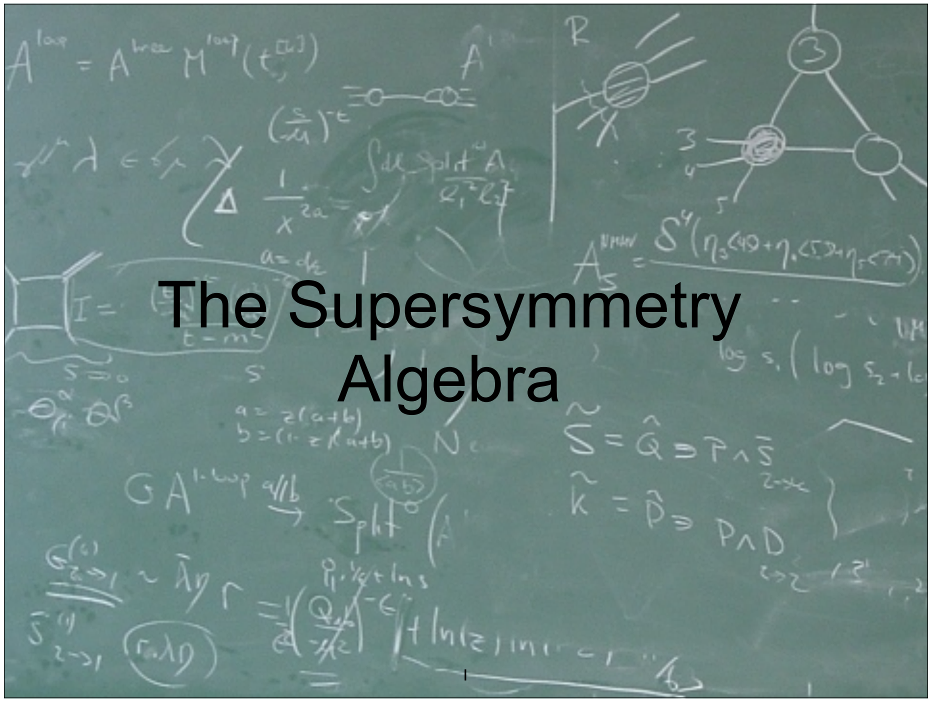


The Supersymmetry Algebra

Use the free-field representation to compute

$$\{\hat{Q}_\alpha, \hat{Q}^\dagger_{\dot\beta}\} = 2\int d^3p\, d^3q\, w_\alpha(\vec{p}) w^\dagger_{\dot\beta}(\vec{q}) \times \Big[\{\hat{c}^\dagger(\vec{p})\hat{a}(\vec{p}),\, \hat{a}^\dagger(\vec{q})\hat{c}(\vec{q})\} + \{\hat{b}^\dagger(\vec{q})\hat{d}(\vec{q}),\, \hat{d}^\dagger(\vec{q})\hat{b}(\vec{q})\} \Big]$$

$$\{\hat{c}^\dagger\hat{a}, \hat{a}^\dagger\hat{c}\} = \{\hat{a}, \hat{a}^\dagger\}\hat{c}^\dagger\hat{c} + [\hat{c}, \hat{c}^\dagger]\hat{a}^\dagger\hat{a}$$

$$\{\hat{b}^\dagger\hat{d}, \hat{d}^\dagger\hat{b}\} = \{\hat{b}, \hat{b}^\dagger\}\hat{d}^\dagger\hat{d} + [\hat{d}, \hat{d}^\dagger]\hat{b}^\dagger\hat{b}$$

$a,\ b =$ fermion
$c, d =$ boson

$$\{\hat{Q}_\alpha, \hat{Q}^\dagger_{\dot\beta}\} = 2\int d^3p\, \underbrace{w_\alpha(\vec{p}) w^\dagger_{\dot\beta}(\vec{p})}_{= \sigma^\mu_{\alpha\dot\beta} p_\mu} \Big[\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) + \cdots + \hat{d}^\dagger(\vec{p})\hat{d}(\vec{p})\Big]$$

$$= 2\sigma^\mu_{\alpha\dot\beta}\hat{P}_\mu$$

$\hat{P}_\mu =$ 4-momentum operator

$$\boxed{\{\hat{Q}_\alpha, \hat{Q}^\dagger_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}\hat{P}_\mu}$$

Similarly,

$$\{\hat{Q}_\alpha, \hat{Q}_\beta\} = 0 \qquad \{\hat{Q}^\dagger_{\dot\alpha}, \hat{Q}^\dagger_{\dot\beta}\} = 0$$

$$[\hat{P}_\mu, \hat{Q}_\alpha] = 0 \qquad [\hat{P}_\mu, \hat{Q}^\dagger_{\dot\alpha}] = 0$$

This is the famous ($\mathcal{N} = 1$) SUSY algebra.

Drop the hats from now on...

Consequences of SUSY Algebra:

$$\hat{P}^\mu = \bar{\sigma}^{\mu\dot\alpha\beta}\{\hat{Q}^\dagger_{\dot\alpha}, \hat{Q}_\beta\}$$

$$\hat{P}^0 = \hat{Q}^\dagger_1\hat{Q}_1 + \hat{Q}_1\hat{Q}^\dagger_1 + \hat{Q}^\dagger_2\hat{Q}_2 + \hat{Q}_2\hat{Q}^\dagger_2$$

For any state $|\psi\rangle$

$$\langle\psi|\hat{P}^0|\psi\rangle = \|\hat{Q}^\dagger_1|\psi\rangle\|^2 + \|\hat{Q}_1|\psi\rangle\|^2 + \|\hat{Q}^\dagger_2|\psi\rangle\|^2 + \|\hat{Q}_2|\psi\rangle\|^2 \geq 0$$

If SUSY is unbroken, the vacuum state is SUSY invariant:

$$\hat{Q}_\alpha|0\rangle = 0, \qquad \hat{Q}^\dagger_{\dot\alpha}|0\rangle = 0$$

$$\Rightarrow\ \hat{P}^\mu|0\rangle = 0$$

In particular, the vacuum energy vanishes.

A clue to the cosmological constant problem?

Requires supergravity...

Massless 1-particle states:

$|\vec{p}, \lambda\rangle \qquad \lambda = \hat{p}\cdot\vec{S} = \text{helicity}$

Choose frame $p^\mu = (E, 0, 0, E) \qquad E > 0$

$$\{Q_1, Q_1^\dagger\} = 0$$

$$\{Q_2, Q_2^\dagger\} = 4E$$

This is the algebra of one fermionic creation and annihilation operator $Q_2^\dagger$, Q_2 [$Q_1^\dagger$, Q_1 act trivially].

$$[Q_\alpha, \hat{p}\cdot\vec{S}] = [Q_\alpha, M^{12}] = -(\sigma^{12})_\alpha{}^\beta Q_\beta$$

$$[Q_2, \hat{p}\cdot\vec{S}] = +\tfrac{1}{2}Q_2 \qquad [Q_2^\dagger, \hat{p}\cdot\vec{S}] = -\tfrac{1}{2}Q_2^\dagger$$

$\Rightarrow$ Q_2 ($Q_2^\dagger$) acts as raising (lowering) operator for helicity.

Irreducible 1-particle representations:

$$\begin{matrix} |\vec{p}, \lambda\rangle \\ |\vec{p}, \lambda + \tfrac{1}{2}\rangle \end{matrix} \quad \overset{CPT}{\longleftrightarrow} \quad \begin{matrix} |\vec{p}, -\lambda\rangle \\ |\vec{p}, -\lambda - \tfrac{1}{2}\rangle \end{matrix}$$

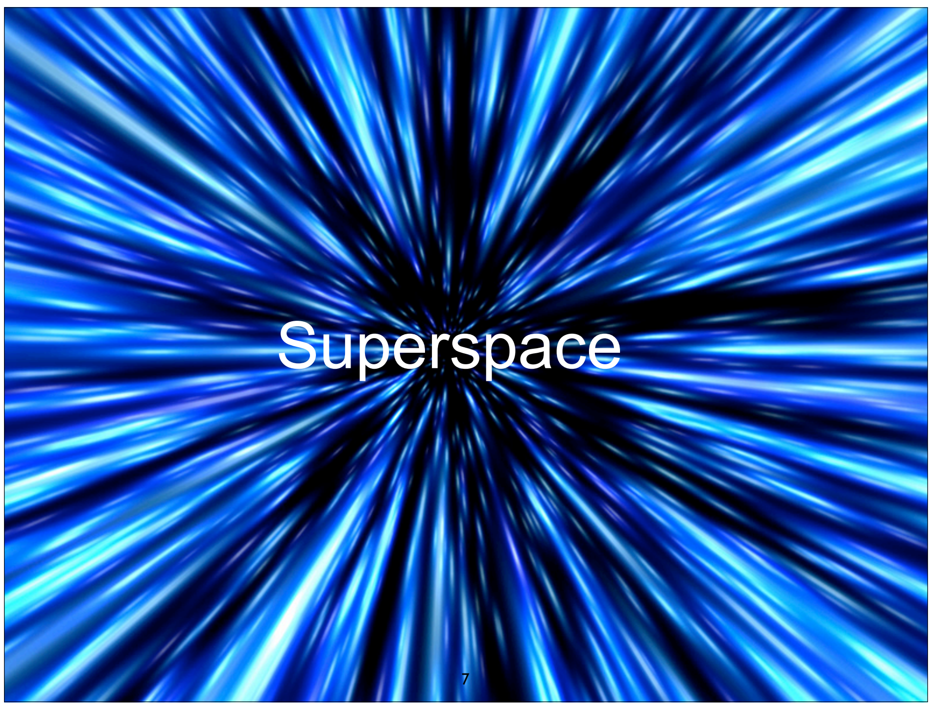
$$\lambda = 0: \quad \begin{matrix} |\vec{p}, 0\rangle & CPT|\vec{p}, 0\rangle & \longleftrightarrow \text{ complex scalar} \\ |\vec{p}, \tfrac{1}{2}\rangle & |\vec{p}, -\tfrac{1}{2}\rangle & \longleftrightarrow \text{ Weyl fermion} \end{matrix}$$

This is the *chiral multiplet*.

$$\lambda = \tfrac{1}{2}: \quad \begin{matrix} |\vec{p}, \tfrac{1}{2}\rangle & |\vec{p}, -\tfrac{1}{2}\rangle & \longleftrightarrow \text{ Weyl fermion} \\ |\vec{p}, 1\rangle & |\vec{p}, -1\rangle & \longleftrightarrow \text{ massless gauge field} \end{matrix}$$

This is the *massless vector multiplet*.

These are the multiplets that describe massless particles of spin ≤ 1.



A good pedagogical discussion for this subject:

D. Bertonini, J. Thaler, Z. Thomas, "Super-tricks for Superspace" (TASI 2012 lectures), `arXiv:1302.6229`.

Warning: although this uses the same spinor conventions as we do, but uses a non-conventional definition of the SUSY generators. It is easy to translate between them:

$$Q_\alpha^{(\text{them})} = -Q_\alpha^{(\text{us})}$$

$$P_\mu^{(\text{them})} = -P_\mu^{(\text{us})}$$

Lorentz transformations act naturally on spacetime:

$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$

SUSY acts naturally on *superspace*.

superspace = $\{(x^\mu, \theta^\alpha, \bar\theta^{\dot\alpha})\}$

$\theta^\alpha,\ \bar\theta^{\dot\alpha} =$ anticommuting "coordinates"

$\bar\theta^{\dot\alpha} = \theta^{\dagger\dot\alpha} = (\theta^\alpha)^\dagger$

$$\{\theta^\alpha, \theta^\beta\} = 0 \qquad \{\bar\theta^{\dot\alpha}, \bar\theta^{\dot\beta}\} = 0$$

$\Rightarrow \quad \theta^1\theta^1 = \theta^2\theta^2 = 0,$ *etc.*

The natural variables for SUSY quantum field theory are therefore *superfields* = functions of superspace.

Superfields are defined by Taylor expanding in θ, $\bar\theta$.

θ, $\bar\theta$ anticommute $\Rightarrow$ expansion contains finitely many terms.

Function of one real anticommuting variable θ:

$f(\theta) = a + b\theta \qquad \theta^2 = 0$

Superfield = function of θ^α, $\bar\theta^{\dot\alpha}$:

Highest component $= \theta^1\theta^2\bar\theta^1\bar\theta^2$

Simplify expansion using identities

$\theta_\alpha\theta_\beta = 2 \times 2$ antisymmetric matrix $\propto \epsilon_{\alpha\beta}$

$$\theta_\alpha\theta_\beta = \tfrac{1}{2}\epsilon_{\alpha\beta}\theta\theta \qquad \theta^\alpha\theta^\beta = -\tfrac{1}{2}\epsilon^{\alpha\beta}\theta\theta \qquad \theta\theta = \theta^\alpha\theta_\alpha$$

$$\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = \tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta \qquad \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = -\tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta \qquad \bar\theta\bar\theta = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}$$

General (scalar) superfield:

$$\begin{aligned} S(x,\theta,\bar\theta) = A(x) &+ \theta^\alpha\psi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha}(x) \\ &+ \theta\theta B(x) + \bar\theta\bar\theta C(x) \\ &+ \theta\sigma^\mu\bar\theta V_\mu(x) \\ &+ (\bar\theta\bar\theta)\theta^\alpha\lambda_\alpha(x) + (\theta\theta)\bar\theta_{\dot\alpha}\bar\eta^{\dot\alpha}(x) \\ &+ (\theta\theta)(\bar\theta\bar\theta)D(x) \end{aligned}$$

$S \longleftrightarrow (A, \psi_\alpha, \bar\chi_{\dot\alpha}, B, C, V_\mu, \lambda_\alpha, \bar\eta^{\dot\alpha}, D)$

$\theta^\alpha,\ \bar\theta^{\dot\alpha} =$ algebraic placeholders

Analogy: complex numbers:

$z = x + iy \qquad i^2 = -1$

$z \longleftrightarrow (x, y) \qquad i =$ placeholder

$z_1 + z_2 \longleftrightarrow (x_1 + x_2,\ y_1 + y_2)$

$z_1 z_2 \longleftrightarrow (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$

Superfields naturally add and multiply together:

$$S_1(x,\theta,\bar\theta) + S_2(x,\theta,\bar\theta) = A_1 + A_2 + \theta^\alpha(\psi_{1\alpha} + \psi_{2\alpha}) + \cdots$$

$$S_1 + S_2 \longleftrightarrow (A_1 + A_2,\ \psi_{1\alpha} + \psi_{2\alpha},\ \ldots)$$

$$S_1(x,\theta,\bar\theta)\, S_2(x,\theta,\bar\theta) = A_1A_2 + \theta^\alpha(A_1\psi_{2\alpha} + A_2\psi_{1\alpha}) + \cdots$$

$$S_1S_2 \longleftrightarrow (A_1A_2,\ A_1\psi_{2\alpha} + A_2\psi_{1\alpha},\ \ldots)$$

Spacetime translations generated by derivative operator:

$$\phi(x) \mapsto \phi(x-a) = e^{ia^\mu P_\mu}\phi(x) \qquad \boxed{P_\mu = i\partial_\mu}$$

$$\delta\phi(x) = ia^\mu P_\mu \phi(x)$$

Define SUSY transformation of superfields:

$$\boxed{\delta S(x,\theta,\bar\theta) = i(a^\mu P_\mu + \xi^\alpha Q_\alpha + \bar\xi_{\dot\alpha}\bar Q^{\dot\alpha})S(x,\theta,\bar\theta)}$$

$Q_\alpha, \bar Q^{\dot\alpha} =$ derivative operators

$(\xi^\alpha Q_\alpha)^\dagger = \bar\xi_{\dot\alpha}\bar Q^{\dot\alpha}$

Derivative operators: in addition to $\partial_\mu = \dfrac{\partial}{\partial x^\mu}$,

define $\dfrac{\partial}{\partial\theta^\alpha}, \dfrac{\partial}{\partial\bar\theta_{\dot\alpha}}$:

$$\boxed{\frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha{}^\beta} \qquad \boxed{\frac{\partial}{\partial\theta^\alpha}\bar\theta_{\dot\beta} = 0}$$

$$\boxed{\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\bar\theta_{\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta}} \qquad \boxed{\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\theta^\beta = 0}$$

Careful about signs:

$$\frac{\partial}{\partial\theta^\alpha}(\psi^\beta\theta^\gamma) = \frac{\partial}{\partial\theta^\alpha}(-\theta^\gamma\psi^\beta) = -\delta^\alpha{}_\gamma\psi^\beta$$

Anticommute spinors to the left before acting with $\dfrac{\partial}{\partial\theta^\alpha}$

Exercise: Show that $\dfrac{\partial}{\partial\theta_\alpha} = -\epsilon^{\alpha\beta}\dfrac{\partial}{\partial\theta^\beta}$

Define SUSY generators acting on superfields:

$$\boxed{\begin{aligned} iQ_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar\theta)_\alpha\partial_\mu \\ i\bar Q^{\dot\alpha} &= \frac{\partial}{\partial\bar\theta_{\dot\alpha}} + i(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu \end{aligned}}$$

Check that they satisfy SUSY algebra:

$$\boxed{\begin{aligned} &\{Q_\alpha, \bar Q_{\dot\beta}\} = 2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu \\ &\{Q_\alpha, Q_\beta\} = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0 \end{aligned}}$$

Remember $P_\mu = i\partial_\mu$

SUSY covariant derivatives:

$S =$ superfield $\Rightarrow$ $\partial_\mu S =$ superfield?

Does $\partial_\mu S$ transform like a superfield?

$$\begin{aligned} \delta(\partial_\mu S) &= \partial_\mu(\delta S) = \partial_\mu\left[i(\xi Q + \bar\xi\bar Q)S\right] \\ &= i(\xi Q + \bar\xi\bar Q)\partial_\mu S \end{aligned}$$

Works because $[\partial_\mu, Q_\alpha] = [\partial_\mu, \bar Q_{\dot\alpha}] = 0$

$$\Leftrightarrow \; [P_\mu, Q_\alpha] = [P_\mu, \bar Q_{\dot\alpha}] = 0$$

On the other hand,

$$\frac{\partial}{\partial\theta^\alpha}S \neq \text{superfield}$$

$$\left\{\frac{\partial}{\partial\theta^\alpha}, \bar Q_{\dot\beta}\right\} \neq 0$$

Define SUSY covariant derivatives that anticommute with Q_α, $\bar{Q}_{\dot\alpha}$:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\bar\theta)_\alpha\partial_\mu \qquad iQ_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar\theta)_\alpha\partial_\mu$$

$$\bar{D}^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} - i(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu \qquad i\bar{Q}^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} + i(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu$$

$$0 = \{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot\beta}\} = \{\bar{D}_{\dot\alpha}, Q_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\}$$

$$\{D_\alpha, \bar{D}_{\dot\beta}\} = 2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu$$

$$0 = \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\}$$

$D_\alpha S =$ superfield:

$$\delta(D_\alpha S) = D_\alpha(\delta S) = D_\alpha\left[i(\xi Q + \bar\xi\bar Q)S\right]$$
$$= +i(\xi Q + \bar\xi\bar Q)D_\alpha S$$

$+$ sign because $\{D_\alpha, \xi^\beta\} = \{D_\alpha, \bar\xi_{\dot\beta}\} = 0$

$$e.g.\ \frac{\partial}{\partial\theta^\alpha}(\xi^\beta\theta^\gamma) = -\frac{\partial}{\partial\theta^\alpha}(\theta^\gamma\xi^\beta) = -\delta^\gamma{}_\alpha\xi^\beta$$

Chiral superfields:

SUSY covariant derivative allows us to construct simpler superfields with fewer component fields.

Define *chiral superfield* Φ by condition

$$\bar{D}_{\dot\alpha}\Phi(x,\theta,\bar\theta) = 0$$

$\Rightarrow$ Φ is independent of something?

Change variables in superspace: $(x^\mu, \theta, \bar\theta) \to (y^\mu, \theta, \bar\theta)$

$$y^\mu = x^\mu + i\underbrace{\bar\theta\bar\sigma^\mu\theta}_{= -\theta\sigma^\mu\bar\theta} \qquad \text{note } [\theta] = [\bar\theta] = -\tfrac{1}{2}$$

Work out D_α, $\bar{D}_{\dot\alpha}$ in terms of new variables:

$$\bar{D}^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} - i(\bar\sigma^\mu\theta)^{\dot\alpha}\frac{\partial}{\partial x^\mu}$$
$$= \frac{\partial}{\partial\bar\theta_{\dot\alpha}} + \underbrace{\frac{\partial y^\mu}{\partial\bar\theta_{\dot\alpha}}}_{= i(\bar\sigma^\mu\theta)^{\dot\alpha}}\frac{\partial}{\partial y^\mu} - i(\bar\sigma^\mu\theta)^{\dot\alpha}\underbrace{\frac{\partial y^\nu}{\partial x^\mu}}_{= \delta^\nu{}_\mu}\frac{\partial}{\partial y^\nu}$$
$$= \frac{\partial}{\partial\bar\theta_{\dot\alpha}}$$

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\bar\theta)_\alpha\frac{\partial}{\partial x^\mu}$$
$$= \frac{\partial}{\partial\theta^\alpha} + \underbrace{\frac{\partial y^\mu}{\partial\theta^\alpha}}_{= -i(\sigma^\mu\theta)_\alpha}\frac{\partial}{\partial y^\mu} - i(\sigma^\mu\bar\theta)_\alpha\underbrace{\frac{\partial y^\nu}{\partial x^\mu}}_{= \delta^\nu{}_\mu}\frac{\partial}{\partial y^\nu}$$
$$= \frac{\partial}{\partial\theta^\alpha} - 2i(\sigma^\mu\bar\theta)_\alpha\frac{\partial}{\partial y^\mu}$$

Summarize:

$$\overline{D}^{\dot\alpha} = \frac{\partial}{\partial \overline{\theta}_{\dot\alpha}} \qquad D_\alpha = \frac{\partial}{\partial \theta^\alpha} - 2i(\sigma^\mu \overline{\theta})_\alpha \frac{\partial}{\partial y^\mu}$$

$\overline{D}_{\dot\alpha}\Phi = 0 \;\Rightarrow\; \Phi =$ function of (y, θ) (independent of $\overline{\theta}$)

Component fields: "chiral representation"

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta^\alpha \psi_\alpha(y) + \theta\theta F(y)$$

$$\Phi \longleftrightarrow (\phi, \psi_\alpha, F)$$

Can expand to write as function of $(x, \theta, \overline{\theta})$:

$$\phi(y) = \phi(x) + i\partial_\mu \phi(x)\overline{\theta}\overline{\sigma}^\mu \theta + \cdots$$

$$\overline{D}_{\dot\alpha}\Phi^n = 0 \qquad \Rightarrow \quad \Phi^n = \text{chiral superfield}$$

In fact, for any function $f(\Phi)$

$$\overline{D}_{\dot\alpha} f(\Phi) = 0 \quad \Rightarrow \quad f(\Phi) = \text{chiral superfield}$$

To get a chiral superfield, $f(\Phi)$ cannot depend on $\Phi^\dagger$.
That is, $f(\Phi)$ must be a *holomorphic* function of Φ.
This has far-reaching implications, as we will see below.

Note that

$$D_\alpha \Phi^\dagger = 0$$

Superfields satisfying this constraint are called *anti-chiral*.

The properties of anti-chiral superfields can be worked out by complex conjugating the results for chiral superfields.

Can use D_α, $\overline{D}_{\dot\alpha}$ to define component fields by projection:

$$\phi = \Phi\big|_{\theta,\overline{\theta}=0} \equiv \Phi|$$

$$\psi_\alpha = \frac{1}{\sqrt{2}}\frac{\partial}{\partial\theta^\alpha}\Phi| = \frac{1}{\sqrt{2}} D_\alpha \Phi|$$

extra terms vanish for $\theta, \overline{\theta} = 0$

$$F = -\tfrac{1}{4}\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}\Phi| = -\tfrac{1}{4}DD\Phi|$$

Use algebra of derivative operators to compute SUSY transformation of component fields:

$$\delta\phi = i(\xi Q + \overline{\xi}\overline{Q})\Phi|$$
$$= (\xi D + \overline{\xi} D)\Phi| \;=\; \underbrace{\frac{1}{\sqrt{2}}}\,\xi\psi$$

because we defined $Q_\alpha = \sqrt{2}\int d^3x J^0_\alpha$

$$\begin{aligned}
\delta\psi_\alpha &= \frac{i}{\sqrt{2}}(\xi Q + \overline{\xi}\overline{Q}) D_\alpha \Phi| \\
&= \frac{i}{\sqrt{2}} D_\alpha(\xi Q + \overline{\xi}\overline{Q})\Phi| \\
&= \frac{1}{\sqrt{2}} D_\alpha(\xi D + \overline{\xi}\overline{D})\Phi| \\
&= \frac{1}{\sqrt{2}}\left[-\tfrac{1}{2}\xi_\alpha DD\Phi| - 2i(\sigma\overline{\xi})_\alpha \partial_\mu \Phi|\right] \\
&= \sqrt{2}\xi_\alpha F - i\sqrt{2}(\sigma\overline{\xi})_\alpha \partial_\mu \phi
\end{aligned}$$

Similarly,

$$\delta F = -i\sqrt{2}\overline{\xi}\overline{\sigma}^\mu \partial_\mu \psi \qquad \text{(exercise)}$$

Summarize:

$$\phi = \Phi|$$
$$\psi_\alpha = \frac{1}{\sqrt{2}} D_\alpha \Phi|$$
$$F = -\tfrac{1}{4} DD\Phi|$$

$$\delta\phi = \frac{1}{\sqrt{2}} \xi\psi$$
$$\delta\psi_\alpha = \sqrt{2}\xi_\alpha F - i\sqrt{2}(\sigma\bar{\xi})_\alpha \partial_\mu \phi$$
$$\delta F = -i\sqrt{2}\bar{\xi}\bar{\sigma}^\mu \partial_\mu \psi$$

Write SUSY invariant Lagrangians using D_α, $\bar{D}_{\dot{\alpha}}$:

$$\mathcal{L}_D = \frac{1}{16} D^2 \bar{D}^2 K| \qquad K = K^\dagger = \text{real superfield} \;\Rightarrow\; \mathcal{L}_D^\dagger = \mathcal{L}_D$$

Here we use shorthand

$$D^2 = DD = D^\alpha D_\alpha \qquad \bar{D}^2 = \bar{D}\bar{D} = \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \qquad \textit{etc.}$$

$\mathcal{L}_D$ is called a "D-term." Reason:

$$K = \cdots + \theta^2 \bar{\theta}^2 D_K \qquad D_K = \text{highest component field of } K$$

50% of the symbols in SUSY are some form of the letter D...

$$[\mathcal{L}_D] = 4 \;\Rightarrow\; [K] = 2$$

$$\delta\mathcal{L}_D = \frac{i}{16} D^2 \bar{D}^2 (\xi Q + \bar{\xi}\bar{Q}) K|$$
$$= \frac{i}{16} (\xi Q + \bar{\xi}\bar{Q}) D^2 \bar{D}^2 K|$$
$$= \frac{1}{16} (\xi D + \bar{\xi}\bar{D}) D^2 \bar{D}^2 K| \quad = \text{total derivative}$$

Reason: $D_\alpha D^2 = 0$

$$\bar{D}_{\dot{\alpha}} D^2 \bar{D}^2 K = \underbrace{[\bar{D}_{\dot{\alpha}}, D^2]}_{\propto\, \partial_\mu} D^2 K$$

$$[\bar{D}_{\dot{\alpha}}, D^2] = -4i\sigma^\mu_{\alpha\dot{\alpha}} D^\alpha \partial_\mu \qquad [D_\alpha, \bar{D}^2] = 4i\sigma^\mu_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} \partial_\mu$$

Exercise: Show that if K is a chiral superfield, then $\mathcal{L}_D$ is a total derivative.

Can use similar ideas to construct SUSY invariant from a chiral superfield:

$$\mathcal{L}_F = -\frac{1}{4} D^2 W| + \text{h.c.} \qquad W = \text{chiral superfield} = \cdots \theta^2 \bar{\theta}^2 F_W \qquad \text{“F term”}$$

$$\Rightarrow\; \mathcal{L}_F^\dagger = \mathcal{L}_F$$

$$\delta\mathcal{L}_F = -\frac{i}{4} D^2 (\xi Q + \bar{\xi}\bar{Q}) W|$$
$$= -\frac{i}{4} (\xi Q + \bar{\xi}\bar{Q}) D^2 W|$$
$$= -\frac{1}{4} (\xi D + \bar{\xi}\bar{D}) D^2 W| = \text{total derivative}$$

$$D_\alpha D^2 = 0 \qquad \bar{D}_{\dot{\alpha}} D^2 W = [\bar{D}_{\dot{\alpha}}, D^2] W \propto \partial_\mu$$

$$[\mathcal{L}_F] = 4 \;\Rightarrow\; [W] = 3$$

In the literature, it is common to use the notation

$$\mathcal{L}_D = \int d^4\theta\, K \qquad \mathcal{L}_F = \int d^2\theta\, W + \text{h.c.}$$

This arises because integration and differentiation are identical for anticommuting variables:

$$\int d\theta\,(a + b\theta) = b = \frac{\partial}{\partial\theta}(a + b\theta)$$

We will use this notation with the understanding that it is defined by

$$\boxed{\int d^4\theta\, K = \frac{1}{16} D^2 \overline{D}^2 K|} \qquad \boxed{\int d^2\theta\, W = -\frac{1}{4} D^2 W|}$$

We now have all the tools we need to write interacting SUSY invariant Lagrangians!

$$\Phi = \text{chiral superfield}$$

$$= \phi(y) + \theta\psi(y) + \theta^2 F(y)$$

$$[\phi] = 1 \;\Rightarrow\; [\psi] = \tfrac{3}{2},\; [F] = 2$$

$$\Rightarrow\; [\Phi] = 1$$

Write the most general SUSY invariant Lagrangian with dimensionless couplings:

$$\mathcal{L} = \int d^4\theta\, \Phi^\dagger\Phi + \left(\int d^2\theta\, \frac{\lambda}{3}\Phi^3 + \text{h.c.}\right)$$

$$[d^4\theta] = 2, \quad [d^2\theta] = 1$$

$$\int d^4\theta\, \Phi^\dagger\Phi = \frac{1}{16} D^2\overline{D}^2(\Phi^\dagger\Phi)| \;= \frac{1}{16} D^2\left[(\overline{D}^2\Phi^\dagger)\Phi\right]\Big|$$

$$= \frac{1}{16}\Big[D^2\overline{D}^2\Phi^\dagger|\,\Phi|$$

$$+ 2D^\alpha\overline{D}^2\Phi^\dagger|\,D_\alpha\Phi|$$

$$+ \overline{D}^2\Phi|\,D^2\Phi|\Big]$$

$$D^2\overline{D}^2\Phi^\dagger| = [D^2, \overline{D}^2]\Phi^\dagger = -16\Box\phi^\dagger \qquad \boxed{[D^2, \overline{D}^2] = -16\Box}$$

$$D_\alpha\overline{D}^2\Phi^\dagger| = [D_\alpha, \overline{D}^2]\Phi^\dagger| = 4\sqrt{2}(\sigma^\mu\partial_\mu\psi^\dagger)_\alpha$$

$$\int d^4\theta\, \Phi^\dagger\Phi = -(\Box\phi^\dagger)\phi + \psi i\sigma^\mu\partial_\mu\psi^\dagger + F^\dagger F$$

$$\boxed{\int d^4\theta\, \Phi^\dagger\Phi = \partial^\mu\phi^\dagger\partial_\mu\phi + \psi^\dagger i\overline{\sigma}^\mu\partial_\mu\psi + F^\dagger F}$$

Evaluate F term for general function of Φ:

$$\int d^2\theta\, W(\Phi) = -\frac{1}{4} D^2 W(\Phi)|$$

$$= -\frac{1}{4}\left[W'(\Phi)D^2\Phi + W''(\Phi)D^\alpha\Phi D_\alpha\Phi\right]\Big|$$

$$= W'(\phi)F - \tfrac{1}{2}W''(\phi)\psi\psi$$

$$\boxed{\int d^2\theta\, W(\Phi) = W'(\phi)F - \tfrac{1}{2}W''(\phi)\psi\psi}$$

$$W(\Phi) = \text{superpotential}$$

For our theory,

$$\mathcal{L} = \int d^4\theta\, \Phi^\dagger \Phi + \left(\int d^2\theta \frac{\lambda}{3} \Phi^3 + \text{h.c.} \right)$$
$$= \partial^\mu \phi^\dagger \partial_\mu \phi + \psi^\dagger i \bar{\sigma}^\mu \partial_\mu \psi + F^\dagger F$$
$$+ \lambda \phi^2 F - \tfrac{1}{2} \lambda \phi \psi \psi + \text{h.c.}$$

We recognize kinetic terms for ϕ, ψ and Yukawa coupling.

Note no derivatives act on F, and $\mathcal{L}$ is quadratic on F. We say that F is an *auxiliary field*.

The significance of this is that we can integrate out F exactly:

$$|F^\dagger + \lambda\phi^2|^2 = F^\dagger F + (\lambda \phi^2 F + \text{h.c.}) + |\lambda\phi^2|^2$$

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi + \psi^\dagger i \bar{\sigma}^\mu \partial_\mu \psi$$
$$- \tfrac{1}{2}(\lambda \phi\psi\psi + \text{h.c.}) + |F^\dagger + \lambda\phi^2|^2 - |\lambda\phi^2|^2$$

Write functional integral:

$$Z = \int [d\phi]\,[d\psi]\,[dF]\, e^{i\int \mathcal{L}}$$

Integral over F is trivial:

$$\int [dF]\, e^{i\int |F^\dagger + \lambda\phi^2|^2} = \int [dX]\, e^{i\int |X|^2} \qquad X = F + (\lambda\phi^2)^\dagger$$

This change of variables is a simple shift, and therefore has trivial Jacobian.

The functional integral over F is therefore independent of other fields, and does not affect correlation functions.

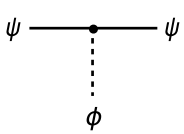
⇒ integrating out F gives

$$\mathcal{L} \to \partial^\mu \phi^\dagger \partial_\mu \phi + \psi^\dagger i \bar{\sigma}^\mu \partial_\mu \psi$$
$$- \tfrac{1}{2}(\lambda \phi\psi\psi + \text{h.c.}) - |\lambda\phi^2|^2$$

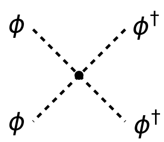
scalar potential!

SUSY relates the Yukawa coupling and the quartic scalar coupling!

Schematically,



$= \lambda$



$= |\lambda|^2$

This is exactly the structure we described in the first lecture for the Higgs-top/stop couplings.

Consider a general theory with N chiral superfields

$$\Phi^a \qquad a = 1, \ldots, N$$

$$\boxed{\mathcal{L} = \int d^4\theta\, \Phi^\dagger_a \Phi^a + \left(\int d^2\theta\, W(\Phi) + \text{h.c.} \right)}$$

After integrating out auxiliary fields F^a

$$\boxed{\mathcal{L}_{\text{int}} = \frac{1}{2} \frac{\partial^2 W(\phi)}{\partial\phi^a \partial\phi^b} \psi^a \psi^b - \left(\frac{\partial W}{\partial \phi^a} \right)^\dagger \frac{\partial W}{\partial \phi^a}}$$

The most general renormalizable superpotential is

$$\boxed{W(\Phi) = \kappa_a \Phi^a + \tfrac{1}{2} m_{ab} \Phi^a \Phi^b + \tfrac{1}{3} \lambda_{abc} \Phi^a \Phi^b \Phi^c}$$

$$[W] = 3 \qquad [\Phi] = 1$$

Note that integrating out F^a is equivalent to imposing its equation of motion

$$F_a^\dagger = \frac{\partial W(\phi)}{\partial \phi^a}$$

We can therefore write the F-term potential as

$$V_F = F_a^\dagger F_a$$

Note that $V_F \geq 0$, and unbroken SUSY requires

$$\langle F^a \rangle = 0 \qquad \text{for all } a$$

Non-renormalization theorem:

The UV divergences of SUSY theories are very constrained. We will show that the coupling constants in the superpotential are not renormalized.

The UV divervences of a QFT can be parameterized by local terms in the 1PI effective action that are relevant or marginal:

$$\Gamma[\Phi] = \text{1PI effective action} = \int d^4x\, \mathcal{L}_{\text{1PI}}[\Phi]$$

$$\mathcal{L}_{\text{1PI}} = \int d^4\theta\, (\delta Z)^a{}_b \Phi_a^\dagger \Phi^b + \int d^2\theta \left(\delta\kappa_a \Phi^a + \tfrac{1}{2}\delta m_{ab}\Phi^a\Phi^b + \tfrac{1}{3}\delta\lambda_{abc}\Phi^a\Phi^b\Phi^c \right) + \text{h.c.} + \text{finite (and non-local)}$$

$\Lambda = \text{UV cutoff}$

$$\delta Z \sim \ln\Lambda$$

$$\delta\kappa \sim \Lambda^2 + \Lambda m + \kappa \ln\Lambda$$

$$\delta m \sim \Lambda + m\ln\Lambda$$

$$\delta\lambda \sim \ln\Lambda$$

The UV divergent terms must respect the symmetries of the original theory.

This is true as long as the UV regulator that preserves the symmetries in question.

In the present class of theories we can use *e.g.* Pauli-Villars or higher derivative regulator to regulate the theory while preserving SUSY.

A very powerful technique is to promote the couplings κ, m, λ to background chiral superfields that transform under SUSY.

This generalized Lagrangian is SUSY invariant as long as we keep κ, m, λ inside the superspace integrals.

$$\mathcal{L}_{\text{int}} = \int d^2\theta \left(\kappa_a \Phi^a + \tfrac{1}{2} m_{ab}\Phi^a\Phi^b + \tfrac{1}{3}\lambda_{abc}\Phi^a\Phi^b\Phi^c \right) + \text{h.c.}$$

The Lagrangian is also invariant under a $U(N)$ symmetry

$$\Phi^a \mapsto U^a{}_b \Phi^b$$

$$\kappa_a \mapsto (U^{-1})^b{}_a \kappa_b$$

$$m_{ab} \mapsto (U^{-1})^c{}_a (U^{-1})^d{}_b m_{cd}$$

$$\lambda_{abc} \mapsto (U^{-1})^d{}_a (U^{-1})^e{}_b (U^{-1})^f{}_c \lambda_{def}$$

That is, all quantities transform according to their index structure.

The coefficients $\delta Z, \delta\kappa, \delta m, \delta\lambda$ must be functions of κ, m, λ that respect this $U(N)$ symmetry.

SUSY also requires the coefficients $\delta\kappa, \delta m, \delta\lambda$ to be holomorphic functions of κ, m, λ (*i.e.* independent of $\kappa^\dagger, m^\dagger, \lambda^\dagger$).

Claim: most general allowed form is

$$\delta\lambda_{abc} = c_\lambda \lambda_{abc} \ln\Lambda$$

$$\delta m_{ab} = c_m m_{ab} \ln\Lambda$$

$$\delta\kappa_a = c_\kappa \kappa_a \ln\Lambda$$

$$c_\lambda, c_m, c_\kappa = \text{independent of couplings}$$

Note there are no couplings with upper $U(N)$ indices we can use to contract indices. If not for holomorphy, we could use

$$\kappa^{\dagger a},\ m^{\dagger ab},\ \lambda^{\dagger abc}$$

[For $N = 1$, this follows from $U(1)$ symmetry, since κ, m, λ have different charges.]

Also, we cannot have divergent term such as $m_{ab} \propto \kappa_a\kappa_b$ because of dimensional analysis.

$$[\kappa_a] = 2, \quad [m_{ab}] = 1, \quad [\lambda_{abc}] = 0$$

Because $\delta\kappa, \delta m, \delta\kappa$ are linear in the couplings, they can be computed in perturbation theory. But all loop diagrams have at least 2 powers of the couplings.

$$\mathcal{L}_{\text{int}} = \lambda_{abc}\phi^a\psi^b\psi^c + \text{h.c.}$$
$$- \sum_a \left|\kappa_a + m_{ab}\phi^b + \lambda_{abc}\phi^b\phi^c\right|^2$$

$$\Rightarrow c_\kappa, c_m, c_\lambda = 0$$

$$\Rightarrow \delta\kappa, \delta m, \delta\lambda = 0 \qquad \text{QED}$$

This is a symmetry argument
$\Rightarrow$ valid beyond perturbation theory.

The coefficient δZ is nonzero:

$$\delta Z^a{}_b = c_Z \lambda^{\dagger acd}\lambda_{bcd} \ln\Lambda + \underbrace{O(\lambda^4)}_{\sim (\lambda^\dagger\lambda)^2 \text{ by } U(N) \text{ invariance}}$$

$$c_Z = -\frac{1}{4\pi^2}$$

δZ cannot depend on κ or m by dimensional analysis.

Treat Λ dependence using standard renormalization theory.

For simplicity, focus on the case of one chiral superfield Φ:

$$\mathcal{L} = \int d^4\theta\, \Phi^\dagger\Phi$$
$$+ \int d^2\theta \left(\kappa\Phi + \tfrac{1}{2}m\Phi^2 + \tfrac{1}{3}\lambda\Phi^3\right) + \text{h.c.}$$

Cancel Λ dependence in the 1PI effective action by adding a counterterm to the Lagrangian:

$$\Delta\mathcal{L}_{\text{ct}} = \int d^4\theta \left[-c_Z\lambda^\dagger\lambda \ln\frac{\Lambda}{\mu} + O(\lambda^4)\right]\Phi^\dagger\Phi$$

This eliminates the dependence on Λ, at the price of introducing dependence on the renormalization scale μ.

Renormalized Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{R}} + \Delta\mathcal{L}_{\text{ct}}$$

$$\mathcal{L}_{\text{R}} = \int d^4\theta\, Z_{\text{R}}(\mu)\Phi^\dagger\Phi$$
$$+ \int d^2\theta \left(\kappa\Phi + \tfrac{1}{2}m\Phi^2 + \tfrac{1}{3}\lambda\Phi^3\right) + \text{h.c.}$$

$$\mu\frac{d}{d\mu}\ln Z_{\text{R}} = -c_Z\lambda^\dagger\lambda + O(\lambda^4) \qquad \text{wavefunction RG equation}$$

Define canonically normalized fields

$$\Phi = [Z_R(\mu)]^{-1/2}\,\hat{\Phi}$$

Note $Z_R \sim 1 + \lambda^\dagger\lambda \;\neq$ chiral superfield

$\Rightarrow \hat{\Phi}$ is a chiral superfield only in the case where λ is a constant (independent of $x, \theta, \bar{\theta}$).

For this case, we can write

$$\mathcal{L}_R = \int d^4\theta\, \hat{\Phi}^\dagger \hat{\Phi} + \int d^2\theta \left[\kappa_R(\mu)\hat{\Phi} + \tfrac{1}{2} m_R(\mu)\hat{\Phi}^2 + \tfrac{1}{3}\lambda_R(\mu)\hat{\Phi}^3\right] + \text{h.c.}$$

$$\kappa_R(\mu) = [Z_R(\mu)]^{-1/2}\kappa$$

$$m_R(\mu) = [Z_R(\mu)]^{-1} m$$

$$\lambda_R(\mu) = [Z_R(\mu)]^{-3/2}\lambda$$

We see that the physical couplings are multiplicatively renormalized:

$$\mu\frac{d}{d\mu}\kappa_R = -\tfrac{1}{2}c_Z\lambda_R^2\kappa_R + O(\lambda_R^4)$$

$$\mu\frac{d}{d\mu}m_R = -c_Z\lambda_R^2 m_R + O(\lambda_R^4) \qquad (\lambda_R = \text{real})$$

$$\mu\frac{d}{d\mu}\lambda_R = -\tfrac{3}{2}c_Z\lambda_R^3 + O(\lambda_R^4)$$

One implication of this is that if a superpotential coupling is set to zero at some scale, it remains zero at all scales, whether or not there is an enhanced symmetry.

<u>SUSY Gauge Theory</u>

Guiding princple: gauge invariance

Suppose ϕ, ψ are components of a chiral superfield Φ with charge q under a $U(1)$ gauge group:

$$\psi_\alpha(x) \mapsto e^{iq\alpha(x)}\psi_\alpha(x) \qquad \psi_\alpha(x) \mapsto e^{iq\alpha(x)}\psi_\alpha(x)$$

$\alpha(x) =$ gauge transformation parameter

$q =$ charge (*e.g.* ± 1)

Generalize to superspace:

$$\boxed{\Phi \mapsto e^{qg\Omega}\Phi} \qquad \alpha = \operatorname{Im}\Omega|$$

Transformed superfield = chiral $\Rightarrow \Omega =$ chiral superfield

Kinetic term is not invariant:

$$\int d^4\theta\,\Phi^\dagger\Phi \mapsto \int d^4\theta\, e^{qg(\Omega+\Omega^\dagger)}\,\Phi^\dagger\Phi$$

$(\Omega^\dagger = -\Omega \Rightarrow \Omega =$ independent of x.)

Make kinetic term gauge invariant by introducing a real superfield V transforming as

$$\boxed{V \mapsto V - \tfrac{1}{2}(\Omega + \Omega^\dagger)} \qquad V^\dagger = V$$

$$\Rightarrow \int d^4\theta\,\Phi^\dagger e^{2qV}\Phi = \text{gauge invariant}$$

Define components by projection:

$$C = V|$$
$$\chi_\alpha = D_\alpha V|$$
$$B = D^2 V|$$
$$A^\mu = \frac{i}{4}\bar{\sigma}^{\mu\dot{\alpha}\beta}[\bar{D}_{\dot{\alpha}}, D_\beta]V| \qquad A_\mu^\dagger = A_\mu$$
$$\lambda_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V|$$
$$D = \frac{1}{32}\{D^2, \bar{D}^2\}V| \qquad D^\dagger = D$$

Compute gauge transformation of these components.

Define components of Ω:

$$\omega + i\alpha = \Omega| \qquad \omega, \theta = \text{real}$$
$$\eta_\alpha = D_\alpha \Omega|$$
$$E = D^2\Omega|$$

Gauge transformation:

$$\delta C = -\tfrac{1}{2}(\Omega + \Omega^\dagger)| = -\omega$$
$$\delta\chi_\alpha = D_\alpha\left[-\tfrac{1}{2}(\Omega + \Omega^\dagger)\right]\Big| = -\tfrac{1}{2}\eta_\alpha$$
$$\delta A^\mu = \frac{i}{4}\bar{\sigma}^{\mu\dot{\alpha}\beta}\bar{D}_{\dot{\alpha}}D_\beta\left[-\tfrac{1}{2}(\Omega + \Omega^\dagger)\right]\Big| + \text{h.c.}$$
$$= \partial^\mu\alpha \longleftarrow \text{conventional gauge transformation!}$$
$$\delta\lambda_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha\left[-\tfrac{1}{2}(\Omega + \Omega^\dagger)\right]\Big| = 0$$

To understand δA_μ and $\delta\lambda_\alpha$, note that

$$\bar{D}_{\dot{\alpha}}D_\beta(\Omega + \Omega^\dagger) = \bar{D}_{\dot{\alpha}}D_\beta\Omega$$
$$= \{\bar{D}_{\dot{\alpha}}, D_\beta\}\Omega$$
$$= 2i\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu\Omega$$
$$\bar{D}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}}D_\beta(\Omega + \Omega^\dagger) = 2i\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu\bar{D}^{\dot{\alpha}}\Omega = 0$$

Summarize:

$$\delta C = -\omega \qquad \delta\chi_\alpha = -\tfrac{1}{2}\eta_\alpha \qquad \delta B = E$$
$$\delta A_\mu = \partial_\mu\alpha \qquad \delta\lambda_\alpha = 0 \qquad \delta D = 0$$

We can use gauge freedom in ω, η_α, E to fix

$$C = 0 \qquad \chi_\alpha = 0 \qquad B = 0$$

(Wess-Zumino gauge)

In Wess-Zumino gauge, the gauge-invariant kinetic term is

$$\int d^4\theta\, \Phi^\dagger e^{2qV}\Phi = D^\mu\phi^\dagger D_\mu\phi + \psi^\dagger i\bar{\sigma}^\mu D_\mu\psi + F^\dagger F$$
$$- \sqrt{2}q\left(\phi^\dagger\lambda\psi + \text{h.c.}\right) + q\phi^\dagger\phi D$$

where

$$D_\mu\phi = (\partial_\mu - iqA_\mu)\phi \qquad D_\mu\psi = (\partial_\mu - iqA_\mu)\psi$$

The form of the kinetic terms is dictated by the residual gauge invariance

$$\phi \mapsto e^{iq\alpha}\phi \qquad \psi \mapsto e^{iq\alpha}\psi \qquad F \mapsto e^{iq\theta}F$$

We see that A_μ is a conventional gauge field.

$$[A_\mu] = 1 \qquad [\lambda] = \tfrac{3}{2} \qquad [D] = 2$$

$A_\mu =$ gauge field

$\lambda =$ propagating spin $\frac{1}{2}$ field

$=$ superpartner of gauge particle

$D =$ auxiliary field

We can get conventional form of the Lagrangian by rescaling $V \to gV$, where g is the gauge coupling:

$$\int d^4\theta\, \Phi^\dagger e^{2qgV}\Phi = D^\mu\phi^\dagger D_\mu\phi + \psi^\dagger i\bar{\sigma}^\mu D_\mu\psi + F^\dagger F - \sqrt{2}qg\left(\phi^\dagger\lambda_\alpha\psi^\alpha + \text{h.c.}\right) + qg\phi^\dagger\phi D$$

$$D_\mu\phi = (\partial_\mu - iqgA_\mu)\phi \qquad D_\mu\psi = (\partial_\mu - iqgA_\mu)\psi$$

To write gauge kinetic term, use superfield whose lowest component is λ_α:

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V \quad = \text{gauge invariant}$$

$\bar{D}_{\dot\beta} W_\alpha = 0$ *i.e.* $W_\alpha =$ chiral

W_α has nothing to do with superpotential W (sorry!)

$[W_\alpha] = \frac{3}{2}$

$\Rightarrow$ we can write dimension-4 gauge- and SUSY-invariant term

$$\int d^2\theta\, W^\alpha W_\alpha = -\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\tau}F_{\mu\nu}F_{\rho\tau} - 2\lambda^\dagger i\bar{\sigma}^\mu\partial_\mu\lambda + D^2$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L}_{\text{gauge}} = \int d^2\theta \left(\frac{1}{4g^2} - \frac{i\Theta}{32\pi^2}\right) W^\alpha W_\alpha + \text{h.c.}$$

$$= -\frac{1}{4g^2}F^{\mu\nu}F_{\mu\nu} + \frac{\Theta}{64\pi^2}\epsilon^{\mu\nu\rho\tau}F_{\mu\nu}F_{\rho\tau} + \frac{1}{g^2}\lambda^\dagger i\bar{\sigma}^\mu\partial_\mu\lambda + \frac{1}{2g^2}D^2$$

$g =$ gauge coupling

$\epsilon^{\mu\nu\rho\tau}F_{\mu\nu}F_{\rho\tau} =$ total derivative

$\Theta =$ vacuum angle

Θ term is a total derivative, and does not give any observable effects for a $U(1)$ gauge theory.

It plays an important role in non-abelian gauge theories.

To get canonically normalized kinetic terms, write (neglecting Θ term)

$$\mathcal{L}_{\text{gauge}} = \int d^2\theta\, \tfrac{1}{4}W^\alpha W_\alpha + \text{h.c.} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \lambda^\dagger i\bar{\sigma}^\mu\partial_\mu\lambda + \tfrac{1}{2}D^2$$

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V$$

D-term Potential

The component field D appears quadratically in $\mathcal{L}$ and without derivatives. It is therefore an auxiliary field and can be integrated out exactly.

Consider a general theory with N chiral superfields and one $U(1)$ gauge group:

$$\Phi^a \mapsto e^{g q_a \Omega} \Phi^a \qquad a = 1, \ldots, N$$

We have rescaled $V \to gV$ so that gauge fields are canonically normalized.

$$\begin{aligned}\mathcal{L} = &\int d^4\theta \sum_a \Phi_a^\dagger e^{2 g q_a V} \Phi^a \\ &+ \int d^2\theta \, \tfrac{1}{4} W^\alpha W_\alpha + \text{h.c.} \\ &+ \int d^2\theta \, W(\Phi) + \text{h.c.}\end{aligned}$$

$$= gD \sum_a q_a \phi_a^\dagger \phi^a + \tfrac{1}{2} D^2 + \text{independent of } D$$

$$= \tfrac{1}{2}\left(D + g q_a \phi_a^\dagger \phi^a\right)^2 - \tfrac{1}{2} g^2 \left(\sum_a q_a \phi_a^\dagger \phi^a\right)^2 + \cdots$$

Integrating out D generates potential

$$\boxed{V_D = \tfrac{1}{2} g^2 \left(\sum_a q_a \phi_a^\dagger \phi^a\right)^2}$$

Integrating out D is equivalent to imposing its equation of motion

$$D = g \sum_a q_a \phi_a^\dagger \phi_a$$

We can therefore write the D-term potential as

$$V_F = \tfrac{1}{2} D^2$$

Note that $V_D \geq 0$, and unbroken SUSY requires

$$\langle D \rangle = 0$$

Exercise: Consider SUSY QED, a $U(1)$ SUSY gauge theory with 2 chiral superfields $\Phi_\pm$ with gauge charge ± 1.

The superpotential is

$$W = m \Phi_+ \Phi_-$$

Work out the scalar potential for this model.

Show that the only minimum of the potential is at $\langle \phi_+ \rangle = \langle \phi_- \rangle = 0$.

Exercise: Consider the same theory with the addition of a chiral superfield S that is neutral under the gauge group.

The superpotential is

$$W = \lambda S \Phi_+ \Phi_- + \frac{\kappa}{3} S^3$$

Show that this theory has a minimum of the potential for any value of $\langle S \rangle$.

Non-Abelian Gauge Theory

The formulation of SUSY non-Abelian gauge theory follows the same steps as Abelian gauge theory, with some technical complications.

$\Phi =$ chiral superfield in fundamental representation of $SU(N)$ gauge group

$$\Phi^a \mapsto (e^{\Omega_A T_A})^a{}_b \Phi^b \qquad \begin{aligned} &a, b = 1, \ldots, N \\ &(T_A)^a{}_b = SU(N) \text{ generator} \\ &\qquad A = 1, \ldots, N^2 - 1 \\ &\operatorname{tr}(T_A T_B) = \tfrac{1}{2}\delta_{AB} \end{aligned}$$

Kinetic term is not gauge invariant:

$$\int d^4\theta\, \Phi^\dagger \Phi \mapsto \int d^4\theta\, \Phi^\dagger e^{\Omega_A^\dagger T_A} e^{\Omega_A T_A} \Phi$$

Introduce one gauge superfield for each gauge generator:

$$V = V_A T_A \qquad V_A^\dagger = V_A$$

Invariant kinetic term:

$$\int d^4\theta\, \Phi^\dagger e^{2V} \Phi = \text{gauge invariant}$$

$$e^{2V} \mapsto e^{-\Omega^\dagger} e^{2V} e^{-\Omega} \qquad \Omega = \Omega_A T_A$$

$$\Rightarrow\ V_A \mapsto V_A - \tfrac{1}{2}(\Omega_A + \Omega_A^\dagger) + \cdots$$

To define components with simple gauge transformation properties, note that

$$e^{-2V} D_\alpha e^{2V} \mapsto e^{\Omega} e^{-2V} e^{\Omega^\dagger} D_\alpha (e^{-\Omega^\dagger} e^{2V} e^{-\Omega}$$

$$= e^{\Omega}(e^{-2V} D_\alpha e^{2V}) e^{-\Omega} + e^{\Omega} D_\alpha e^{-\Omega}$$

Looks like spinor version of

$$A_\mu \mapsto e^{i\theta_A T_A} A_\mu e^{-i\theta_A T_A} + e^{i\theta_A T_A} \partial_\mu e^{-i\theta_A T_A}$$

Define component fields

$$A^\mu = \frac{i}{8} \bar\sigma^{\mu\dot\alpha\beta} \bar D_{\dot\alpha} (e^{-2V} D_\alpha e^{2V})| + \text{h.c.}$$

$$\lambda_\alpha = -\frac{1}{4} \bar D^2 (e^{-2V} D_\alpha e^{2V})|$$

$$D = \frac{1}{64} D^\alpha \bar D^2 (e^{-2V} D_\alpha e^{2V})| + \text{h.c.}$$

Other components vanish in Wess-Zumino gauge.

Work out gauge invariant kinetic term in terms of component fields:

$$\int d^4\theta\, \Phi^\dagger e^{2V} \Phi = D^\mu \phi^\dagger D_\mu \phi + \psi^\dagger i \bar\sigma^\mu D_\mu \psi + F^\dagger F + \sqrt{2}(\phi^\dagger T_A \lambda_A \psi + \text{h.c.}) + D_A \phi^\dagger T_A \phi$$

$$D_\mu \phi = \partial_\mu \phi - i A_{\mu A} T_A \phi$$

To write gauge kinetic term, note that

$$\bar{D}^2(e^{-2V}D_\alpha e^{2V}) \mapsto e^{\Omega}\bar{D}^2(e^{-2V}D_\alpha e^{2V})e^{-\Omega} + e^{\Omega}\underbrace{\bar{D}^2 D_\alpha e^{-\Omega}}$$

$$= [\bar{D}^2, D_\alpha]e^{-\Omega}$$

$$\propto \partial_\mu \bar{D}_{\dot{\alpha}} e^{-\Omega} = 0$$

$$\boxed{W_\alpha = -\frac{1}{8}\bar{D}^2(e^{-2V}D_\alpha e^{2V})} = \text{chiral superfield}$$

$$W_\alpha \mapsto e^{\Omega} W_\alpha e^{-\Omega}$$

$$W_\alpha = W_{\alpha A} T_A$$

Write invariant Lagrangian:

$$\mathcal{L} = \int d^2\theta \left(\frac{1}{4g^2} - \frac{i\Theta}{32\pi^2}\right) W_A^\alpha W_{\alpha A} + \text{h.c.}$$

$$+ \int d^4\theta \Phi^\dagger e^{2V_A T_A}\Phi$$

$$= -\frac{1}{4g^2} F_A^{\mu\nu} F_{\mu\nu A} - \frac{i\Theta}{64\pi^2}\epsilon^{\mu\nu\rho\tau} F_{\mu\nu A} F_{\rho\tau A}$$

$$+ \frac{1}{g^2}\lambda_A^\dagger i\bar{\sigma}^\mu (D_\mu\lambda)_A + \frac{1}{2g^2} D_A D_A$$

$$+ D^\mu\phi^\dagger D_\mu\phi + \psi^\dagger i\bar{\sigma}^\mu D_\mu\psi + F^\dagger F + D_A\phi^\dagger T_A\phi$$

$$(D_\mu\lambda)_A = \partial_\mu\lambda_A + f_{ABC}A_{\mu B}\lambda_C \qquad [T_A, T_B] = i f_{ABC} T_C$$

Integrate out auxiliary fields D_A:

$$\Rightarrow \boxed{V_D = \frac{g^2}{2}\left(\sum_A \phi^\dagger T_A \phi\right)^2}$$

If there are chiral superfields Φ^i, $i = 1, \ldots, n$, we have

Exercise: Consider $SU(N)$ SUSY QCD with one flavor. This is the theory of two chiral superfields Q and $\tilde{Q}$ transforming in the fundamental and antifundamental representation.

$T_A =$ generators of fundamental representation

$-T_A^T =$ generators of antifundamental representation

Write the gauge invariant kinetic term for Q and $\tilde{Q}$ and work out the D-term potential.