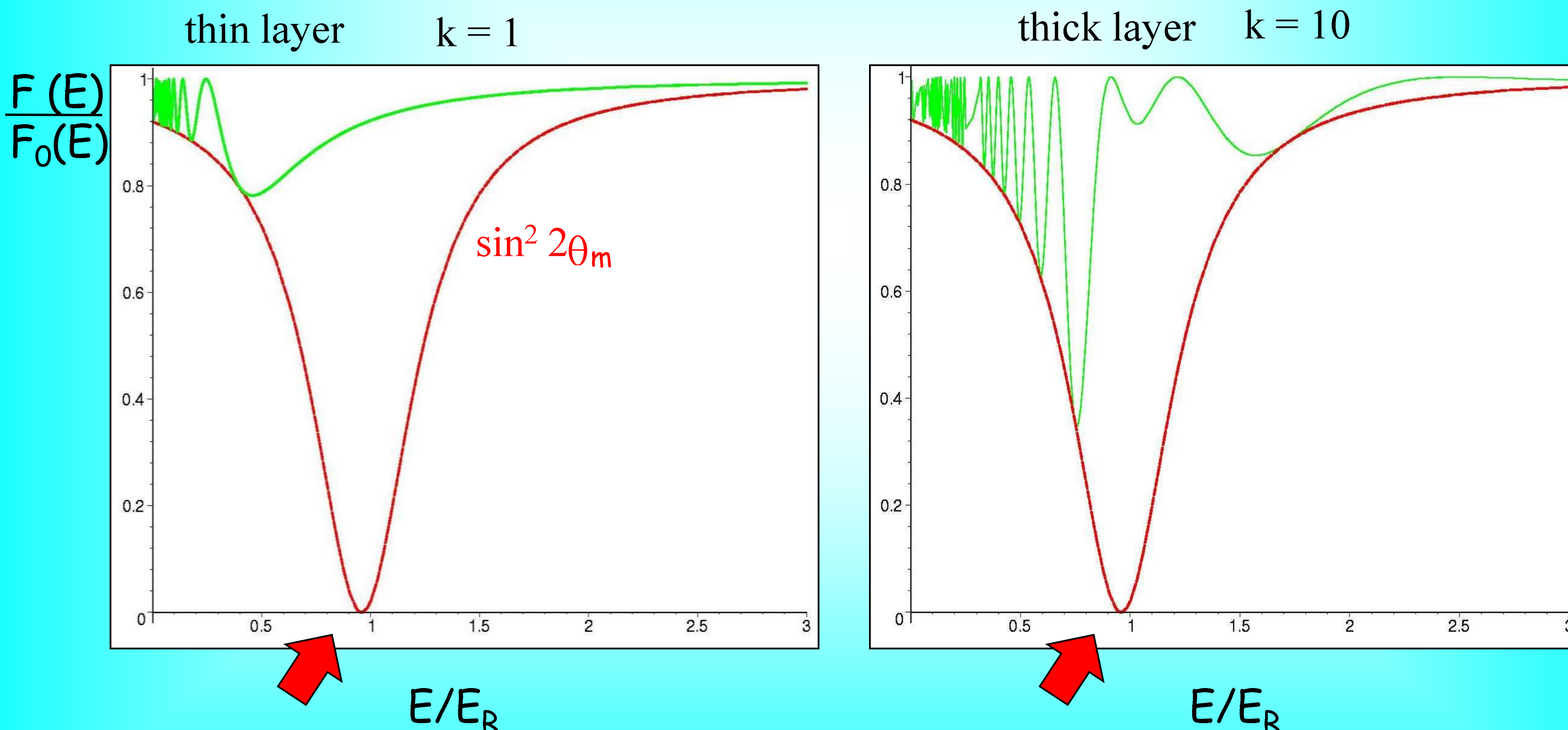
Layer of length L $k = \pi L / l_0$

ν

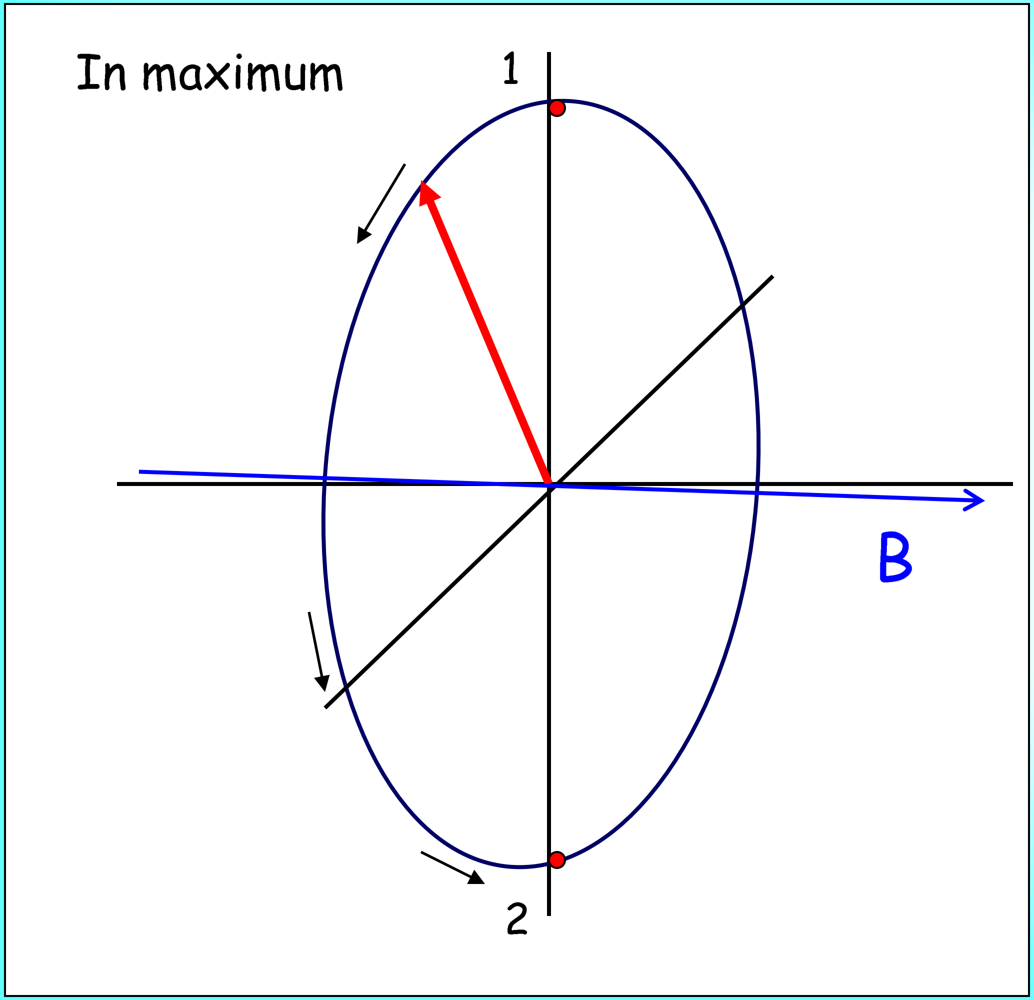
thin layer $k = 1$ thick layer $k = 10$



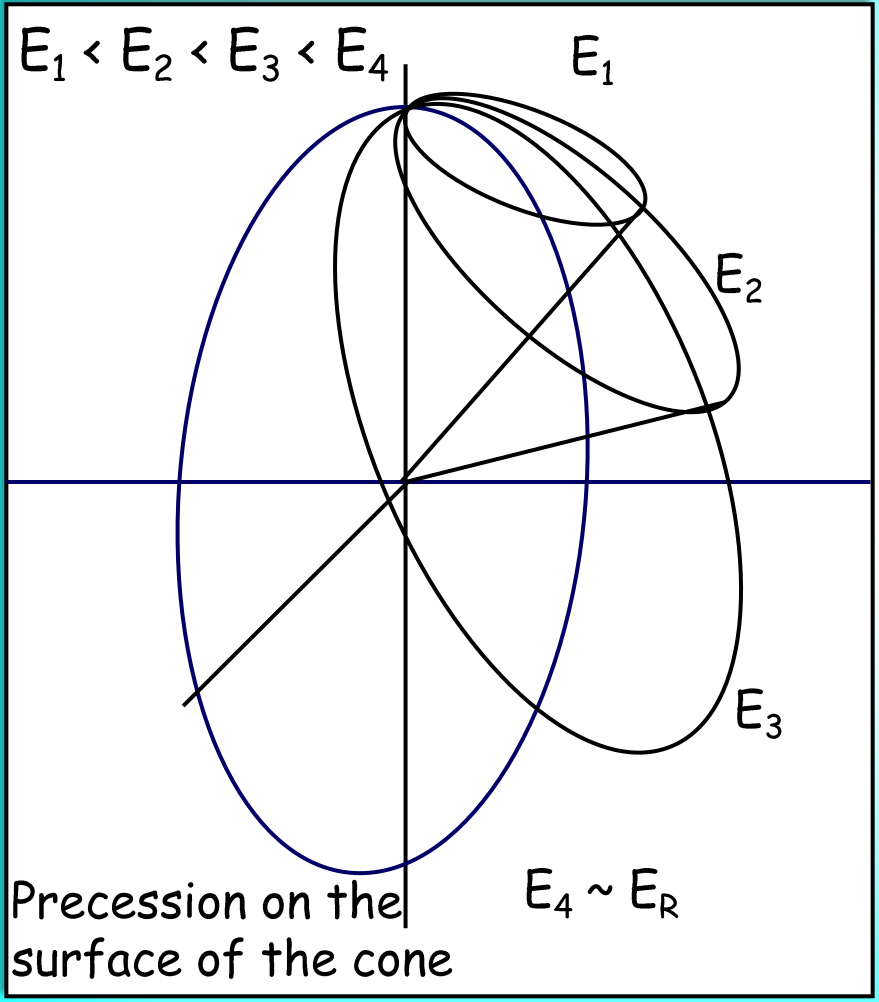
Small mixing $\sin^2 2\theta = 0.08$



Resonance enhancement

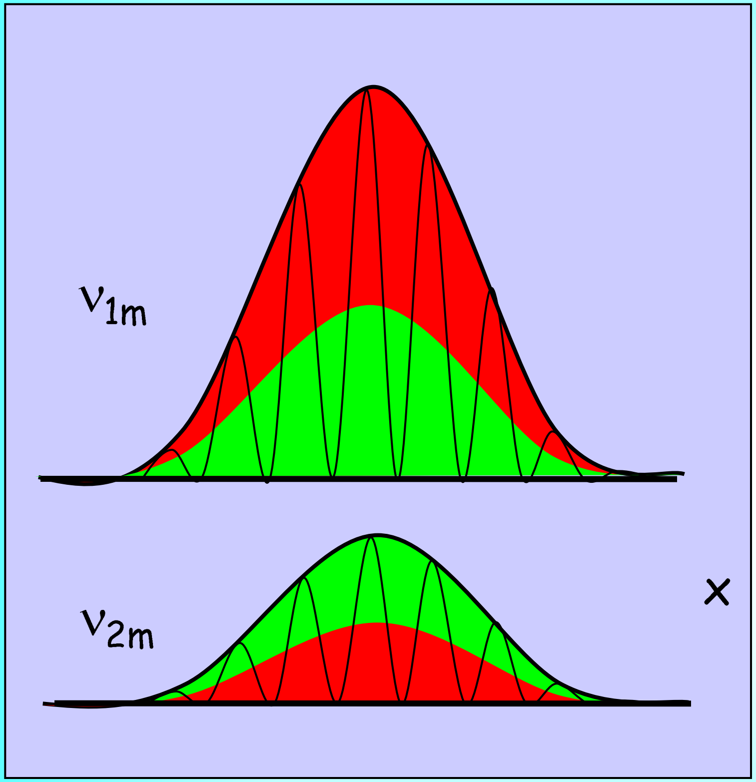


Graphic representation



Resonance enhancement

Oscillations in matter



$H_0 \rightarrow H = H_0 + V$

$\nu_k \rightarrow \nu_{mk}$

eigenstates of H_0 — eigenstates of H

$\theta \rightarrow \theta_m (n)$

Resonance - maximal mixing in matter - oscillations with maximal depth

$\theta_m = \pi/4$

Resonance condition:

$$V = \cos 2\theta \frac{\Delta m^2}{2E}$$

Constant density medium: the same dynamics

Mixing changed
phase difference changed