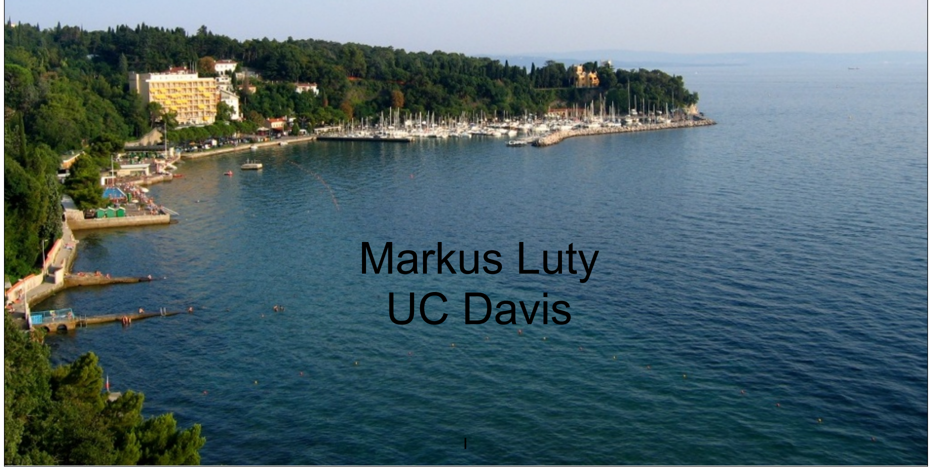
ICTP Lectures on Supersymmetry



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Topics

- The hierarchy problem
- The supersymmetry algebra
- Superspace
- The minimal supersymmetric standard model (MSSM)
- Soft supersymmetry breaking
- Experimental searches for supersymmetry

The Hierarchy Problem

Effective Field Theory

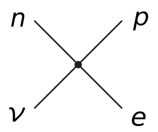
Effective theory = approximate description of physics valid in a limited dynamical range.

Example: the Navier-Stokes equations describe fluids on length scales large compared to atomic distances

Is the Standard Model an effective field theory? If so, at what scale does it break down?

Until very recently, the theoretical description of weak interactions required new physics at the TeV scale:

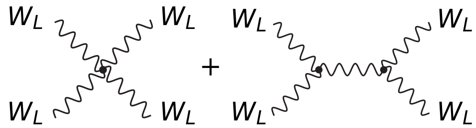
1930s: Fermi theory



$\sim G_F E^2$

$\Rightarrow$ breaks down for $E \gtrsim G_F^{-1/2} \sim$ TeV

1980s: W, Z bosons, no Higgs



$$\sim \frac{g^2 E^2}{m_W^2} \sim G_F E^2$$

2012: Standard Model with Higgs

Can be consistently extrapolated all the way to the Planck scale. *No guarantee of new physics!*

The Standard Model many important phenomena unexplained $\Rightarrow$ new physics beyond the Standard Model.

Experimental facts:

- Neutrino masses
- Dark matter
- Cosmological density perturbations
- Baryogenesis

Theoretically motivated:

- Grand unification
- Origin of fermion masses and mixing
- Naturalness of the electroweak scale

Only naturalness requires new physics at the TeV scale.

Consider a coupling constant λ with mass dimenson n

$$[\lambda] = n \qquad \lambda = M^n \qquad M = \text{mass scale}$$

Treat λ as a perturbation:

$$\mathcal{A}(E) \sim \underbrace{\mathcal{A}_0(E)}_{= O(\lambda^0)} \left[1 + \underbrace{\left(\frac{M}{E}\right)^n}_{= O(\lambda^1)} + \cdots \right] \qquad E = \text{physical energy scale}$$

$n > 0$: perturbation theory breaks down at small E
relevant coupling

$n < 0$: perturbation theory breaks down at large E
irrelevant coupling

$n = 0$: perturbation theory good at all E*
marginal coupling

* ln E dependence at loop level

There are an infinite number of irrelevant couplings:

$$[\phi] = [A_\mu] = 1, \qquad [\psi] = \tfrac{3}{2}$$

$$\Delta\mathcal{L} = \frac{1}{M^{64}} (\bar{\psi}\psi)^{12} \Box^9 \phi^{14} + \cdots$$

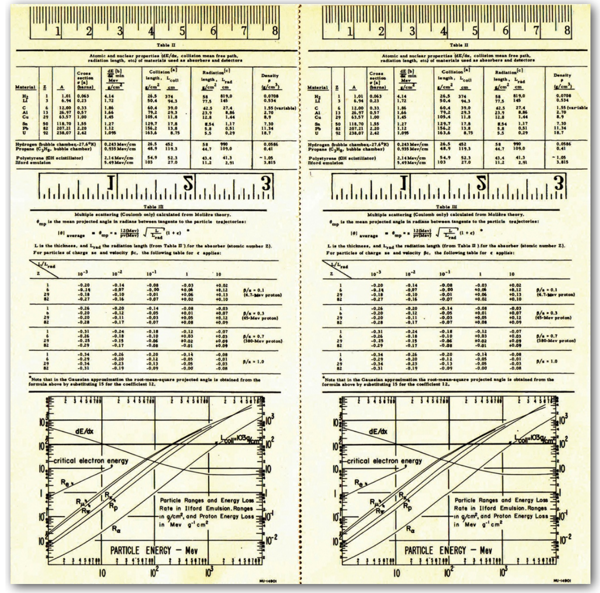
Assume $M \gg$ TeV $\Rightarrow$ effects of irrelevant operators suppressed at low energies.

This naturally occurs if these operators are generated by integrating out new physics (particles) with mass scale $M \gg$ TeV.

Effective theory at low energies parameterized by a finite number of marginal and relevant couplings. [K. Wilson]

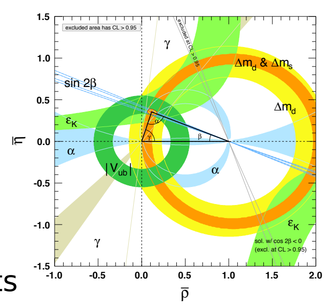
The Standard Model is the most general effective Lagrangian containing all relevant and marginal couplings of the experimentally observed elementary particles compatible with Lorentz symmetry and $\underbrace{\text{gauge invariance}}_{\text{needed for spin 1}}$.

	$SU(3)_C$	$SU(2)_W$	$U(1)_Y$	
Q_L	3	2	$\frac{1}{6}$	$\times 3$
u_R	3	1	$\frac{2}{3}$	
d_R	3	1	$-\frac{1}{3}$	
L_L	1	2	$-\frac{1}{2}$	
e_R	1	1	-1	
H	1	2	$\frac{1}{2}$	



First Particle Data Group wallet card (1958)

This effective field theory has an amazing amount of predictive power, and agrees with all experiments performed to date.



- Weak decays
- Quark mixing, *CP* violation
- No flavor-changing neutral currents
- Baryon and lepton number symmetry

Is the standard model the perfect effective field theory?

The Standard Model contains one relevant coupling:

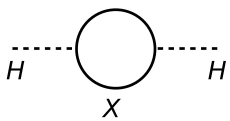
$$\mathcal{L}_{\rm SM} = -m_H^2 H^\dagger H + \cdots \qquad\qquad m_H^2 < 0$$

$$m_h^2 = -2m_H^2 = \text{physical Higgs mass}$$

Dimensional analysis suggests that $m_H^2 \sim M^2 \gg \text{TeV}$.

Is this a problem?

X = any particle with mass $M_X \gg$ TeV



$$\Rightarrow \; \Delta m_H^2 \sim \underbrace{\frac{y^2}{16\pi^2} M_X^2}$$

finite large correction
to Higgs mass

Expect new physics at scales $M \gg$ TeV:

$M_{\nu_R} \sim 10^{14}$ GeV

$M_{\rm GUT} \sim 10^{16}$ GeV

$M_{\rm Pl} \sim 10^{19}$ GeV

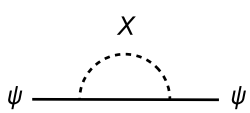
$m_H^2 \simeq -(88\ {\rm GeV})^2$ requires large unexplained cancellation
"hierarchy problem"

Possible explanations:

- Δm_H^2 forbidden by symmetry
- Higgs compositeness
- Quantum gravity at TeV scale
  (large extra dimensions)
- Antrhopic selection
- Relaxation models
  Graham, Kaplan, Rajendran arXiv:1504.07551

⋮
?

Small fermion masses are natural because there is an additional chiral symmetry as $m_\psi \to 0$:



$$\Rightarrow \; \Delta m_\psi \sim \underbrace{\frac{y^2}{16\pi^2} m_\psi}_{\lesssim\, m_\psi}$$

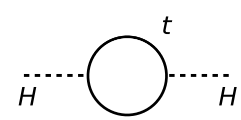
But scalar mass term $H^\dagger H$ is invariant under all symmetries
. . . except SUSY!

$H \leftrightarrow \tilde{H}$ = Higgsino
= fermion partner of the Higgs

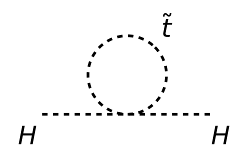
SUSY $\Rightarrow m_H = m_{\tilde{H}}$
Chiral symmetry $\Rightarrow m_{\tilde{H}} = 0$

$\Rightarrow m_H^2$ insensitive to UV scales

Bose-Fermi symmetry not observed in nature $\Rightarrow$ SUSY broken
Nontrivial cancelations among diagrams:



$$\Delta m_H^2 = \frac{3y_t^2}{8\pi^2}\left[-\Lambda^2 + 6m_t^2 \ln\Lambda + \cdots\right]$$



$$\Delta m_H^2 = \frac{3y_t^2}{8\pi^2}\left[\Lambda^2 - 6m_{\tilde{t}}^2 \ln\Lambda + \cdots\right]$$

quadratic sensitivity to UV
scales cancels

$\tilde{t}$ = stop
= scalar partner of the top

$$\Delta m_H^2 = -\frac{3y_t^2}{8\pi^2} \times 6(m_{\tilde{t}}^2 - m_t^2)\ln\Lambda + \cdots$$

$m_{\tilde{t}} \lesssim$ TeV $\Rightarrow$ mild logarithmic sensitivity to UV scales.

$\longleftrightarrow$

Bose-Fermi Symmetry in Quantum Mechanics

The Supersymmetric Simple Harmonic Oscillator

The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction. — Sidney Coleman

Simplest example of supersymmetry in quantum mechanics:
Define in terms of creation/annihilation operators:

$\boxed{H_b = \hbar\omega_b b^\dagger b}$ (subtract 0-point energy)

b is for "boson"

$\boxed{[b, b^\dagger] = 1}$ $[b, b] = [b^\dagger, b^\dagger] = 1$

States:

$\boxed{b|0\rangle = 0}$ $\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle}$ $\Rightarrow \langle n|m\rangle = \delta_{nm}$

$\boxed{H_b|n\rangle = n(\hbar\omega_b)|n\rangle}$

Fermionic simple harmonic oscillator:

$\boxed{H_f = \hbar\omega_f f^\dagger f}$ $H_f^\dagger = H_f$

f is for "fermion"

$\boxed{\{f, f^\dagger\} = 1}$ $\{f, f\} = \{f^\dagger, f^\dagger\} = 0$

$\{A, B\} = AB + BA =$ anticommutator

States:

$\boxed{f|0\rangle = 0}$ $\boxed{|1\rangle = f^\dagger|0\rangle}$

$(f^\dagger)^2|0\rangle = 0 \Rightarrow$ 2-state system
(Pauli exclusion principle)

$\boxed{H_f|n\rangle = n(\hbar\omega_f)|n\rangle}$ $n = 0, 1$

Combine bosonic and fermionic oscillators:

$\boxed{H = H_b + H_f}$

$[b, f] = [b, f^\dagger] = [b^\dagger, f] = [b^\dagger, f^\dagger] = 0$

$b|0\rangle = f|0\rangle = 0$

$\boxed{|n, 0\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle}$ $\boxed{|n, 1\rangle = f^\dagger|n, 0\rangle}$

Label states: $|n_b, n_f\rangle$ $n_b = \#$ of bosons $= 0, 1, 2, \ldots$
$n_f = \#$ of fermions $= 0, 1$

For $\omega_b = \omega_f$, this system has Bose-Fermi symmetry

Spectrum of energy levels:

$$\boxed{H|n_b, n_f\rangle = (n_b + n_f)(\hbar\omega)|n_b, n_f\rangle} \qquad \omega = \omega_b = \omega_f$$

E

$\vdots$

3ω — $|3,0\rangle,\ |2,1\rangle$

2ω — $|2,0\rangle,\ |1,1\rangle$

ω — $|1,0\rangle,\ |0,1\rangle$

0 — $|0,0\rangle = \text{ground state}$
$= |0\rangle$

Degeneracies are the sign of a symmetry...

Generator of symmetry:

$$\boxed{Q = b^\dagger f + f^\dagger b}$$

$$Q|n_b, n_f\rangle = |n_b - 1, n_f + 1\rangle + |n_b + 1, n_f - 1\rangle$$

$$\boxed{[Q, H] = 0} \qquad \text{(definition of symmetry in QM)}$$

$$\boxed{Q^2 = b^\dagger b + f^\dagger f}$$
$$= (\hbar\omega)^{-1} H$$

Q is “square root” of H

Implies that zero point energy cancels due to symmetry:

$Q|0\rangle = 0$ (ground state is invariant)

$\Rightarrow\ H|0\rangle = 0$

Note: free field field theory = one harmonic oscillator for each $\vec{p}$.

Example:

$$\mathcal{L} = \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi + \tfrac{1}{2}\partial^\mu\phi_i\partial_\mu\phi_i - \tfrac{1}{2}m^2\phi_i\phi_i$$

$\Psi = \text{Dirac fermion}$

$\phi_i = \text{real scalar} \qquad i = 1, \ldots, 4$

Note: same mass for fermion, boson.

Gives a spectrum with Bose-Fermi degeneracy: for each $\vec{p}$ there are 4 fermionic and 4 bosonic states with energy $\sqrt{\vec{p}^2 + m^2}$.

In fact, this theory has non-minimal ($\mathcal{N} = 2$) supersymmetry. To get theory with minimal supersymmetry need minimal fermion: Weyl spinor.

Weyl Fermions

Note on conventions:

$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$

Spinor index notation is that of Dreiner, Haber, Martin, **Phys. Rep.** 464 (2010) (`arXiv:0812.1594`.) This should be consulted for additional details and results.

These conventions are used by a majority of researchers in SUSY phenomenology.

Conventions should be conventional.

—Markus Luty

Weyl fermions are the minimal spin $\frac{1}{2}$ field in 4D. They are the basic building blocks for all theories of fermions.

Start with Dirac representation:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

$$\Rightarrow \ \Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = SO(3,1) \text{ generators}$$

Defines Dirac spinor representation: under infinitesmal Lorentz transformations

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

$$\delta\Psi = -\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\Psi \qquad \Psi = \text{Dirac spinor}$$

Dirac representation is universal: exists for all spacetime dimensions, any metric signature.

Finite transformations:

$$\Psi \mapsto \underbrace{e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}}_{= S(\Lambda)}\Psi \qquad \bar{\Psi} = \Psi^\dagger\gamma^0 \mapsto \bar{\Psi}\underbrace{e^{+\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}}_{= [S(\Lambda)]^{-1}}$$

Index notation:

$$\Psi^a \mapsto S^a{}_b \Psi^b \qquad \bar{\Psi}_a \mapsto \bar{\Psi}_b (S^{-1})^b{}_a$$

$$a, b = 1, \ldots, 4 = \text{Dirac spinor index}$$

Dirac matrices have index structure $(\gamma^\mu)^a{}_b$

$(\gamma^\mu)^a{}_b$ is an *invariant tensor*: it is invariant when transformed according to its index structure.

Spacetime metric is the canonical example of this:

$$\eta^{\mu\nu} = \underbrace{\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma}}_{\text{Lorentz transformation of } \eta^{\mu\nu}}$$

For γ^μ:

$$(\gamma^\mu)^a{}_b = \underbrace{\Lambda^\mu{}_\nu S^a{}_c (\gamma^\nu)^c{}_d (S^{-1})^d{}_b}_{\text{Lorentz transformation of } (\gamma^\mu)^a{}_b} \qquad \gamma^\mu = \Lambda^\mu{}_\nu S \gamma^\nu S^{-1}$$

$\Rightarrow$ can form Lorentz tensors by contracting spinor indices:

$$\bar{\Psi}_a \Psi^a \qquad \bar{\Psi}_a (\gamma^\mu)^a{}_b \Psi^b \qquad \cdots$$

$$(\gamma^5)^a{}_b = \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)^a{}_b = \text{invariant tensor}$$

$\Rightarrow$ additional Lorentz tensors:

$$\bar{\Psi}_a (\gamma^5)^a{}_b \Psi^b \qquad \bar{\Psi}_a (\gamma^\mu\gamma^5)^a{}_b \Psi^b \qquad \cdots$$

Weyl basis for Dirac matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} \sigma^\mu &= (1, \vec{\sigma}) \\ \bar{\sigma}^\mu &= (1, -\vec{\sigma}) \end{aligned}}$$

$\vec{\sigma} =$ Pauli matrices

Note: $\bar{\sigma}^\mu \neq (\sigma^\mu)^\dagger$ or $(\sigma^\mu)^*$

$$\Rightarrow \ \Sigma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}$$

$=$ block diagonal

$\Rightarrow$ Dirac representation is *reducible*

$$\boxed{\begin{aligned} \sigma^{\mu\nu} &= \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \\ \bar{\sigma}^{\mu\nu} &= \frac{i}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu) \end{aligned}}$$

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$\psi_L =$ left-handed Weyl spinor

$\psi_R =$ right-handed Weyl spinor

$$\left.\begin{aligned} \delta\psi_L &= -\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\psi_L \\ \delta\psi_R &= -\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\psi_R \end{aligned}\right\} \text{ different reps of } SO(3,1)$$

Index notation:

$(\psi_L)_\alpha$ $\quad$ $\alpha = 1, 2, =$ Weyl spinor index

$(\psi_R)^{\dot{\alpha}}$ $\quad$ $\dot{\alpha} = 1, 2, =$ dotted Weyl spinor index

$$\delta(\psi_L)_\alpha = -\frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta(\psi_L)_\beta$$

$$\delta(\psi_R)^{\dot{\alpha}} = -\frac{i}{2}\omega_{\mu\nu}(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}(\psi_R)^{\dot{\beta}}$$

Finite transformations:

$$(\psi_L)_\alpha \mapsto \underbrace{\left(e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}\right)_\alpha{}^\beta}_{= S_\alpha{}^\beta(\Lambda)} (\psi_L)_\beta$$

$$(\psi_R)^{\dot{\alpha}} \mapsto \underbrace{\left(e^{-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}}\right)^{\dot{\alpha}}{}_{\dot{\beta}}}_{= \bar{S}^{\dot{\alpha}}{}_{\dot{\beta}}(\Lambda)} (\psi_R)^{\dot{\beta}}$$

Define transformation for general tensors with upper/lower dotted/undotted Weyl spinor indices:

$$T^{\alpha\cdots\dot{\beta}\cdots}_{\gamma\cdots\dot{\delta}\cdots} \mapsto S_\gamma{}^{\gamma'}\bar{S}^{\dot{\beta}}{}_{\dot{\beta}'}(S^{-1})_{\alpha'}{}^{\alpha}(\bar{S}^{-1})^{\dot{\delta}'}{}_{\dot{\delta}} \cdots T^{\alpha'\cdots\dot{\beta}'\cdots}_{\gamma'\cdots\dot{\delta}'\cdots}$$

Invariant tensors:

$$\sigma^\mu_{\alpha\dot{\beta}} = \underbrace{\Lambda^\mu{}_\nu S_\alpha{}^\gamma \sigma^\nu_{\gamma\dot{\delta}} (\bar{S}^{-1})^{\dot{\delta}}{}_{\dot{\beta}}}_{\text{Lorentz transformation of } \sigma^\mu_{\alpha\dot{\beta}}}$$

$$\bar{\sigma}^{\mu\dot{\alpha}\beta} = \Lambda^\mu{}_\nu(\bar{S}^{-1})^{\dot{\alpha}}{}_{\dot{\gamma}}\bar{\sigma}^{\nu\dot{\gamma}\delta}(S^{-1})_\delta{}^\beta$$

Also:

$$\boxed{\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

$$\epsilon_{\alpha\beta} = \underbrace{S_\alpha{}^\gamma S_\beta{}^\delta \epsilon_{\gamma\delta}}_{\text{Lorentz transformation of } \epsilon_{\alpha\beta}}$$

$$\epsilon^{\dot{\alpha}\dot{\beta}} = \bar{S}^{\dot{\alpha}}{}_{\dot{\gamma}}\bar{S}^{\dot{\beta}}{}_{\dot{\delta}}\epsilon^{\dot{\gamma}\dot{\delta}} \quad \textit{etc.}$$

Summarize: invariant tensors

$$\boxed{\begin{array}{llll} \sigma^\mu_{\alpha\dot\beta} & \bar\sigma^{\mu\dot\alpha\beta} & & \\ \epsilon^{\alpha\beta} & \epsilon_{\alpha\beta} & \epsilon^{\dot\alpha\dot\beta} & \epsilon_{\dot\alpha\dot\beta} \end{array}}$$

can be used to form invariants by contracting indices.

Proof of invariance identities follows from identities on 2×2 matrices. For example, invariance of $\epsilon_{\alpha\beta}$:

$$0 \overset{?}{=} \delta\epsilon_{\alpha\beta} = -\frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\mu})_\alpha{}^{\alpha'}\epsilon_{\alpha'\beta} - \frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\mu})_\beta{}^{\beta'}\epsilon_{\alpha\beta'}$$

$$\Leftrightarrow 0 = \sigma^{\mu\nu}\epsilon + \sigma^{\mu\nu}\epsilon^T \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Leftrightarrow \epsilon\sigma^{\mu\nu}\epsilon^T = -\sigma^{\mu\nu}$$

Follows from $\epsilon\sigma^\mu\epsilon^T = (\bar\sigma^\nu)^*$ Exercise: check this.

Complex conjugation relates dotted/undotted spinor indices:

$$\boxed{\left[(\sigma^{\mu\nu})_\alpha{}^\beta\right]^* = (\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\alpha}}$$

$\Rightarrow$ makes sense to write

$$(\psi_\alpha)^\dagger = \psi^\dagger_{\dot\alpha} \qquad \textit{etc.}$$

Any spinor Lagrangian can be written entirely in terms of L Weyl spinors.

Given ψ_α, can define Weyl spinors with any index structure:

$$\boxed{\begin{array}{l} \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \\ \psi^\dagger_{\dot\alpha} = (\psi_\alpha)^\dagger \\ \psi^{\dagger\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\psi^\dagger_{\dot\alpha} \end{array}}$$

Invariant Lagrangians (finally!)

Most general quadratic Lagrangian for a Weyl spinor ψ_α:

$$\boxed{\mathcal{L} = \psi^\dagger_{\dot\alpha} i\bar\sigma^{\mu\dot\alpha\beta}\partial_\mu\psi_\beta - \tfrac{1}{2}\left(\epsilon^{\alpha\beta}\psi_\alpha\psi_\beta + \text{h.c.}\right)}$$

The mass term is a *Majorana mass term*.

Note that it breaks any $U(1)$ symmetry acting on ψ.

Nonzero mass term requires anticommuting spinor fields:

$$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} \qquad \psi_\alpha\psi_\beta = -\psi_\beta\psi_\alpha$$

Canonical quantization: quantum fermion fields obey anticommutation relations, $\hbar \to 0$ limit gives anticommuting classical spinor fields.

Path integral quantization: fermion path integral is over anticommuting fields.

Check $\mathcal{L}^\dagger = \mathcal{L}$ (needed for Hermitian quantum Hamiltonian)

To agree with Hermitian conjugation of operators, complex conjugation of classical anticommuting spinors must be defined to reverse the order of spinors:

$$(\psi_\alpha\chi_\beta)^\dagger = \chi^\dagger_{\dot\beta}\psi^\dagger_{\dot\alpha} \qquad \text{(no change of sign)}$$

With this rule, we have

$$(\epsilon^{\alpha\beta}\psi_\alpha\psi_\beta)^\dagger = \epsilon^{\dot\alpha\dot\beta}\psi^\dagger_{\dot\beta}\psi^\dagger_{\dot\alpha} = -\epsilon^{\dot\alpha\dot\beta}\psi^\dagger_{\dot\alpha}\psi^\dagger_{\dot\beta}$$

$$(\psi^\dagger_{\dot\alpha} i\bar\sigma^{\mu\dot\alpha\beta}\partial_\mu\psi_\beta)^\dagger = -i\partial_\mu\psi^\dagger_{\dot\beta}\underbrace{(\bar\sigma^{\mu\dot\beta\alpha})^*}_{=\bar\sigma^{\mu\dot\beta\alpha}}\psi_\beta \qquad (\bar\sigma^\mu)^\dagger = \bar\sigma^\mu$$

$$= +\psi^\dagger_{\dot\beta} i\bar\sigma^{\mu\dot\beta\alpha}\partial_\mu\psi_\alpha \qquad \text{(integrate by parts)}$$

$\Rightarrow \mathcal{L}^\dagger = \mathcal{L}$.

Expressions look cleaner when spinor indices are implicit:

$$\psi^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\psi_\beta = \psi^\dagger\bar\sigma^\mu\psi$$

$$\chi^\alpha\sigma^\mu_{\alpha\dot\beta}\chi^{\dagger\dot\beta} = \chi\sigma^\mu\chi^\dagger$$

In general, omit summed indices

$${}^{\alpha}{}_{\alpha} \qquad \text{and} \qquad {}_{\dot\alpha}{}^{\dot\alpha}$$

Example:

$$\chi\psi = \chi^\alpha\psi_\alpha = \epsilon^{\alpha\beta}\chi_\beta\psi_\alpha = -\underbrace{\epsilon^{\alpha\beta}\psi_\alpha}_{=-\epsilon^{\beta\alpha}\psi_\alpha = -\psi^\beta}\chi_\beta$$

$$= \psi^\beta\chi_\beta$$

$$= +\psi\chi$$

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \qquad etc.$$

Exercise:

Consider theory of a single massless Weyl fermion ψ_α with the Lagrangian given above.

(a) Show that the equation of motion is the *Weyl equation*

$$i\bar\sigma^{\mu\dot\alpha\beta}\partial_\mu\psi_\beta = 0$$

Multiply on the left by $\sigma^\nu\partial_\nu$ to show that the Weyl equation implies the massless Klein-Gordon equation:

$$\Box\psi_\alpha = 0$$

(b) Consider the most general plane wave solution

$$\psi_\alpha(x) = w_\alpha(p)e^{-ip\cdot x}$$

By going to the frame $p^\mu = (E, 0, 0, E)$, show that there is a unique solution for $w_\alpha(p)$.

(c) The most general operator solution to the Weyl equation is

$$\hat\psi_\alpha(x) = \int \frac{d^3p}{(2\pi)^{3/2}(2|\vec p|)^{1/2}}\Big[\hat a(\vec p)w_\alpha(p)e^{-ip\cdot x} + \hat b^\dagger(\vec p)w_\alpha(p)e^{+ip\cdot x}\Big]$$

$p^0 = |\vec p|$

Imposing the anticommutation relations

$$\{\hat a(\vec p), \hat a^\dagger(\vec p\,')\} = \{\hat b(\vec p), \hat b^\dagger(\vec p\,')\} = \delta^3(\vec p - \vec p\,')$$

$$\{\hat a(\vec p), \hat a(\vec p\,')\} = \{\hat b(\vec p), \hat b(\vec p\,')\} = \{\hat a(\vec p), \hat b^\dagger(\vec p\,')\} = 0$$

compute the equal-time anticommutator

$$\{\hat\psi_\alpha(t,\vec x), \hat\psi^\dagger_{\dot\beta}(t,\vec y)\}$$

You will need the identity

$$w_\alpha(p)w^\dagger_{\dot\beta}(p) = \sigma^\mu_{\alpha\dot\beta}p_\mu \qquad \text{(fixes normalization of } u_\alpha(p))$$

which you can verify in the standard frame.

(d) Show that the canonical momentum is

$$\pi^\alpha = \frac{\partial\mathcal{L}}{\partial\dot\psi_\alpha} = -i\psi^\dagger_{\dot\beta}i\bar\sigma^{0\dot\beta\alpha}$$

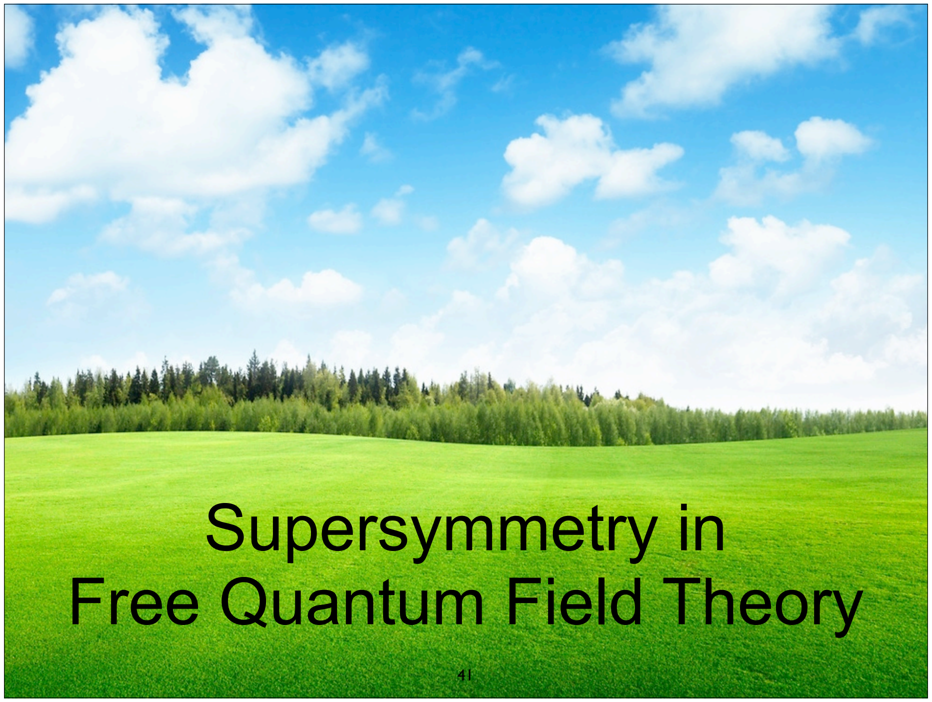
(Remember that ψ_α is a classical anticommuting field.)

(e) Show that the anticommutation relation you derived above is equivalent to the canonical anticommutation relation

$$\{\hat\pi^\alpha(t,\vec x), \hat\psi_\beta(t,\vec y)\} = -i\delta^\alpha{}_\beta\delta^3(\vec x - \vec y)$$

This exercise shows that a Weyl fermion has 2 propagating degrees of freedom.

Note: many textbook treatments of the Dirac equation change the sign of π^α and the canonical anticommutation relations to get the correct commutation relations for the creation and annihilation operators.



Simplest theory with a chance of Bose-Fermi symmetry:

ψ_α = L Weyl fermion

ϕ = complex scalar (2 degrees of freedom)

$$\mathcal{L} = \psi^\dagger i\bar{\sigma}^\mu \partial_\mu \psi + \partial^\mu \phi^\dagger \partial_\mu \phi$$

$m = 0$ for now

Note this preserves $U(1)$ symmetry

$$\psi_\alpha \mapsto e^{i\theta\psi}, \qquad \phi \mapsto e^{i\theta}\phi$$

Write most general SUSY transformation:

- $\delta\phi \sim \psi$, $\delta\psi \sim \phi$
- Lorentz/spinor indices match
- $U(1)$ invariant

$$\delta\phi = \underbrace{\xi\psi}_{= \xi^\alpha \psi_\alpha}$$

ξ^a = spinor "parameter"

Note: $\xi^\dagger\psi^\dagger$ would violate $U(1)$.

$$[\phi] = 1, \quad [\psi] = \tfrac{3}{2} \quad \Rightarrow \quad \boxed{[\xi] = -\tfrac{1}{2}}$$

$$\delta\psi_\alpha = c_0 \xi_\alpha \phi + c_1 \underbrace{(\sigma^\mu \xi^\dagger)_\alpha}_{= \sigma^\mu_{\alpha\dot{\beta}} \xi^{\dagger\dot{\beta}}} \partial_\mu \phi + c_2 \xi_\alpha \Box\phi + \cdots$$

$$[c_0] = 1 \qquad [c_1] = 0 \qquad [c_2] = -1 \qquad \cdots$$

Require no new dimensionful parameters $\Rightarrow$ c_1 term only.

Compute $\delta\mathcal{L}$:

$$\delta(\partial^\mu \phi^\dagger \partial_\mu \phi) = \partial^\mu \phi^\dagger \partial_\mu (\delta\phi) + \text{h.c.}$$

$$= \partial^\mu \phi^\dagger \xi \partial_\mu \psi + \underbrace{\text{h.c.}}_{\text{depends on } \xi^\dagger}$$

$$\delta(\psi^\dagger i\bar{\sigma}^\mu \partial_\mu \psi) = \delta\psi^\dagger i\bar{\sigma}^\mu \partial_\mu \psi + \text{h.c.}$$

$$= ic^* \partial_\mu \phi^\dagger \xi \sigma^\mu \bar{\sigma}^\nu \partial_\nu \psi + \text{h.c.}$$

$$= -ic^* \partial_\mu \partial_\nu \phi^\dagger \xi \sigma^\mu \bar{\sigma}^\nu \psi + \text{h.c.}$$

Use identity

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu} 1_2$$

$$\Rightarrow \delta(\psi^\dagger i\bar{\sigma}^\mu \partial_\mu \psi_\beta = -ic^* \Box\phi^\dagger \xi\psi + \text{h.c.}$$

$$= ic^* \partial^\mu \phi^\dagger \xi \partial_\mu \psi + \text{h.c.}$$

$\Rightarrow \delta\mathcal{L} = 0$ for $c = -i$.

Summarize: $\boxed{\delta\phi = \xi\psi \qquad \delta\psi = -i\sigma^\mu \xi^\dagger \partial_\mu \phi}$

Noether current:

$$\boxed{J^\mu_\alpha = (\sigma^\nu \bar{\sigma}^\mu \psi)_\alpha \partial_\nu \phi^\dagger}$$

Note: carries extra spacetime (spinor) index $\Rightarrow$ sign of space-time symmetry

Check conservation:

$$\partial_\mu J^\mu_\alpha = \underbrace{(\sigma^\nu \bar{\sigma}^\mu)_\alpha{}^\beta \partial_\mu \psi_\beta}_{=0} \partial_\nu \phi^\dagger + \underbrace{\underbrace{(\sigma^\nu \bar{\sigma}^\mu)_\alpha{}^\beta}_{\to \eta^{\mu\nu}\delta_\alpha{}^\beta} \psi_\beta \partial_\mu \partial_\nu \phi^\dagger}_{\propto \Box\phi = 0}$$

$$\Rightarrow \partial_\mu J^\mu_\alpha = 0 \qquad \text{(on classical solutions)}$$

Noether charge:

$$\boxed{Q_\alpha = \sqrt{2}\int d^3x\, J^0_\alpha} \qquad \text{(normalization is conventional)}$$

$$\boxed{\frac{d}{dt} Q_\alpha = 0} \qquad \text{(on classical solutions)}$$

Quantum theory:

$$\hat{\psi}_\alpha(x) = \int \frac{d^3p}{(2\pi)^{3/2}(2|\vec{p}|)^{1/2}} \Big[\hat{a}(\vec{p}) w_\alpha(p) e^{-ip\cdot x} + \hat{b}^\dagger(\vec{p}) w_\alpha(p) e^{+ip\cdot x}\Big]$$

Note: 2 degrees of freedom in quantum Weyl fermion.

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}(2|\vec{p}|)^{1/2}} \Big[\hat{c}(\vec{p}) e^{-ip\cdot x} + \hat{d}^\dagger(\vec{p}) e^{+ip\cdot x}\Big]$$

$$\Rightarrow \boxed{\hat{Q}_\alpha = \sqrt{2}\int d^3p\, u_\alpha(\vec{p}) \Big[\hat{c}^\dagger(\vec{p})\hat{a}(\vec{p}) + \hat{b}^\dagger(\vec{p})\hat{d}(\vec{p})\Big]}$$

$$\hat{Q}^\dagger_{\dot\alpha} = \sqrt{2}\int d^3p\, u^\dagger_{\dot\alpha}(\vec{p}) \Big[\hat{a}^\dagger(\vec{p})\hat{c}(\vec{p}) + \hat{d}^\dagger(\vec{p})\hat{b}(\vec{p})\Big]$$

fermion particle: $|\psi(\vec{p})\rangle = \hat{a}^\dagger(\vec{p})|0\rangle$

fermion antiparticle: $|\bar{\psi}(\vec{p})\rangle = \hat{b}^\dagger(\vec{p})|0\rangle$

scalar particle: $|\phi(\vec{p})\rangle = \hat{c}^\dagger(\vec{p})|0\rangle$

scalar antiparticle: $|\bar{\phi}(\vec{p})\rangle = \hat{d}^\dagger(\vec{p})|0\rangle$

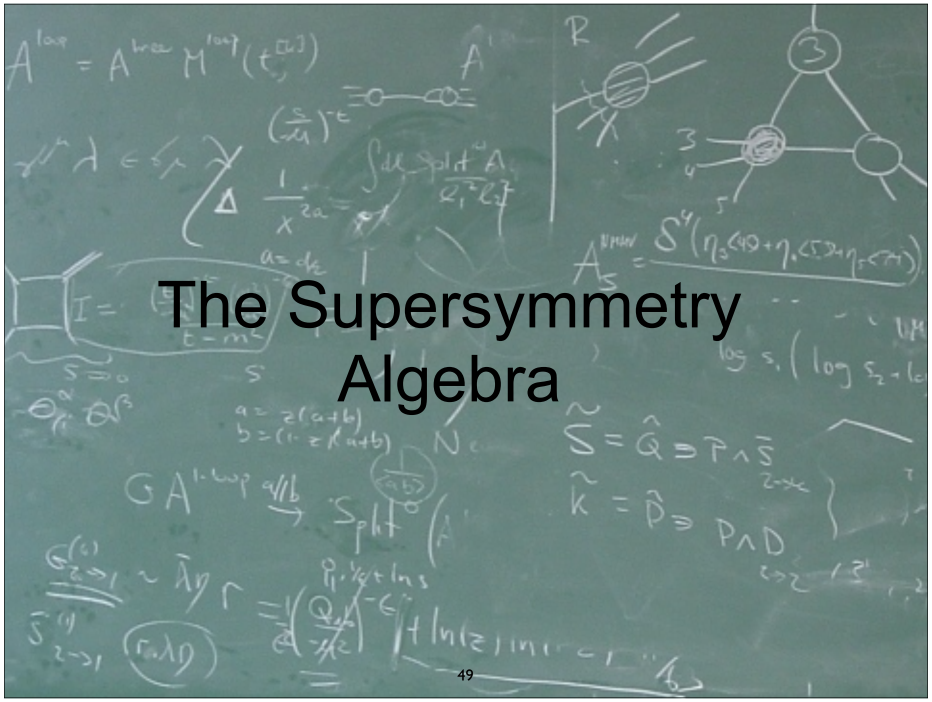
$\hat{Q}_\alpha|\psi(\vec{p})\rangle = \sqrt{2}u_\alpha(\vec{p})|\phi(\vec{p})\rangle \qquad \hat{Q}_\alpha|\bar{\psi}(\vec{p})\rangle = 0$

$\hat{Q}_\alpha|\phi(\vec{p})\rangle = 0 \qquad \hat{Q}_\alpha|\bar{\phi}(\vec{p})\rangle = \sqrt{2}u_\alpha(\vec{p})|\bar{\psi}(\vec{p})\rangle$

$\hat{Q}^\dagger_{\dot\alpha}|\psi(\vec{p})\rangle = 0 \qquad \hat{Q}^\dagger_{\dot\alpha}|\bar{\psi}(\vec{p})\rangle = \sqrt{2}u^\dagger_{\dot\alpha}(\vec{p})|\bar{\phi}(\vec{p})\rangle$

$\hat{Q}^\dagger_{\dot\alpha}|\phi(\vec{p})\rangle = \sqrt{2}u^\dagger_{\dot\alpha}(\vec{p})|\psi(\vec{p})\rangle \qquad \hat{Q}^\dagger_{\dot\alpha}|\bar{\phi}(\vec{p})\rangle = 0$

Summarize: $\bar{\psi} \xrightarrow{Q^\dagger} \bar{\phi} \xrightarrow{Q} \bar{\psi} \qquad \psi \xrightarrow{Q} \phi \xrightarrow{Q^\dagger} \psi$



The Supersymmetry Algebra

Use the free-field representation to compute

$$\{\hat{Q}_\alpha, \hat{Q}^\dagger_{\dot\beta}\} = 2\int d^3p\, d^3q\, u_\alpha(\vec{p})u^\dagger_{\dot\beta}(\vec{q}) \times \Big[\{\hat{c}^\dagger(\vec{p})\hat{a}(\vec{p}),\, \hat{a}^\dagger(\vec{q})\hat{c}(\vec{q})\} + \{\hat{b}^\dagger(\vec{q})\hat{d}(\vec{q}),\, \hat{d}^\dagger(\vec{q})\hat{b}(\vec{q})\}\Big]$$

$$\{\hat{c}^\dagger\hat{a}, \hat{a}^\dagger\hat{c}\} = \{\hat{a}, \hat{a}^\dagger\}\hat{c}^\dagger\hat{c} + [\hat{c}, \hat{c}^\dagger]\hat{a}^\dagger\hat{a}$$

$$\{\hat{b}^\dagger\hat{d}, \hat{d}^\dagger\hat{b}\} = \{\hat{b}, \hat{b}^\dagger\}\hat{d}^\dagger\hat{d} + [\hat{d}, \hat{d}^\dagger]\hat{b}^\dagger\hat{b}$$

$a,\ b =$ fermion
$c,\ d =$ boson

$$\{\hat{Q}_\alpha, \hat{Q}^\dagger_{\dot\beta}\} = 2\int d^3p\ \underbrace{u_\alpha(\vec{p})u^\dagger_{\dot\beta}(\vec{p})}_{=\sigma^\mu_{\alpha\dot\beta}p_\mu}\Big[\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) + \cdots + \hat{d}^\dagger(\vec{p})\hat{d}(\vec{p})\Big]$$

$$= 2\sigma^\mu_{\alpha\dot\beta}\hat{P}_\mu \qquad \hat{P}_\mu = \text{4-momentum operator}$$

$$\boxed{\{\hat{Q}_\alpha, \hat{Q}^\dagger_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}\hat{P}_\mu}$$

$Q \sim$ square root of spacetime translations $\Rightarrow$ SUSY is a space-time symmetry.

Similarly,

$$\boxed{\{\hat{Q}_\alpha, \hat{Q}_\beta\} = 0 \qquad \{\hat{Q}^\dagger_{\dot\alpha}, \hat{Q}^\dagger_{\dot\beta}\} = 0}$$

$$\boxed{[\hat{P}_\mu, \hat{Q}_\alpha] = 0 \qquad [\hat{P}_\mu, \hat{Q}^\dagger_{\dot\alpha}] = 0}$$

This is the famous ($\mathcal{N} = 1$) SUSY algebra.

Drop the hats from now on...

Consequences of SUSY Algebra:

General state can be expanded in eigenstates of $\hat{P}^\mu$ with eigenvalue $p^\mu =$ timelike.

Choose frame $p^\mu = (E, 0, 0, 0) \Rightarrow$

$$\{Q_1, Q_1^\dagger\} = \{Q_2, Q_2^\dagger\} = 2E \qquad (Q_1)^2 = \cdots = (Q_2^\dagger)^2 = 0$$

$$H = \tfrac{1}{2}(Q_1 + Q_1^\dagger)^2 = \tfrac{1}{2}(Q_2 + Q_2^\dagger)^2$$

$= $ square of Hermitian operator

$\Rightarrow \boxed{\langle\psi|H|\psi\rangle \geq 0}$ *i.e.* states have positive energy.

Vacuum state is invariant: $\hat{Q}_\alpha|0\rangle = 0,\ \hat{Q}^\dagger_{\dot\alpha}|0\rangle = 0$

$\Rightarrow \boxed{H|0\rangle = 0}$ (unbroken SUSY)

Vacuum energy vanishes $\Rightarrow$ a clue to cosmological constant problem?

Massless 1-particle states:

$$|\vec{p},\lambda\rangle \qquad \lambda = \hat{p}\cdot\vec{S} = \text{helicity}$$

Choose frame $p^\mu = (E,0,0,E) \qquad E > 0$

$$\{Q_1, Q_1^\dagger\} = 0$$

$$\{Q_2, Q_2^\dagger\} = 4E$$

This is the algebra of one fermionic creation and annihilation operator $Q_2^\dagger$, Q_2 [$Q_1^\dagger$, Q_1 act trivially].

$$[Q_\alpha, \hat{p}\cdot\vec{S}] = [Q_\alpha, M^{12}] = -(\sigma^{12})_\alpha{}^\beta Q_\beta$$

$$[Q_2, \hat{p}\cdot\vec{S}] = +\tfrac{1}{2}Q_2 \qquad [Q_2^\dagger, \hat{p}\cdot\vec{S}] = -\tfrac{1}{2}Q_2^\dagger$$

$\Rightarrow Q_2$ ($Q_2^\dagger$) acts as raising (lowering) operator for helicity.

Irreducible 1-particle representations:

$$\begin{matrix} |\vec{p},\lambda\rangle \\ |\vec{p},\lambda+\tfrac{1}{2}\rangle \end{matrix} \quad \overset{CPT}{\longleftrightarrow} \quad \begin{matrix} |\vec{p},-\lambda\rangle \\ |\vec{p},-\lambda-\tfrac{1}{2}\rangle \end{matrix}$$

$$\lambda = 0: \quad \begin{matrix} |\vec{p},0\rangle & CPT|\vec{p},0\rangle & \longleftrightarrow \text{ complex scalar} \\ |\vec{p},\tfrac{1}{2}\rangle & |\vec{p},-\tfrac{1}{2}\rangle & \longleftrightarrow \text{ Weyl fermion} \end{matrix}$$

This is the *chiral multiplet*.

$$\lambda = \tfrac{1}{2}: \quad \begin{matrix} |\vec{p},\tfrac{1}{2}\rangle & |\vec{p},-\tfrac{1}{2}\rangle & \longleftrightarrow \text{ Weyl fermion} \\ |\vec{p},1\rangle & |\vec{p},-1\rangle & \longleftrightarrow \text{ massless gauge field} \end{matrix}$$

This is the *massless vector multiplet*.

These are the multiplets that describe massless particles of spin ≤ 1.