Axions: Past, Present and Future

ICTP Summer School, 2015

Surjeet Rajendran

The Landscape of Particles

10^{-43} GeV 10^{19} GeV

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Standard model emerges from the electroweak scale ~ 100 GeV

We know there must be new physics somewhere - standard model cannot explain observed facts about the universe such as dark matter or baryogenesis (in addition to other theoretical worries such as the hierarchy problem)

Where is this new physics? How can we glimpse them?

The Landscape of Particles

We see that the standard model itself has a lot of symmetry structures - in particular, broken gauge and global symmetries

It is not unreasonable to think that there might be other global symmetries at some high scale. These symmetries may also be broken spontaneously at some high scale f_a. If so, what are the signatures?

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Good idea to go after them - since they emerge from such a general initial context

Axions and Axion-like-Particles

What is the difference between an “axion” and an “axion-like-particle”?

This is just some bad nomenclature in the field. As we will see in these lectures, **strong dynamics from QCD** can give **mass to a specific kind of goldstone boson**. That goldstone boson is called **the axion or the QCD axion**. In these lectures, we denote the QCD axion by the letter ***a***.

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There are goldstone bosons whose **mass is unaffected by QCD** - these get mass from sources that we don't know about. Such goldstone bosons are called **Axion-like-Particles**. In these lectures, we refer to them by the letter $\boldsymbol{\phi}$.

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All statements I make about axion-like-particles will also apply to the axion. Statements I make about the axion are specific to it because they rely on properties of QCD

Outline

1. The Past
(historical motivation/strong CP problem)

2. The Present
(axion solution and phenomenology, constraints and detection)

3. The Future
(new detection techniques and theoretical applications)

Strong Dynamics and Quantum Mechanics

Historically, axions were introduced not as some generic way in which broken symmetries could interact with the standard model - instead, they were invented to address a specific problem in QCD called the strong CP problem

We will pursue this historical route - it has the virtue of introducing new phenomena, providing an even stronger motivation for the axion.

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To that end, let us consider the U(1) electromagnetic gauge theory. Following Wilson, the Lagrangian should contain all operators that do not violate symmetries

$$\mathcal{L} \supset \underbrace{-F_{\mu\nu}F^{\mu\nu} + eA_\mu}_{\text{Standard}} J^\mu + \underbrace{\theta F_{\mu\nu}F_{\lambda\sigma}\epsilon^{\mu\nu\lambda\sigma}}_{?}$$

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$$S = \int d^4x\mathcal{L} \supset \int d^4x\theta F_{\mu\nu}\tilde{F}^{\mu\nu} = \int d^4x\theta\vec{E}.\vec{B}$$

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Write $\vec{E} = \nabla\Phi$, integrate by parts

$$\int d^4x \vec{E}.\vec{B} = \int d^4x \nabla\phi.\vec{B} = \underbrace{\int d^4x \nabla.\left(\phi\vec{B}\right)}_{\text{surface}} - \underbrace{\int d^4x \phi \nabla.\vec{B}}_{\text{zero}}$$

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This term does not contribute to the classical equations of motion - for these, we take the action, fix the initial and final values of the field and find the path of minimum action that takes you from the initial field value to the final field value

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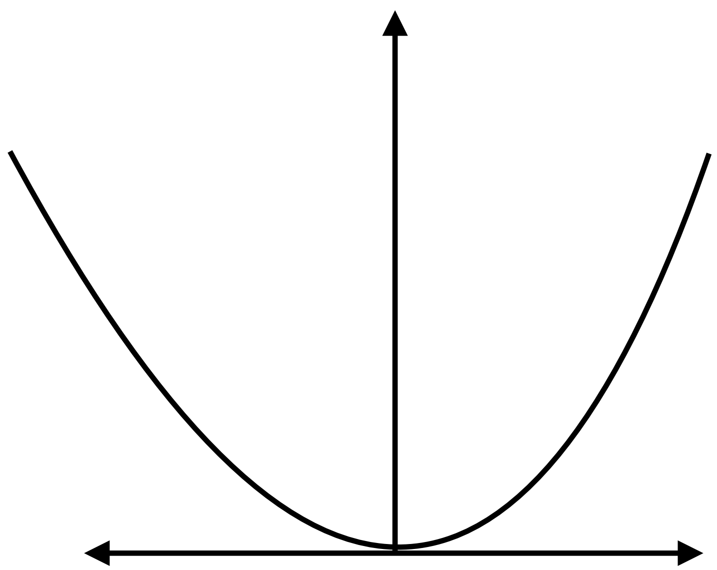
What about quantum mechanically?

Surface Terms in Quantum Mechanics

Before considering field theory, let us play with one particle quantum mechanics

To that end, consider the Lagrangian of a particle undergoing 1D motion along a line in some potential



V(x)

$$\mathcal{L} = \frac{\dot{x}^2}{2} - V\left(x\right)$$

Let V(x) be a potential with just one minimum

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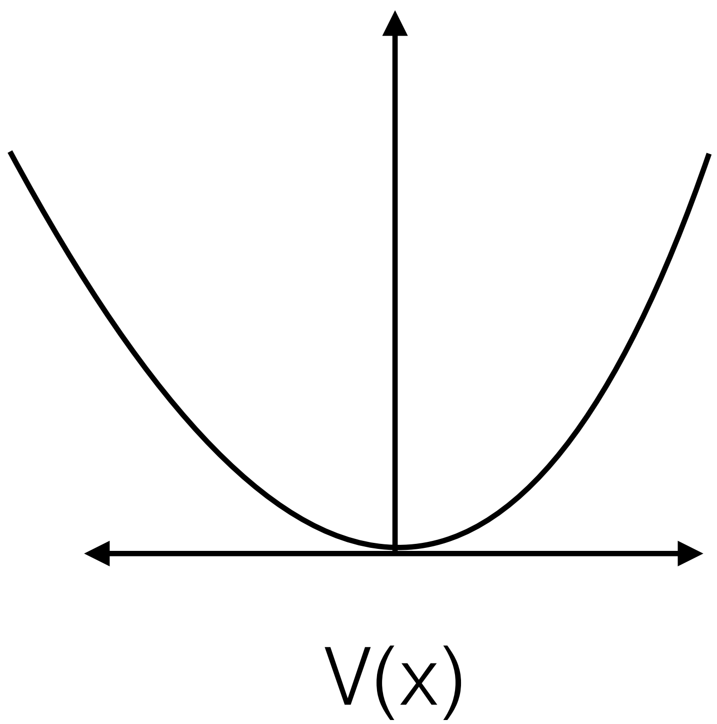

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To this Lagrangian, let us now add a new term

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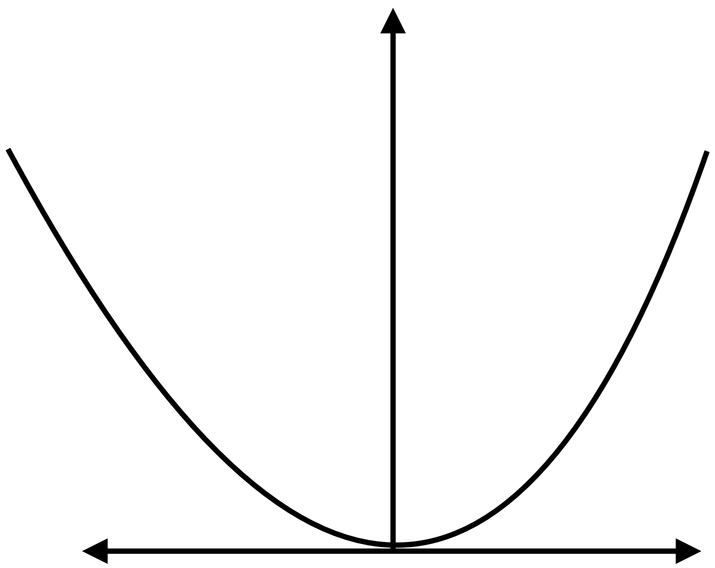

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No effect on classical physics: $\partial_t \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = 0 \implies \ddot{x} + \frac{\partial V}{\partial x} = 0$

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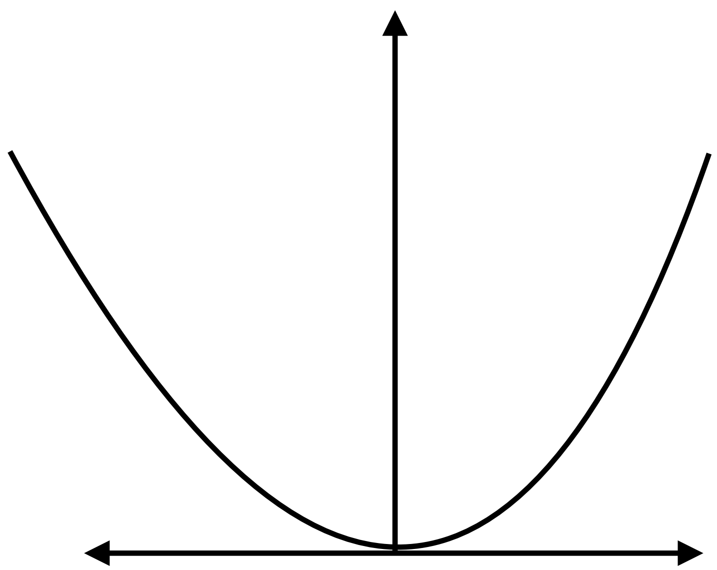

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$S \supset \theta\int dt\dot{x} = \theta(x_f - x_i)$. A constant contribution to all paths - doesn't affect minima

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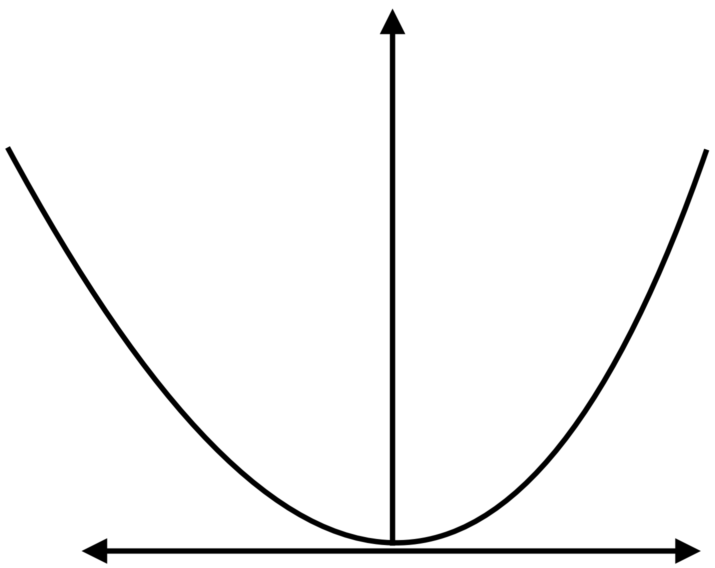
The θ term has no effects on classical physics. What about quantum mechanically? In particular, does it affect the spectrum of the system?



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$$\mathcal{L} = \frac{\dot{x}^2}{2} - V(x) + \theta\dot{x}$$

Calculate Hamiltonian

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} \implies p = \dot{x} + \theta \implies H = p\dot{x} - L = \frac{\dot{x}^2}{2} + V(x)$$

Hamiltonian doesn't depend on θ - the wave functions are all set by the potential V(x) and the kinetic energy piece. By explicit calculation, one can show that θ is irrelevant

Evident also from path integral - depends only on endpoints and so gives same value for all paths

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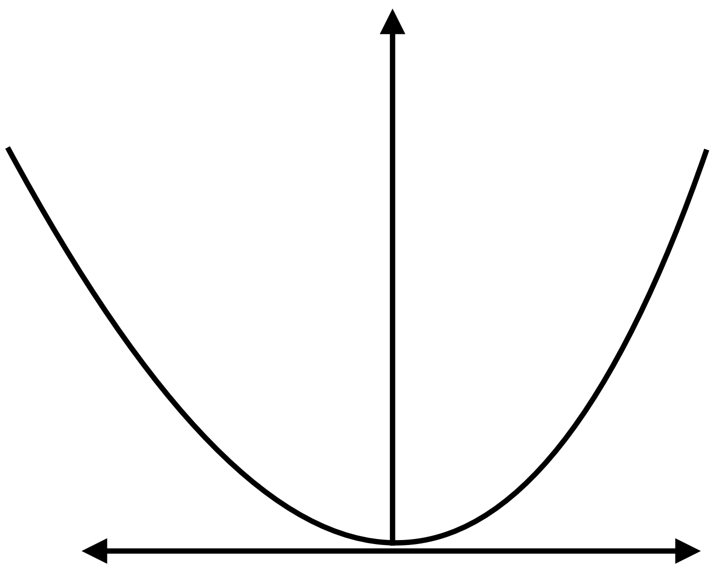

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Is this always true?

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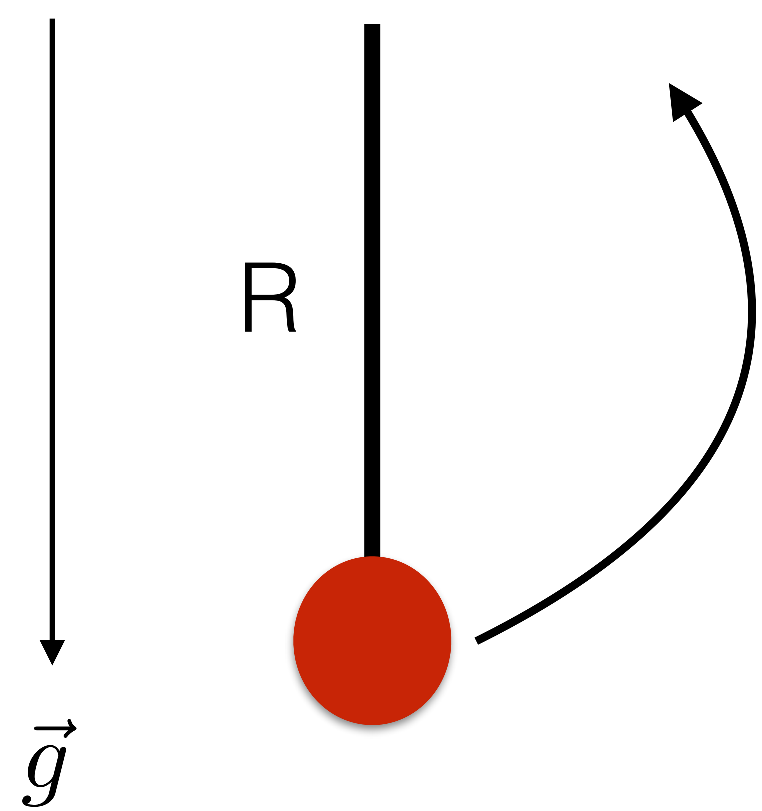
Instead of a particle on a 1D line, consider a pendulum swinging under gravity - assume that the pendulum is suspended from a rigid rod so it can rotate around in a full circle



This pendulum rotates about its pivot - its motion can be characterized by the angle ϕ subtended by it as it rotates. Alternately, we can view it as motion along the 1D x co-ordinate, with the points x and x + 2 π R identified

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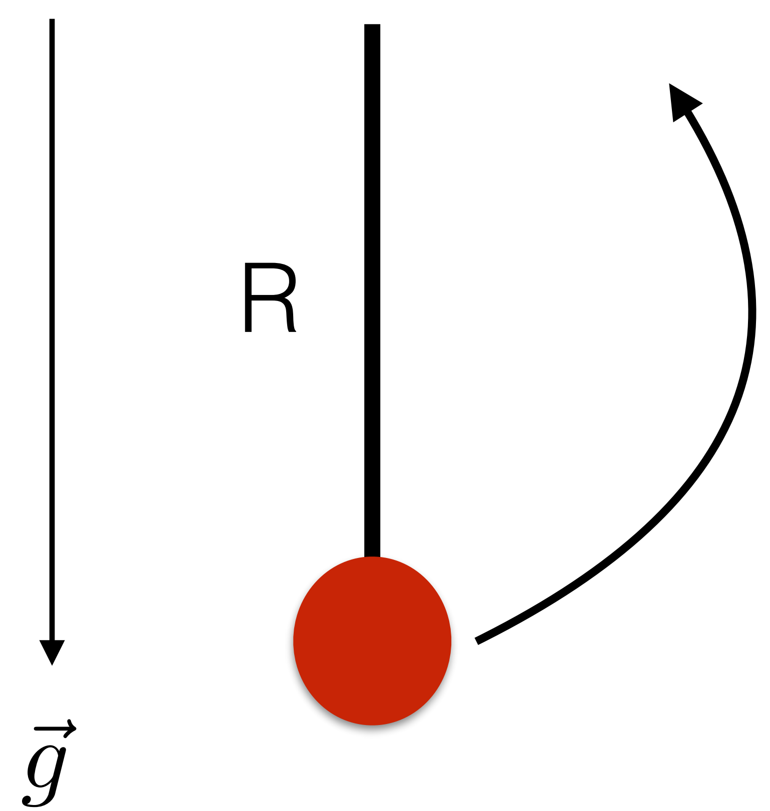

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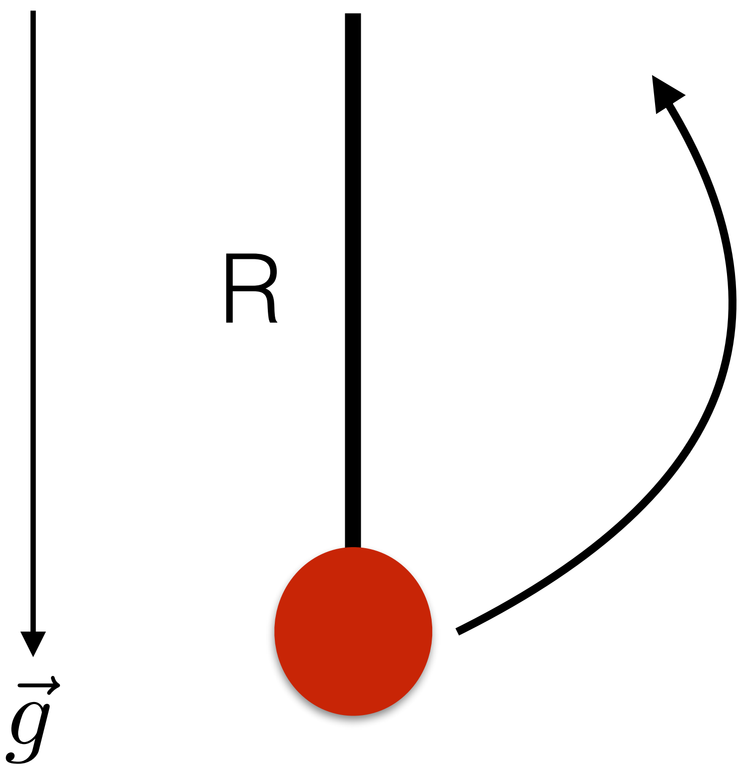

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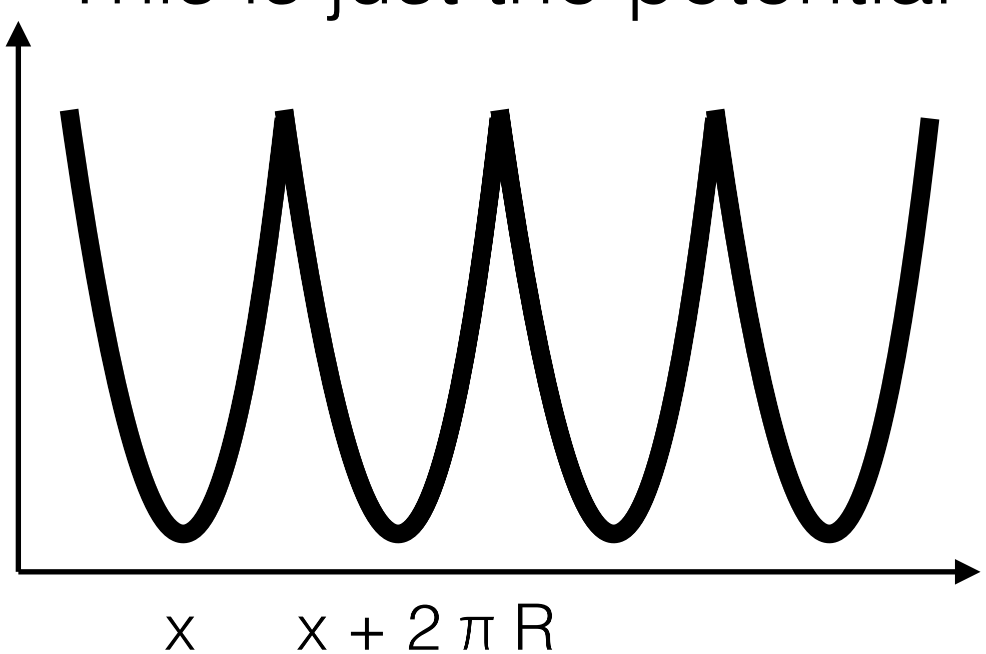
First of all, what does the potential even look like?

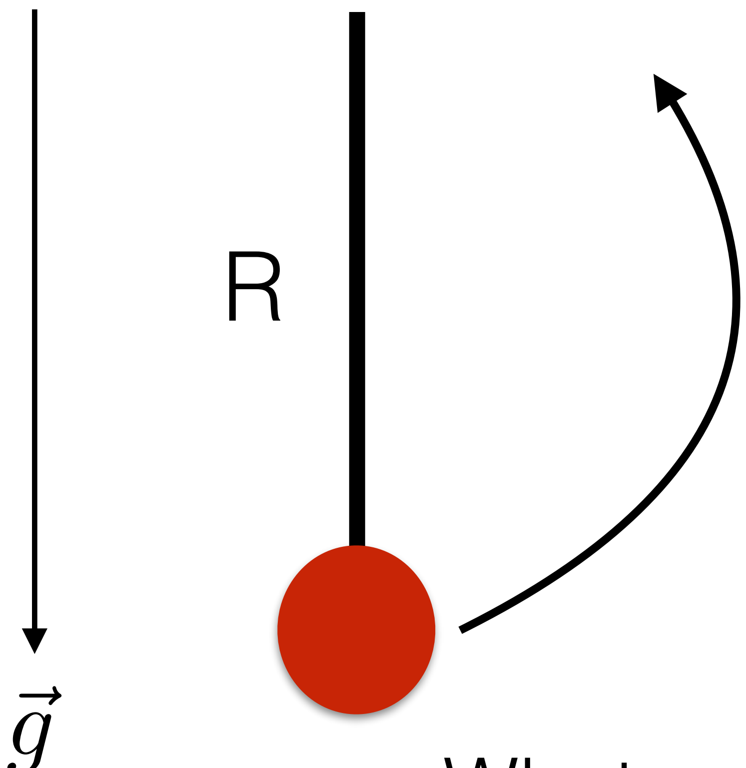
This is just the potential of the pendulum and it is periodic in x

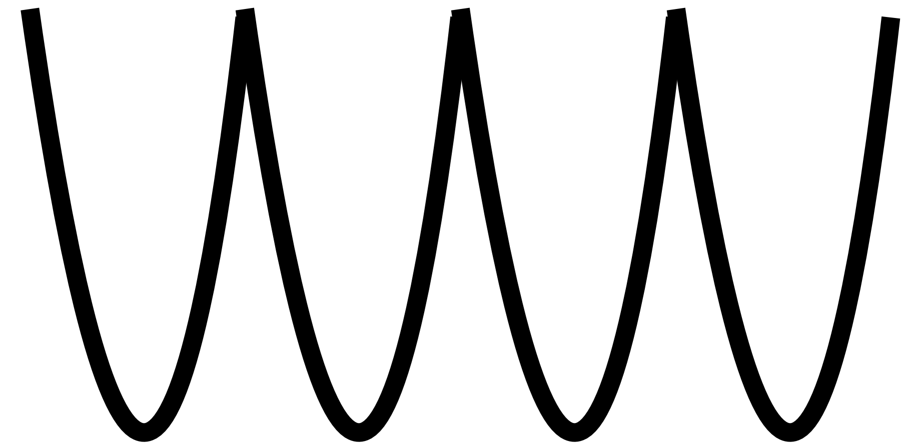


Again, classically, the θ term has no effect - what about quantum mechanically?

Surface Terms in Quantum Mechanics

Rigid Pendulum swinging under gravity





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What are the quantum mechanical effects of θ?

Unlike the previous potential which was basically a particle in a box, the eigenstates of this potential have interesting new features. For one, there is quantum tunneling - the bob tunnels through a full rotation going to the top of the pendulum (the potential barrier) and then dropping down

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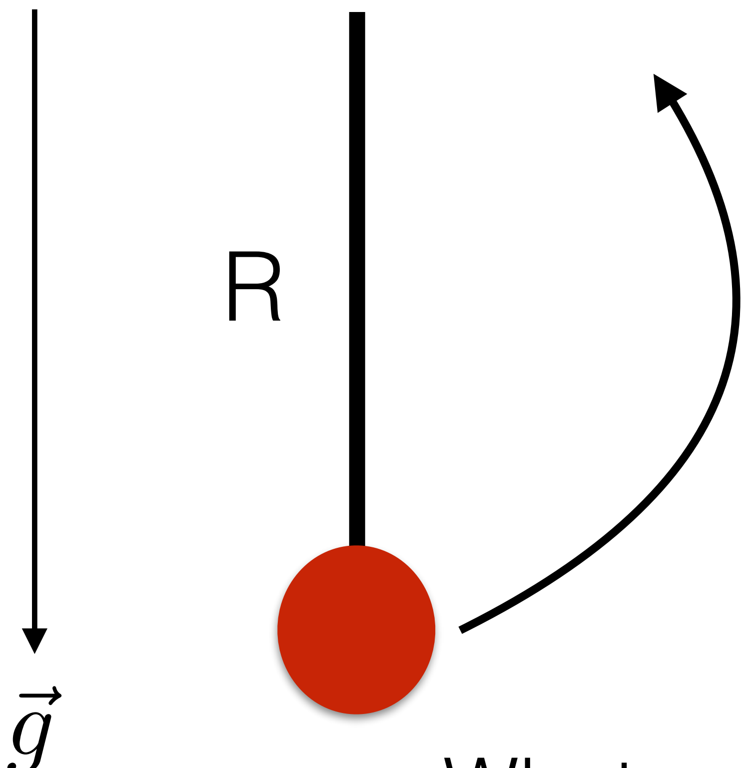

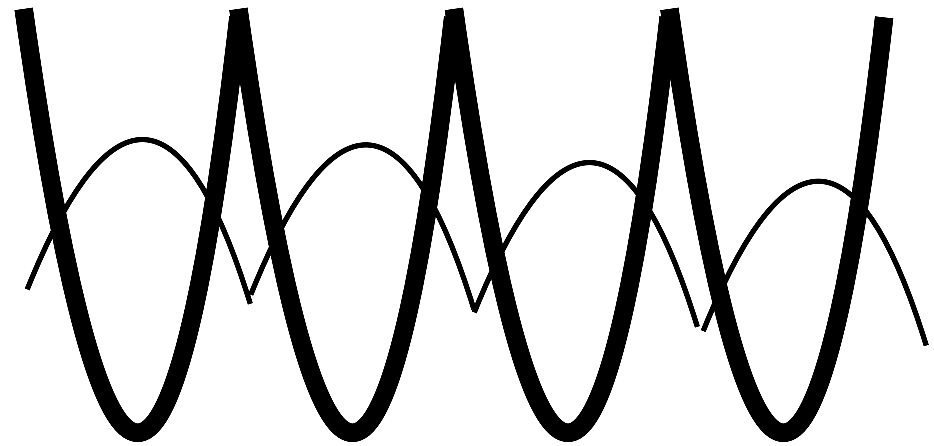

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The ground state is a super-position of states localized at various minima - and the eigen-energies are corrected by tunneling amplitudes

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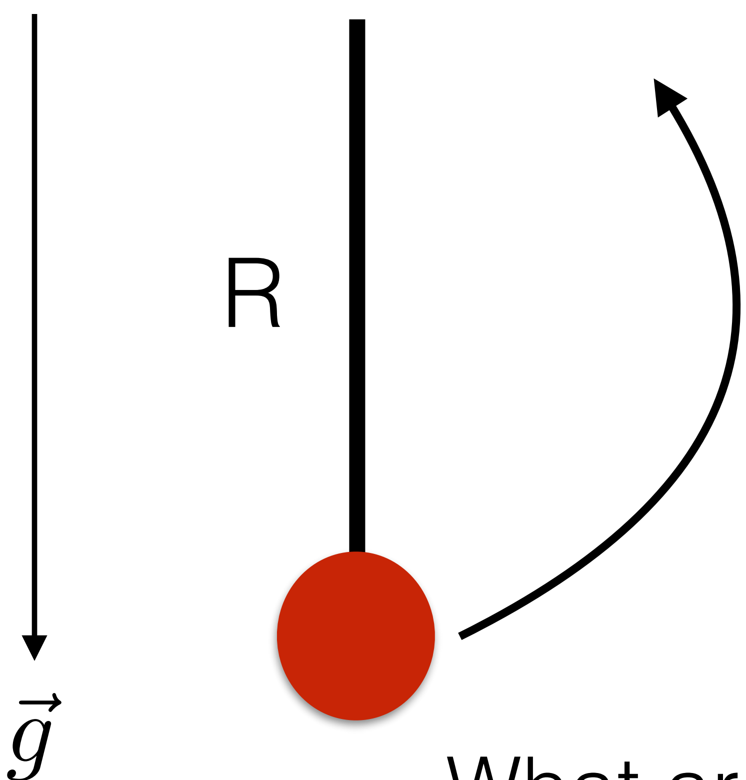

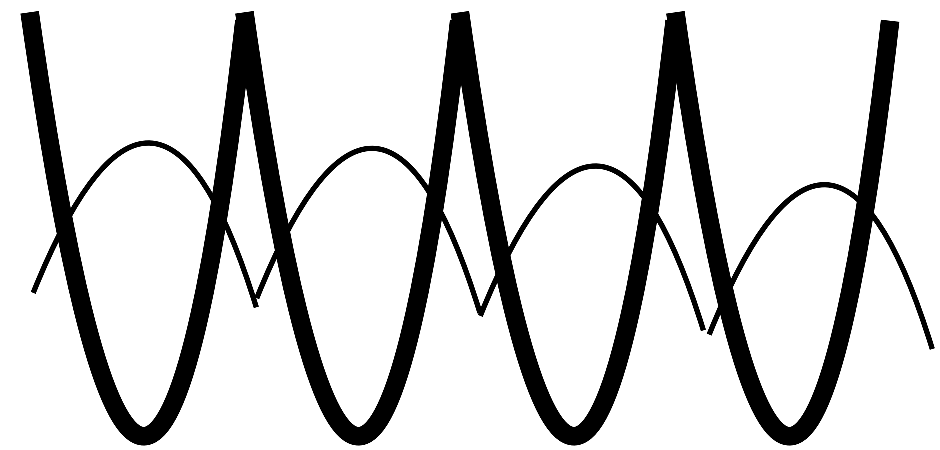

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One could of course use standard tricks like the WKB approximation or the treatment of Bloch-waves in condensed matter systems where periodic potentials are common.

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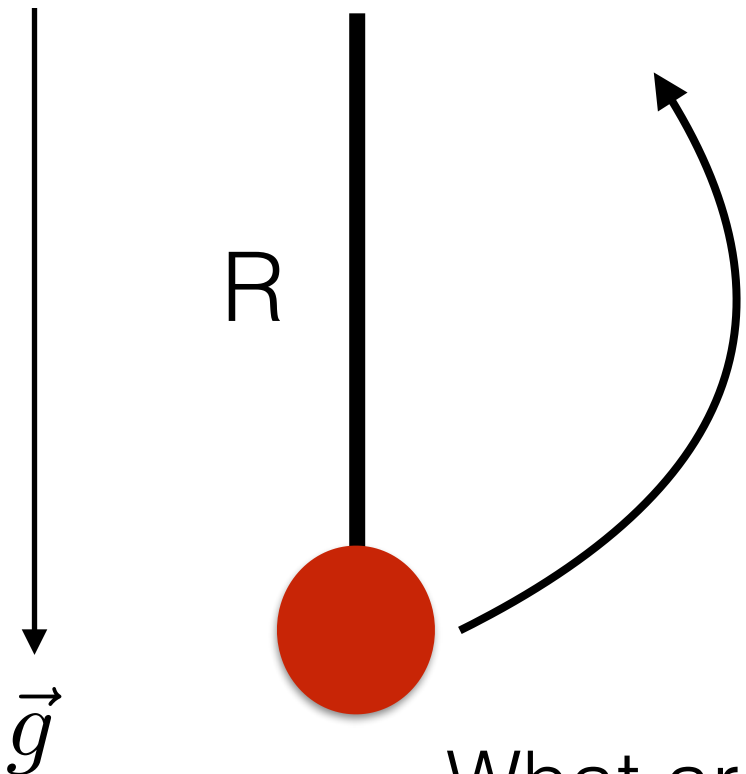

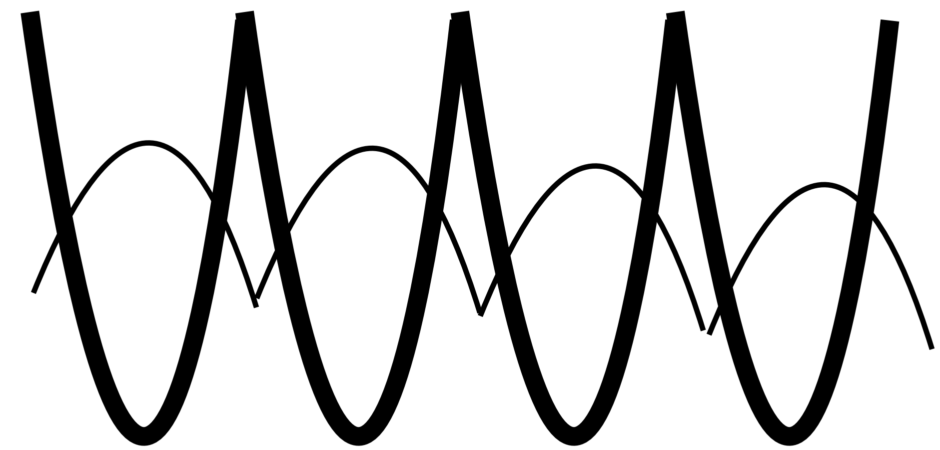

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Instead, we will try a new kind of approximation that is most suited to finding just the energy of the ground state - this is unwieldy for this problem but can be generalized easily to field theory

Ground State Energy Calculation

Given a Hamiltonian H, what technique can we use to obtain its ground state energy (approximately)?

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Consider eigenstates of position $|x_i\rangle$, $|x_f\rangle$

Try to calculate $\langle x_f| e^{-HT} |x_i\rangle$, in the limit that T is very large

$$\langle x_f|e^{-HT}|x_i\rangle = \sum_{n,m}\langle x_f|n\rangle\langle n|e^{-HT}|m\rangle\langle m|x_i\rangle = \sum_n \langle x_f|n\rangle e^{-E_nT}\langle n|x_i\rangle$$

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$$\langle x_f|e^{-HT}|x_i\rangle \approxeq \langle x_f|0\rangle e^{-E_0T}\langle 0|x_i\rangle \text{ for large enough T}$$

This is good, but not yet sufficient since we don't know the ground state wave function.

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Using Feynman's path integral approach

$$\langle x_f | e^{-HT} | x_i \rangle = N \int d\gamma e^{-S_E[\gamma]}$$

That is, we can draw all paths connecting x_i to x_f, evaluate the action on them and then perform the path integral. If this path integral can be approximated, then we have evaluated the right hand side. As we have seen, the left hand side is dominated by the ground state energy - so if the right hand side can be estimated, we are done

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Of course the action that appears here is the Euclidean action since the operator on the left is not the unitary Hamiltonian evolution operator but instead its analytic continuation into imaginary time: t -> i τ

What is this Euclidean action S_E?

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What is this Euclidean action S_E?

$$S = \int dt \left(\frac{\dot{x}^2}{2} - V\left(x\right) + \theta \dot{x} \right)$$

Take the usual action and analytically continue t -> i τ

$$S_E = \int d\tau \left(-\left(\frac{\dot{x}^2}{2} + V\left(x\right) \right) + i\theta \dot{x} \right)$$

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Let us persist and see if we can get new effects. How do we calculate S_E? What kind of approximations can we use?

Ground State Energy Calculation

$$\langle x_f | e^{-HT} | x_i \rangle = N \int d\gamma e^{-S_E[\gamma]}$$

Clearly, paths of minimum Euclidean action dominate this integral - of course, the exact paths of minimum action form a set of zero measure and do not contribute to the integral. But paths near such minima form a set of non-zero measure and dominate this integral

What are the paths that minimize the Euclidean Action?

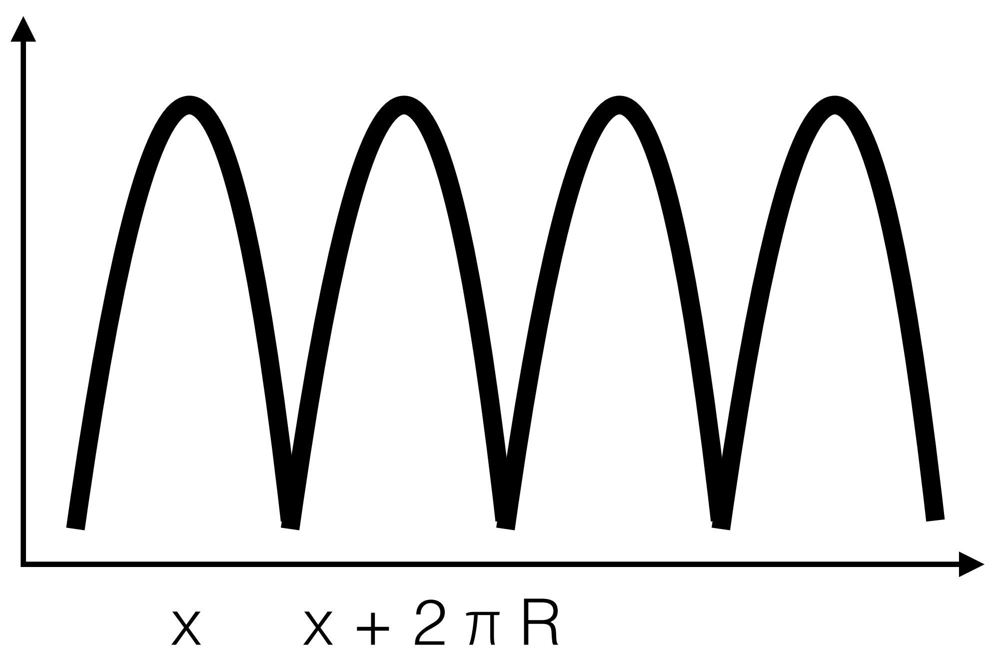
$$S_E = \int d\tau \left(- \left(\frac{\dot{x}^2}{2} + V(x) \right) + i\theta \dot{x} \right)$$

We know the answer to this question - these are simply given by the classical trajectory taken by a particle to go from x_i to x_f. Except, the particle moves in a potential -V(x)

Ground State Energy Calculation

$$\langle x_f | e^{-HT} | x_i \rangle = N \int d\gamma e^{-S_E[\gamma]}$$

$$S_E = \int d\tau \left(- \left(\frac{\dot{x}^2}{2} + V(x) \right) + i\theta\dot{x} \right)$$



Dominated by paths near paths of minimum action. Paths of minimum action are the classical trajectories of a particle that goes from x_i to x_f in a potential -V

Note, we can take $x_f = x_i$ - this does not change any of our approximations. So we are simply asking what are the classical paths that a particle starting at x_i can take so that it comes back to x_i while moving in a potential -V

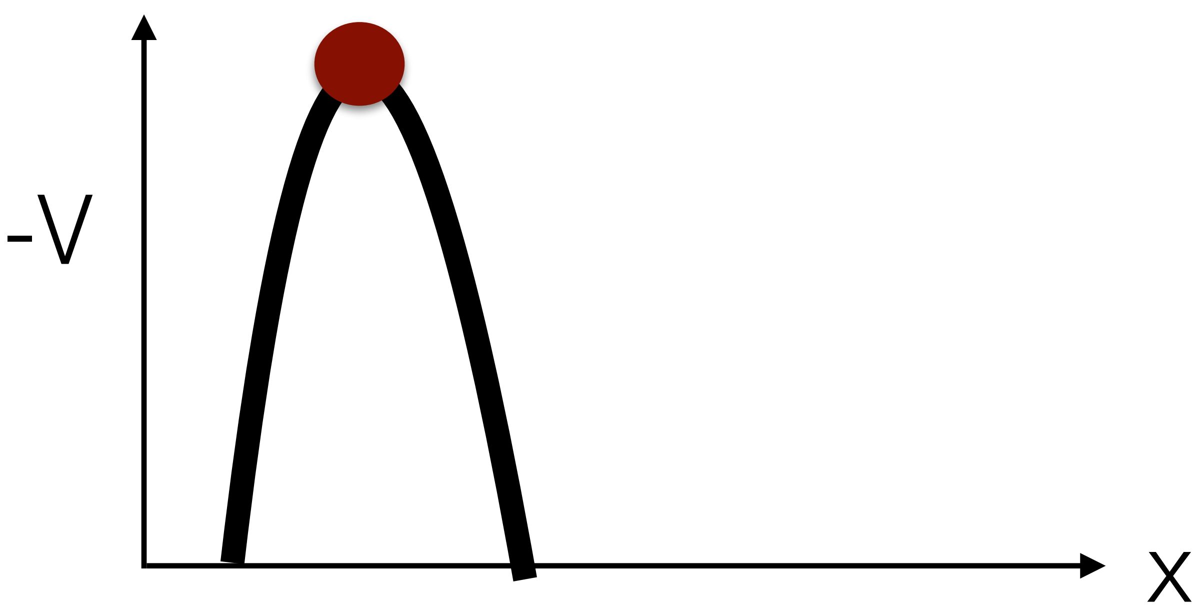
Ground State Energy Calculation

$$\langle x_i|e^{-HT}|x_i\rangle = N\int d\gamma e^{-S[\gamma]} \quad S_E = \int d\tau\left(-\left(\frac{\dot{x}^2}{2} + V\left(x\right)\right) + i\theta\dot{x}\right)$$

Dominated by paths near paths of minimum action. Paths of minimum action are the classical trajectories of a particle that goes from x_i to x_f in a potential -V

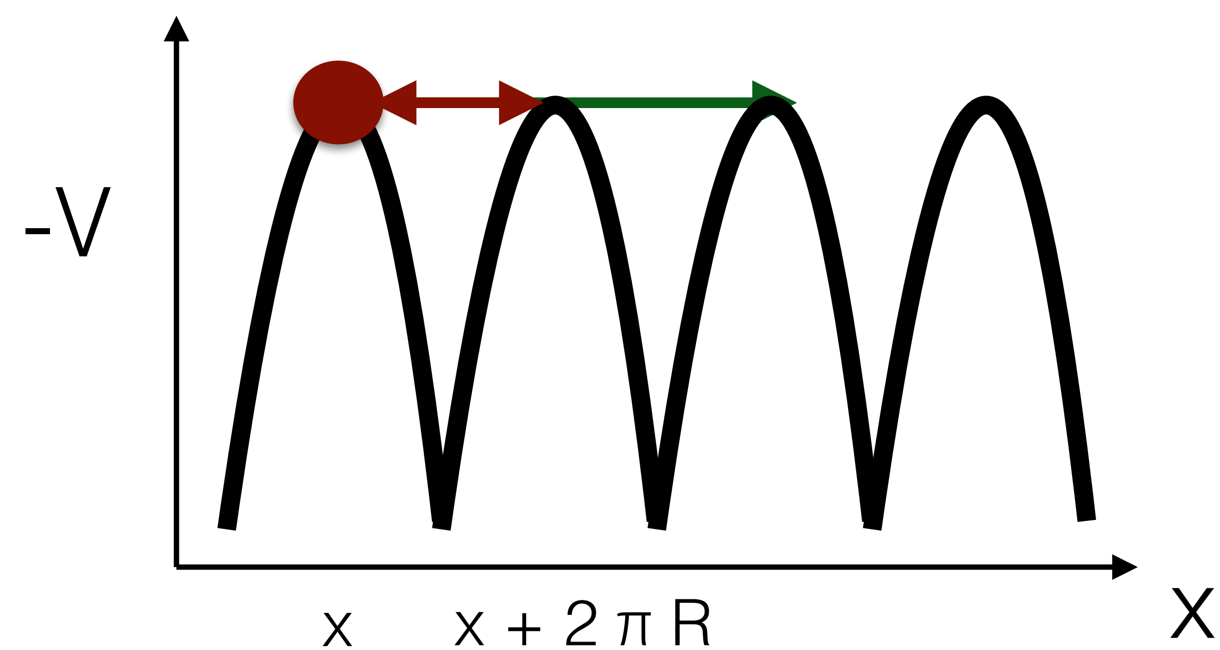
Now we can see a difference between some generic 1D potential we considered earlier and the pendulum

Particle in a Box



Only one path that stays in the same place

Pendulum



Multiple paths - they bounce back and forth

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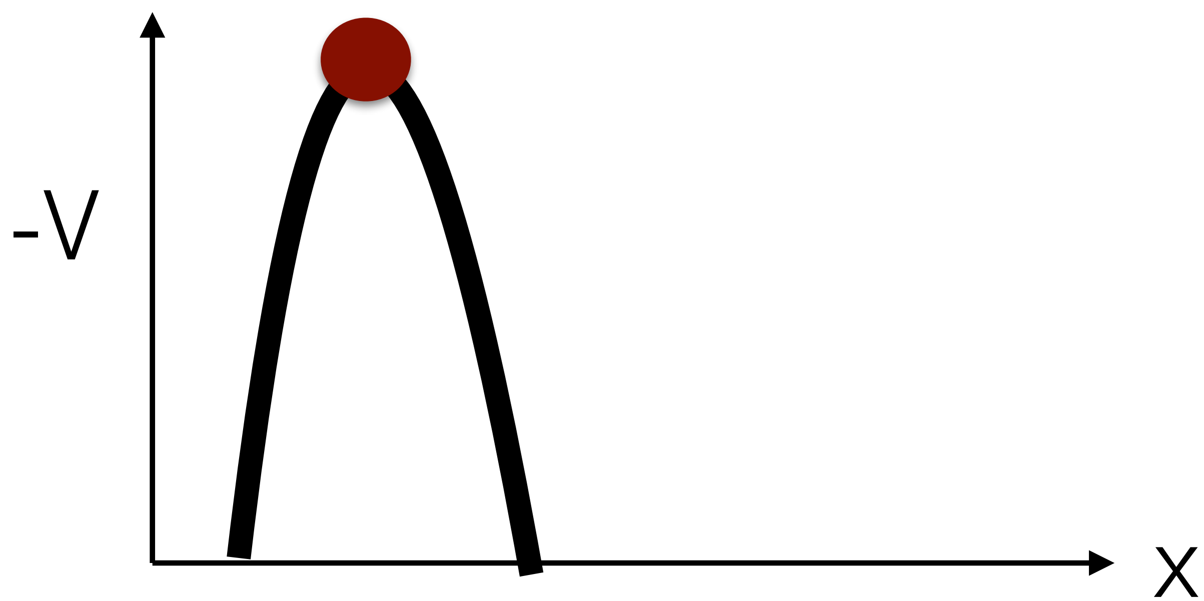

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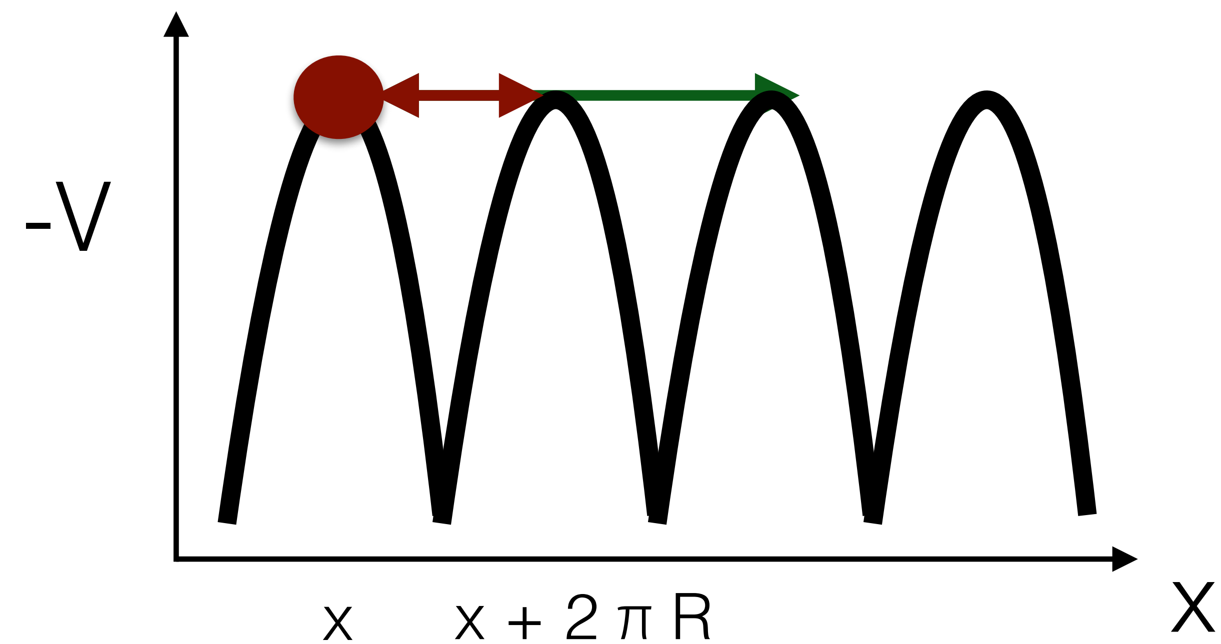


Multiple paths - they bounce back and forth

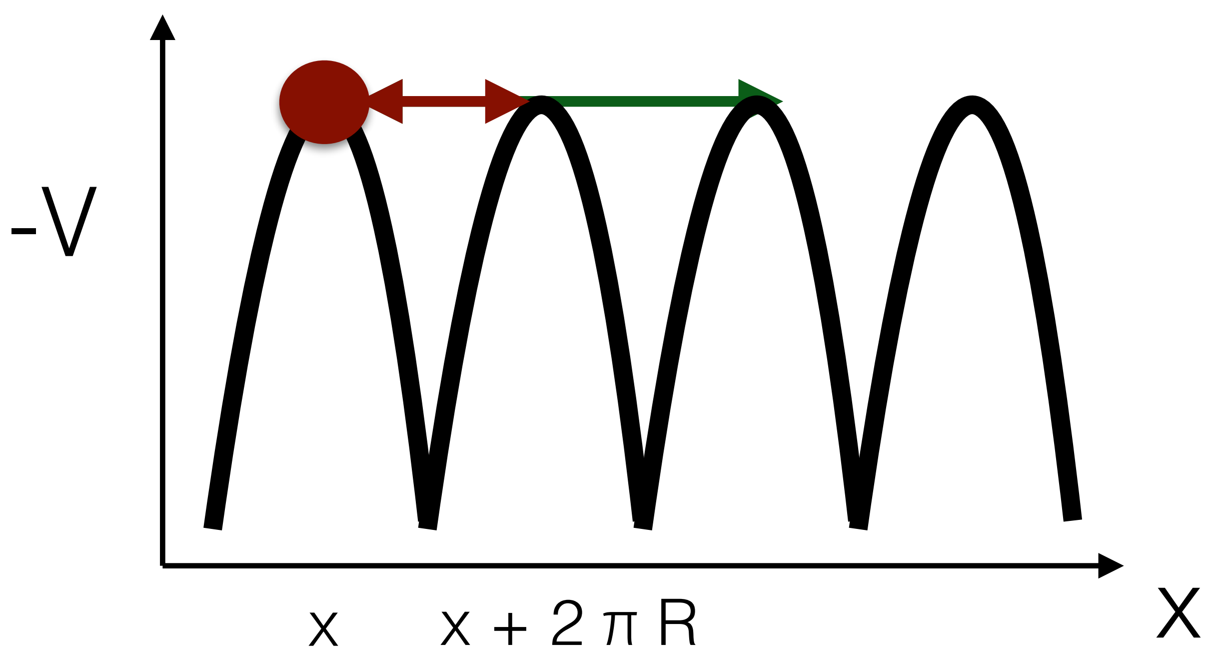
These additional paths are called instantons - they correct the ground state energy by some exponentially small amount. This corresponds to tunneling - which of course was absent for the particle in a box example

Tunneling is intrinsically expected in the case of the pendulum and we expect it to change the energies at some exponentially suppressed way - - this is exactly what these additional (instanton) paths do

Ground State Energy Calculation

$$\langle x_i|e^{-HT}|x_i\rangle = N\int d\gamma e^{-S[\gamma]} \quad S_E = \int d\tau\left(-\left(\frac{\dot{x}^2}{2}+V\left(x\right)\right)+i\theta\dot{x}\right)$$

Pendulum



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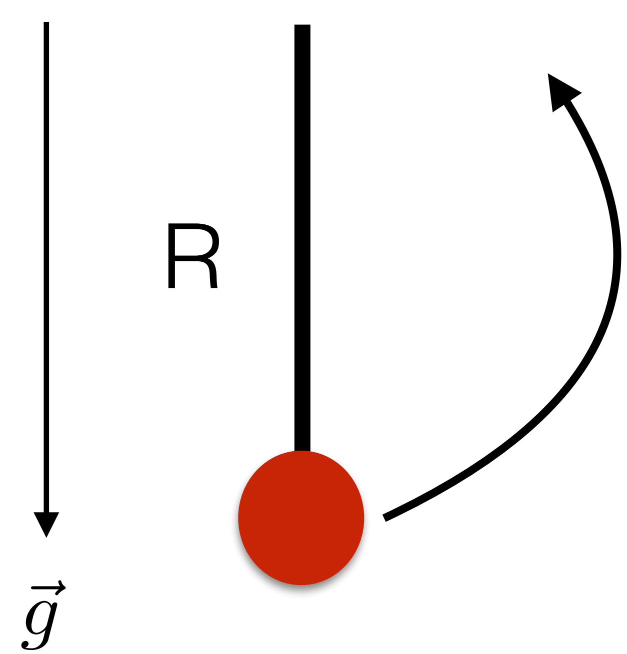

Instanton solutions, characterizing tunneling, correct the ground state energy of this system

What does the θ term do to these instantons?

Notice that there are distinct classes of instantons - there are solutions that go around the circle once, others that go around the circle twice and so on. So in general, there is a winding number associated with an instanton solution.

Ground State Energy Calculation

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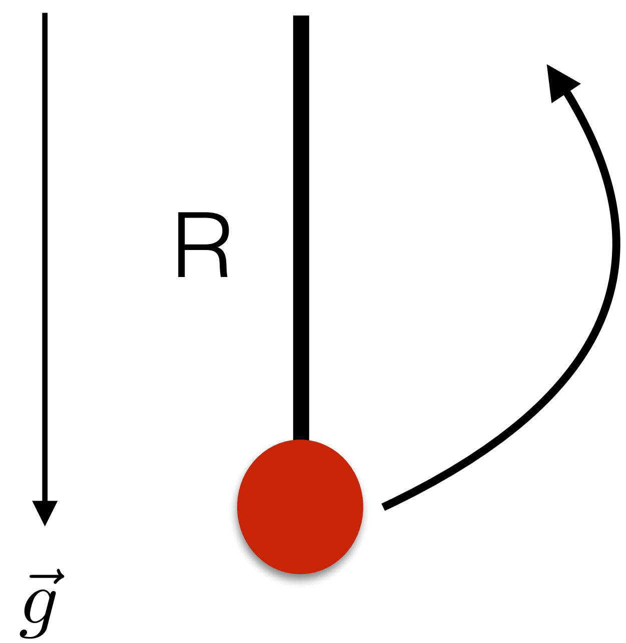

What does the θ term do to these instantons?

There is a winding number associated with each instanton

Now we can deform two instantons that have the same winding number into each other by continuously deforming them - however, we cannot smoothly deform instantons of different winding numbers to each other. These are topologically distinct solutions

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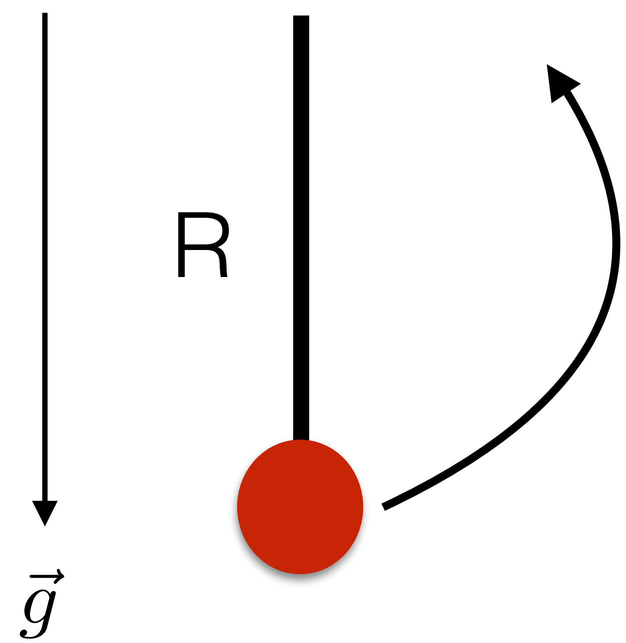

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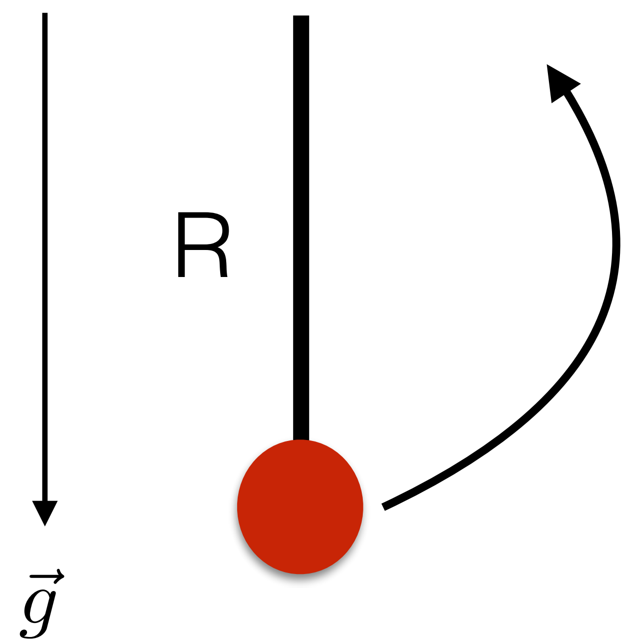
$$\text{Thus} \int d\tau \left(i\theta\dot{x}\right) \propto i\theta n, \text{ where } n = \text{ winding number}$$

So θ term provides a phase telling us how to add instantons of different winding number together - this term is thus absolutely physical!

Such an effect was absent for the particle in a box because there were no instantons there

Calculation Details

$$\langle x_i | e^{-HT} | x_i \rangle = N \int d\gamma e^{-S[\gamma]} \quad S_E = \int d\tau \left(- \left(\frac{\dot{x}^2}{2} + V(x) \right) + i\theta \dot{x} \right)$$



θ term provides a phase telling us how to add instantons of different winding number together - this term is thus absolutely physical!

So we see how the story works - we expect the ground state energy of the pendulum to be corrected by tunneling. This tunneling can be calculated by looking at the Euclidean path integral.

The path integral is dominated by paths that are near the path of minimum action. The path of minimum action is given by the classical solution to the equations of motion of a particle moving along a potential -V

If there are multiple, topologically distinct instantons then the θ term becomes physical - it tells you how to add these instantons together

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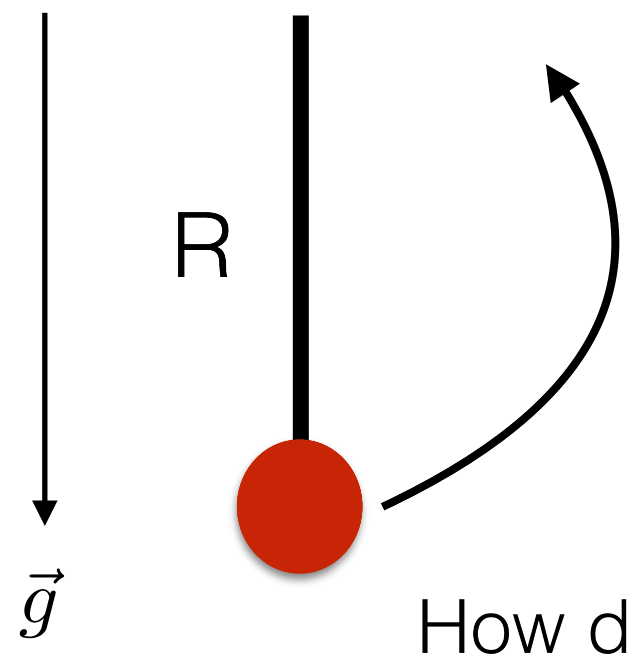


If there are multiple, topologically distinct instantons then the θ term becomes physical - it tells you how to add these instantons together

How does the calculation actually work?

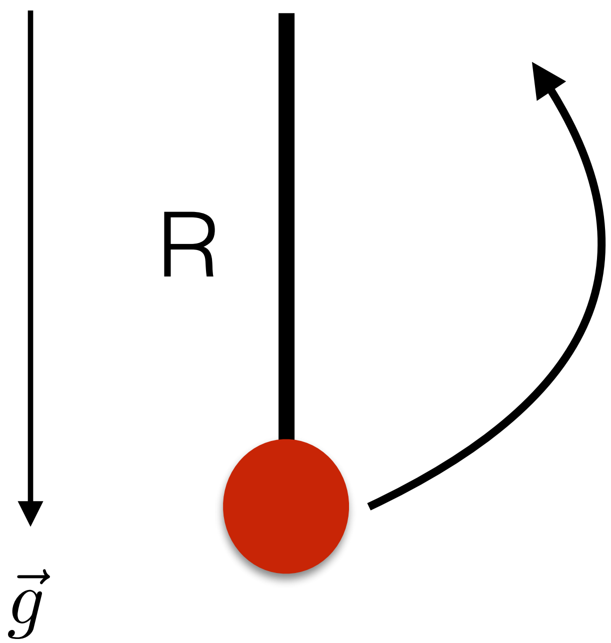
You can look at Coleman's book for the details of the calculations - the basic idea is that once you find the classical instanton solutions X_I, you perturb about them

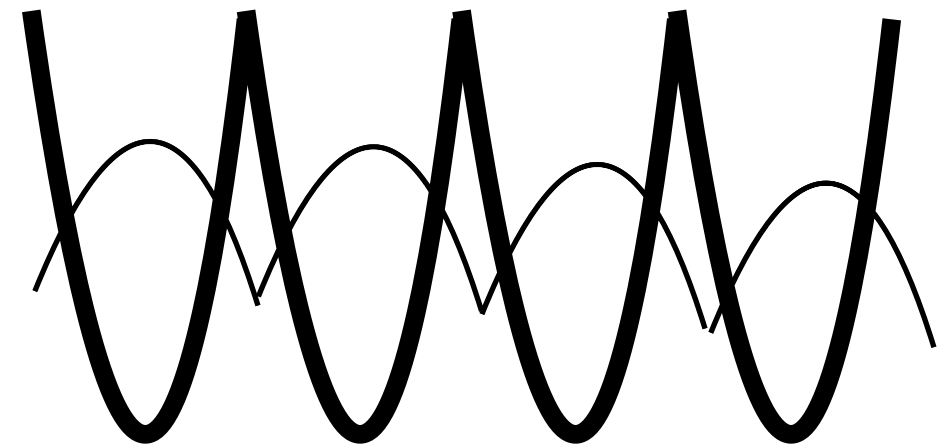
$$x\left(\tau\right) = X_I\left(\tau\right) + \sum_n c_n x_n\left(\tau\right)$$

The x_n are a complete set of orthonormal functions and with this ansatz one calculates the functional integrals

Unlike the original instanton X_I these perturbations are not a set of measure zero and hence contribute (and in fact dominate) the path integral

Effects of θ





$$\mathcal{L} = \frac{\dot{x}^2}{2} - V\left(x\right) + \theta \dot{x}$$

Because of tunneling, for a pendulum under gravity, the surface term θ actually has physical effects

After grunging through the calculations (see Coleman for details), we have the final answer:

$$E = \frac{1}{2}\omega + 2K \cos\theta e^{-S_0}$$

e^{-S_0} barrier penetration factor - proportional to the Euclidean action of instanton with winding number 1

K a complicated pre-factor that comes from functional integration

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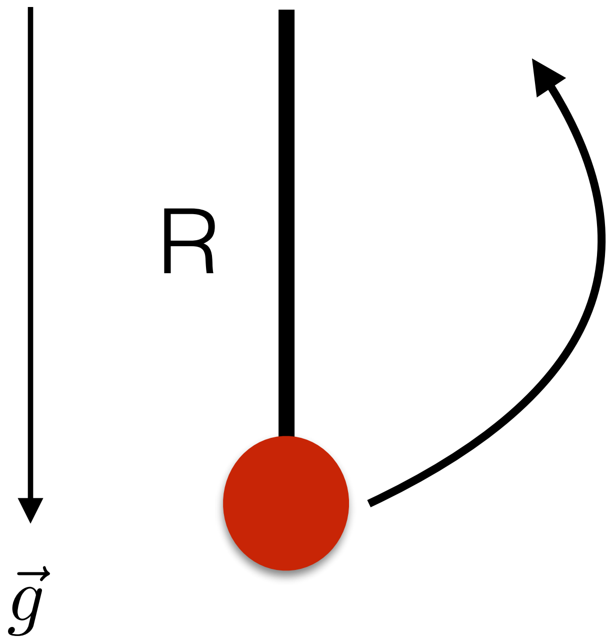

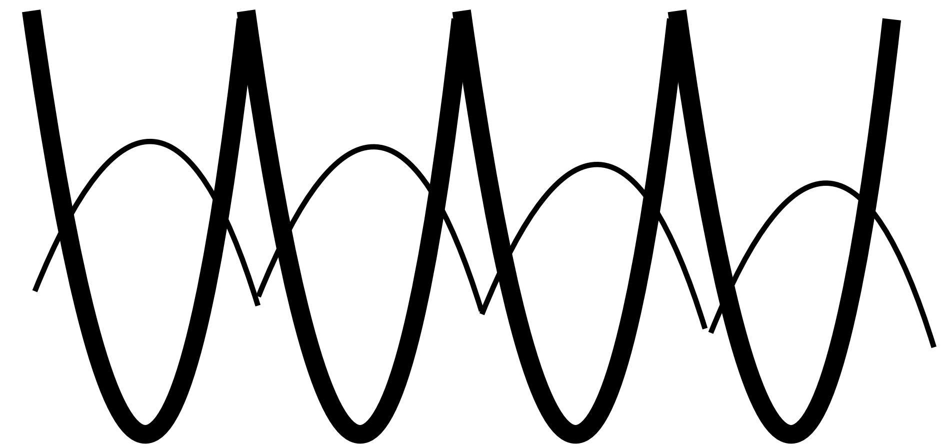

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Key Point: θ actually changes the spectrum - this is non-perturbative

Surface Terms and Quantum Mechanics

What does all this have to do with axions? After all, we said axions were just goldstone bosons of some broken symmetry

Surface Terms and Quantum Mechanics

What does all this have to do with axions? After all, we said axions were just goldstone bosons of some broken symmetry

The point is, we were trying to see why people introduced axions in the first place. To that end, we began by asking why we never considered terms of the form:

$$\mathcal{L} \supset \underbrace{-F_{\mu\nu}F^{\mu\nu} + eA_\mu}_{\text{Standard}} J^\mu + \underbrace{\theta F_{\mu\nu}\tilde{F}^{\mu\nu}}_{?}$$

We said that unless there was a magnetic monopole, the third term was a surface term that had no effects on physics. But as our example with the pendulum demonstrates, these terms can have non-perturbative physical effects, arising from topology

Surface Terms in Gauge Theories

Much like our analysis of the quantum mechanics problem, we can use the Euclidean path integral to calculate the ground state energy of a gauge theory

For now, let us just consider a gauge theory. The theory can be Abelian or non-Abelian. To start with, let us not put in any fermions charged under this gauge group. We will eventually include them, but let us initially ignore them for simplicity.

$$\mathcal{L} \supset -F_{\mu\nu}F^{\mu\nu} + \theta F_{\mu\nu}\tilde{F}^{\mu\nu}$$

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Given this Lagrangian,we can compute the Euclidean Path Integral:

$$\int [dA] e^{-S_E[A]}$$

The integral is of course performed over the vector potential A of the gauge theory

Surface Terms in Gauge Theories

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$$\int [dA]e^{-S_E[A]}$$

Again, this integral will be dominated by perturbations around solutions to the Euclidean equations of motion

The Euclidean equations of motion are of course given by a differential equation:

$$D_\mu F^{\mu\nu} = 0$$

To solve these equations and obtain classical solutions, we have to impose boundary conditions

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What sort of boundary conditions are appropriate?

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What sort of boundary conditions are appropriate?

The action at the end of the day is some integral performed over all space. If this integral diverges (i.e. the solution has infinite energy), then those solutions do not contribute to the path integral

So we should try to find solutions to the equations of motion that have finite energy. These will be our instantons. Of course, as in the prior quantum mechanical case, the path integral will be dominated by perturbations about these instanton solutions

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For the solutions to have finite energy, the **field strength (F)** must vanish at infinity

Key Point: Vanishing of F does not imply that the potential A must vanish at infinity

Surface Terms in Gauge Theories

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What sort of boundary conditions are appropriate?

Need solutions where F vanishes at infinity - but vanishing of F does not mean that the gauge potential A needs to vanish

This means, we seek boundary conditions where the potential is pure gauge out at infinity. It will have non-zero field strength in a finite region and thus contribute as a finite energy solution to the Euclidean Path Integral

That is to say, we look for maps between the boundary of space-time (S^3 if the space-time is R^4) to the gauge group G since the gauge potential is simply a map from this boundary picking up values in G.

Given such a map, we can ask if it can be continuously deformed to another map - if so, we shouldn't count it separately, since it will be part of the functional integral anyway (where we integrate over all perturbations around a classical solution)

Surface Terms in Gauge Theories

$$\int [dA] e^{-S_E[A]}$$

$$D_\mu F^{\mu\nu} = 0$$

What sort of boundary conditions are appropriate?

Look for maps from S^3 to G - ask if these maps can be continuously deformed to one another

This is a math question and can be answered by talking to your friends who know some topology. For example, if the gauge group G is an Abelian U(1) (like electromagnetism), then all maps from the S^3 (the boundary of R^4) to U(1) can be continuously deformed to the identity. So there is only one finite energy solution to the Euclidean equations of motion which is the trivial one where the potential vanishes at infinity.

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This conclusion of course depends on the geometry of space-time. For example, if we lived in 1+1 dimensions, the boundary of R^2 is S^1. In this case, there are non-trivial maps from this boundary S^1 into the U(1) gauge group, which is also S^1. These maps are characterized by a winding number - exactly as in the case of the pendulum

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What sort of boundary conditions are appropriate?

Look for maps from S^3 to G - ask if these maps can be continuously deformed to one another

Turns out that for any SU(N), there are non-trivial maps from S^3 (the boundary of R^4) to the gauge group SU(N).

Each such map picks a boundary condition that leads to a finite energy solution to the Euclidean equations of motion

We characterized maps on S^1 using a winding number - for these higher dimension maps there is something called the Pontryagin index. This index is an integer that performs a similar role - in particular, two maps that can be deformed into one another have the same index

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What about the surface term θ?

$$\mathcal{L} \supset -F_{\mu\nu}F^{\mu\nu} + \theta F_{\mu\nu}\tilde{F}^{\mu\nu}$$

What is the value of the surface term on these instanton solutions?

Surface Terms in Gauge Theories

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Can show that the surface term integrates to the Pontryagin index - a number that characterizes the instanton

Exactly like in the case of the pendulum under gravity, the parameter θ is a phase that tells us how to add different classes of instantons together

Surface Terms in Gauge Theories

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Exactly like in the case of the pendulum under gravity, the parameter θ is a phase that tells us how to add different classes of instantons together

After a lot of grunge (follow Coleman), one can show that much like the case for the pendulum, the ground state energy density of the field theory is corrected by these instanton solutions by terms of the form

$$E\left(\theta\right) \sim K \cos\theta e^{-S_0}$$

Here K is an awful functional determinant and S_0 is the suppression factor associated with an instanton of index 1. **θ has become physical!**

Summary of Surface Term in Gauge Theories

$$\mathcal{L} \supset -F_{\mu\nu}F^{\mu\nu} + \theta F_{\mu\nu}\tilde{F}^{\mu\nu}$$

Much like our initial quantum mechanical pendulum, the ground state energy of this field theory can be calculated using the Euclidean path integral

This integral is dominated by finite energy solutions of the Euclidean field equations - these finite energy solutions are characterized by the fact that they have vanishing field strength at infinity and are thus pure gauge

To understand these solutions, we look at maps from the boundary of the space-time to the gauge group. Depending upon the topology of the space-time, these maps may or may not be trivial. For example, in 1+1 dimensions, there are non-trivial maps from the boundary S^1 to U(1), while no such maps exist from the boundary S^3 (of R^4) to U(1). There are non-trivial maps from S^3 to SU(N)

When there are such non-trivial maps, the surface term θ gives a phase that determines how the different instanton solutions should be added together. The term becomes physical!

Summary of Surface Term in Gauge Theories

$$\mathcal{L} \supset -G_{\mu\nu}G^{\mu\nu} + \theta G_{\mu\nu}\tilde{G}^{\mu\nu}$$

Since we live in R^4, there are no interesting instantons for U(1) electromagnetism. However, there are non-trivial instantons for SU(2) and SU(3)

Going through the calculation of the instanton contributions (follow Coleman again), one can show that the exponential suppression Exp[$-S_0$] associated with the instanton is Exp[$-8\pi^2/g^2$] where g is the gauge coupling

In the standard model, for the SU(2) electroweak gauge group, $g_2 << 1$. So its instanton corrections are negligible. For QCD, g_3 hits strong coupling and hence the exponential suppression disappears

So the question is, what are the effects of θ on QCD?

Next Class: Strong CP problem, Axion solution and its phenomenology