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EFT Methods for Binary Systems (w Rothstein, Ross '04–'12)

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Lecture #1

useful ref: Les Houches lectures arxiv.org/abs/hep-ph/0701129

Motivation: An important class of astrophys. (non-cosmo) sources consists of binary inspirals of compact objects (see M. Branchesi ~~for~~ more detail about sources + exps.)

For isolated objects of mass $\sim M$, compact means

$$r_{phys} = \text{"size"} \gtrsim r_g = \text{Grav. radius} = 2 G_N M \quad (c=1)$$

(aka Schwarzschild radius r_s)

which is typically satisfied for BH or NS w/

~~m_{NS} ... $m_\odot$~~

$$m_{NS} \simeq 1.4\, m_\odot \Rightarrow r_g \sim G m_\odot \sim \text{k}$$

and $r_{phys} \sim 10\,\text{km} \simeq 10\, r_g$.

~~We are interested in calculating waveforms of GWs~~

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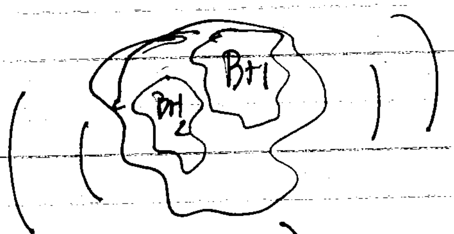
While an isolated compact object is stable in GR, a binary system w/ BH/NS constituents is unstable due to the emission of GW's.

Our goal is to calculate the spectrum of GW's seen by a detector a distance $r \to \infty$ from the "bound state":



$g_{\mu\nu}(x)$

$$g_{\mu\nu}(x^0; |\vec{x}| \to \infty) \to \eta_{\mu\nu} + h_{\mu\nu}(x)$$

(in suitable coords $x^\mu = (x^0, \vec{x})$ at infinity.)

or in frequency space: $h_{\mu\nu}(|\vec{x}| \to \infty, \omega) = \int dt\, e^{-i\omega x^0} h_{\mu\nu}(\vec{x}, x^0)$

If we focus on the "simpler" case of BH/BH binaries, then to do this calculation "all" we need to do is solve the vacuum Einstein eqns of classical GR

$$R_{\mu\nu}[g] = 0$$

subject to suitable initial conditions. Once we found the soln $g_{\mu\nu}(x)$, we go to coords such that asymptotically (at large dist from source) $x^\mu \to (x^0, \vec{x})$, $g_{\mu\nu}(|\vec{x}| \to \infty) = \eta_{\mu\nu} + h_{\mu\nu}$

and then read off the answer). Since $R_{\mu\nu} = 0$ is invariant under const. Weyl transformations

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}, \quad R_{\mu\nu} \to R_{\mu\nu}$$

(dilatations), it follows that the ~~dynamics~~ shape of the GW spectrum as a fn. of freq ω depends on the parameter

$$\epsilon(\omega) = r_s \omega$$

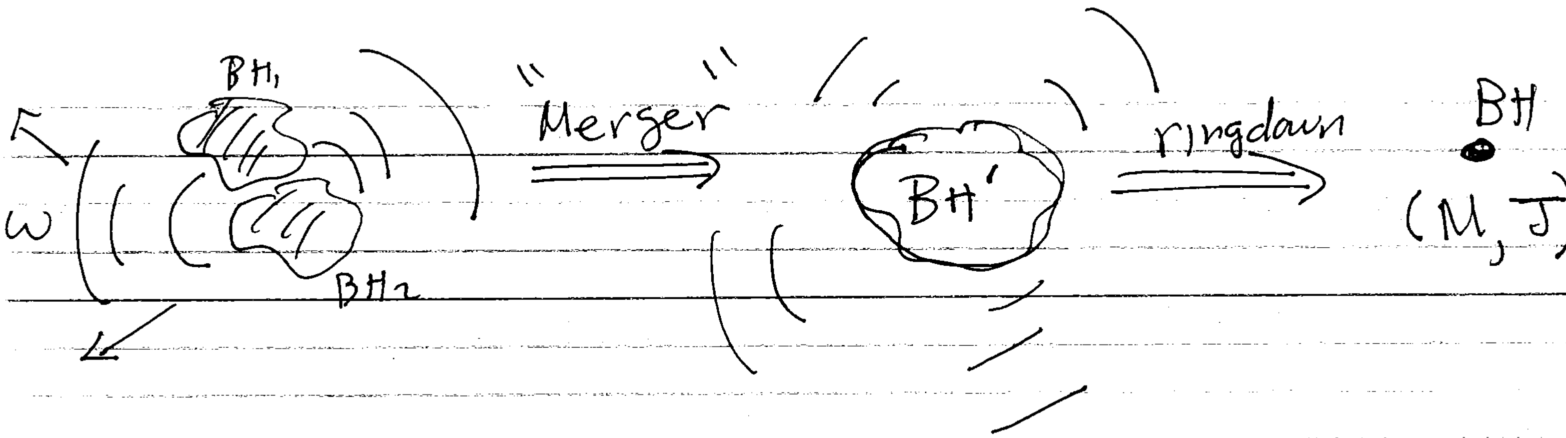
the two regimes are:

(High freq.)
$r_s\omega \sim \mathcal{O}(1)$: In this limit, $\lambda_{GW} \sim 2\pi/\omega$ is of order r_g:

$$\frac{\lambda_{GW}}{r_g} \gtrsim \mathcal{O}(1) \Longrightarrow \text{Non-linear GR}$$

Thus the BH pair is "close together". The system has no symmetries nor small exp. parameter, so this is only tractable by numerical GR.

Despite the highly non-linear nature of ~~GR~~ problem, ~~of the [illegible]~~. It is (almost) certain that in GR, the final state is simple:



The ringdown phase (unlike the non-linear merger) can be analyzed semi-analytically, via BH perturbation theory

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x)$$

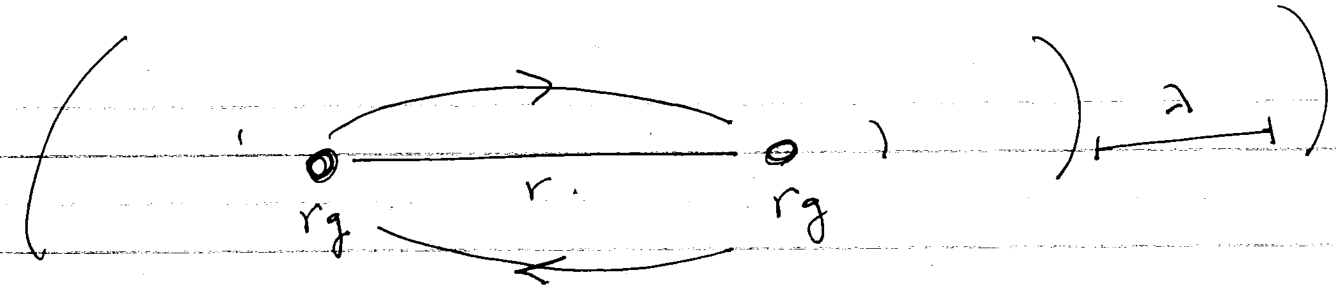
$$R_{\mu\nu}(g) = 0 \Rightarrow \text{"}\Box h_{\mu\nu}(x) = 0\text{"}$$

Regge-Wheeler eqn.

where $\bar{g}_{\mu\nu}(x)$ = Kerr metric for BH w/ (M, J).

Low freq. region : $r_S\omega \ll 1$. In this case GW's have wavelength $\lambda_{GM}/r_g \gg 1$

Thus gravity is weak, and we can do perturbation theory around flat space. The system can be regarded as a set of point masses, approximately interacting via Newtonian gravity:



$r_g \ll r \ll \lambda$

In the Newtonian limit w/ $v \equiv$ typ. velocity

(virial thm) $\frac{GM}{r} \sim v^2 \sim r_g/r \ll 1$

The orbital mechanics is described to an excellent approx by

$$L = \frac{1}{2} m_1 \dot{\vec{x}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{x}}_2^{\,2} + \frac{G m_1 m_2}{r}$$

$(r = |\vec{x}_1 - \vec{x}_2|)$

w/ mainly quadrupolar ($l = 2$) GW emission

$$-\frac{dE}{dt} = \text{mech. energy loss} = P_{GW}$$

w/ $$P_{GW} = \frac{G}{5} \langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle$$ $\langle \cdots \rangle$ = time av.

$$Q_{ij} = \sum_a m_a \left(x_a^i x_a^j - \frac{1}{3} \vec{x}_a^{\,2} \delta^{ij} \right)$$

(For circular orbits; $M = m_1 + m_2$ μ = red. mass = $m_1 m_2 / M$

$$E = -\frac{1}{2} \mu v^2 \qquad P_{GW} = \frac{32}{5} \frac{v^{10}}{G_N} \left(\frac{\mu}{M} \right)^2$$

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where $v = (GM\omega)^{1/3} = \omega r$ (Kepler). Integrating the energy balance eqn. ($q/M \sim O(1)$)

$$\frac{dE}{dt} = -P_{GW} \Longrightarrow \mu \frac{d}{dt}\left[(r_g\omega)^{2/3}\right] \sim \frac{(GM\omega)^{10/3}}{G_N}$$

$$\Rightarrow \dot{\omega} \sim r_g^{-2}(r_g\omega)^{11/3}$$

~~During~~ During this "inspiral" phase w/ $v \ll 1$, the orbital frequency slowly increases in time. We have

$$\Delta t = \int_{\omega_i}^{\omega_f} d\omega \frac{dt}{d\omega} \sim r_g\left[\frac{1}{(r_g\omega_i)^{8/3}} - \frac{1}{(r_g\omega_f)^{8/3}}\right]$$

Plugging in numbers for terrestrial GW detectors, eg LIGO/VIRGO w/ sensitivity in a band

$$\frac{\omega_i}{2\pi} = \nu_i \sim 10\,\mathrm{Hz} \lesssim \nu \lesssim 10^4\,\mathrm{Hz} = \nu_f$$

one finds

$$\Delta t \sim 10\,\mathrm{mins} \cdot \left(\frac{M}{M_\odot}\right)^{-5/3}$$

~~spent~~ of signal in the detector. The number of orbital cycles observed is of order

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$$N = \int_{t_i}^{t_f} dt\, \omega(t) = \int_{\omega_i}^{\omega_f} d\omega\, \frac{\omega}{\dot{\omega}} \sim \frac{1}{(r_g \omega_i)^{5/3}} - \frac{1}{(r_g \omega_f)^{5/3}}$$

$$\sim 10^4 \left(\frac{M}{M_\odot}\right)^{-5/3}.$$

For discovery of a $v \ll 1$ binary event, it is sufficient to use waveforms obtained from linearized GR, w/ Newtonian mechanics + quadrupole radiation.

However, given the large # of orbital cycles, the experiment is sensitive (in principle) to GR corrections to orbital mech + radiation.

$$E \sim \frac{1}{2} \mu v^2 \left[1 + \mathcal{O}(v^2) + \mathcal{O}(v^4) + \mathcal{O}(v^6) + \cdots\right]$$

$$P_{GW} \sim \frac{1}{G_N} v^{10} \left[1 + \mathcal{O}(v^2) + \mathcal{O}(v^3) + \mathcal{O}(v^4) + \mathcal{O}(v^5) + \mathcal{O}(v^6) + \cdots\right] + \mathcal{O}(v^6 \ln v)$$

where we take the expansion parameter to be $v \sim (r_g \omega)^{1/3} \ll 1$ = typ velocity.

Computing corrections in powers of $v^2 \equiv$ "post-Newtonian" expansion of GR has been around since Einstein. Going to $\mathcal{O}(v^4)$ and higher is well motivated by experiment and also by comparison w/ numerical GR.

~~My personal motivation for understanding binary system~~

One reason why this perturbative expansion is difficult is that effects at any given order in $v \ll 1$ can arise from one of several length scales. For the binary problem we have a hierarchy of length scales

$$r_{phys} \gtrsim r_g = 2GM = \text{grav radius}$$

$$r = \text{orbital radius}$$

$$\lambda_{GW} = \text{wavelength}$$

and these scales are correlated:

$$\frac{r_g}{r} \sim \frac{GM}{r} \sim v^2 \ll 1, \qquad \frac{r}{\lambda_{GW}} \sim v \ll 1$$

etc. So at a given (fixed) order in the $v \ll 1$ expansion, distinct physical effects ~~from~~ associated w/ every scale in the problem can arise.

This motivates a formulation of the PN expansion using the tools of Effective Field Theory (EFT). The specific EFT that we will construct (WG

+ Rothstein 2004) has its roots in EFT's for QCD + QED developed in the 1990's (HQET, NRQCD, NRQED) for bound states of heavy particles interacting w/ radiation.

Skematics of EFT

Before discussing the specific EFT, we review the general EFT philosophy.

Units: From now on, I use particle physics units $\hbar = c = 1$ (even though we are doing $\hbar \to 0$ physics)

In these units, every observable ~ (Mass)$^{\text{power}}$.
Note that $[G] = (\text{Mass})^{-2}$, which defines an energy scale, the Planck mass

$$m_{Pl}^2 = 1/32\pi G_N$$

The key idea behind EFT's is decoupling.
To be specific consider a field theory w/

$$\phi(x) = \text{"light fields" (eg } m_\phi = 0)$$

$$\Phi(x) = \text{"heavy fields", mass } M_\Phi \equiv \Lambda$$

"Heavy" means $M_\Phi \gg m_\phi$. Now consider some experiment at energy scale $\omega \ll \Lambda = UV$ scale. In principle, all the observables are generated by a path integral

$$Z = \int D\Phi(x)\, D\phi(x)\, e^{iS[\Phi,\phi]}$$

w/ $S[\Phi,\phi] = \int d^4x\, \mathcal{L}(\Phi,\phi)$ = action functional describing the interactions bet. ϕ, Φ ($S[\Phi,\phi]$ = "Full Theory"). If the experiment only probes scales $\omega \ll \Lambda$, it is natural to do this integral sequentially. This defines an <u>effective</u> <u>Lagrangian</u>

$$e^{iS_{eff}[\phi]} \equiv \int D\Phi(x)\, e^{iS[\Phi,\phi]}$$

for the light fields alone, $S_{eff}(\phi) = \int d^4x\, \mathcal{L}_{eff}(\phi)$

<u>Decoupling</u> is the statement (due to Wilson, 1970s) that this effective Lagrangian is <u>local</u> in the fields $\phi(x)$ for $\omega \ll \Lambda$

$$S_{eff}[\phi] = \sum_i \int d^4x\, c_i\, \mathcal{O}_i(x)$$

$\mathcal{O}_i(x)$ = local operator constructed from $\phi(x)$, $\partial_\mu \phi(x)$

c_i = "Wilson coefficient", coupling constant for $\mathcal{O}_i$

If $\mathcal{O}_i(x)$ has mass dimension $[\mathcal{O}_i] = \Delta_i$ by dim. analysis:

$$[S] = 0 \Rightarrow [\mathcal{L}] = 4 = [c_i \mathcal{O}_i] \Rightarrow [c_i] = 4 - \Delta_i$$

Thus we expect $c_i \sim 1/\Lambda^{\Delta_i - 4}$, (at least for $\underset{=\text{ renorm. scale}}{\mu} \sim \Lambda$)

$$c_i(\mu = \Lambda) = \frac{\alpha_i}{\Lambda^{\Delta_i - 4}}, \quad \alpha_i \sim \mathcal{O}(1)$$

Note that in general, there can be an infinite # of interaction terms $\mathcal{O}_i(x)$. We conclude that short distance (UV) physics can have two types of effects at energies $\omega \ll \Lambda$:

(a) Renormalization of coefs of operators w/ $\Delta \leq 4$ (marginal, relevant)

(b) Generation of an infinite set of "irrelevant" terms w/ $\Delta_i > 4$

Even though the # of terms in $S_{eff}(\phi)$ is infinite, it is still a useful object. This is because:

(1) An operator $\mathcal{O}_i(x)$ contributes to a given observable 1st at relative order

$$\left(\frac{\omega}{\Lambda}\right)^{\Delta_{\mathcal{O}_i} - 4} \ll 1 \quad \text{for } \Delta_i > 4$$

by dim. analysis. The precise rules

for determining which terms come in at a given order in $\omega/\Lambda \ll 1$ depends on the details of $S_{eff}(\phi)$, and is called <u>power counting</u> in EFTs.

(2) Experiments have finite resolution $\epsilon \ll 1$. So for practical purposes, only need to keep terms in $\mathcal{L}_{eff}(\phi)$ up to $\Delta_i < \Delta_{max}$ w/

$$\left(\frac{\omega}{\Lambda}\right)^{\Delta_{max} - 4} \sim \epsilon \ll 1$$

Therefore to make predictions, we may truncate $\mathcal{L}_{eff}$ to include a finite # of terms N consisting of ops. $\mathcal{O}_i(x)$ w/ $\Delta_i < \Delta_{max}$

The EFT $\mathcal{L}_{eff}(\phi)$ is typically (in particle physics, stat. mech, cosmology, ...) used in one of two ways:

(1) <u>Full Theory $S[\phi, \Phi]$ is known</u>:

Then the EFT $S_{eff}(\phi)$ is a convenient bookkeeping device for systematically computing UV effects as an expansion in powers of $(\omega/\Lambda) \ll 1$. Usually easier to do this keeping only light mode $\phi(x)$ rather than both (ϕ, Φ).

(13)

Another advantage of this approach is that it becomes possible to understand non-analytic terms ~~as a consequence~~, of the form

$$\text{Observable} \sim \left(\frac{\omega}{\Lambda}\right)^{p} \left[\ln\left(\frac{\omega}{\Lambda}\right)\right]^{q}$$

as a consequence of renormalization group evolution of the Wilson coefficients of the EFT

$$\mu \frac{d\alpha_i(\mu)}{d\mu} = \beta_i(\{\alpha_j\})$$

from the renormalization scale $\mu = \Lambda = UV$ down to the low energy scale $\mu = \omega$.

Thus a typical EFT calculation consist of integrating out ("matching") at particle thresholds, followed by RG running. Eg

match $\longrightarrow$ Full theory Φ_1, Φ_2, ϕ $\quad \mu = \Lambda_1$

RG $\downarrow$ $\quad$ EFT_1 : $S_{eff}[\phi, \Phi_2]$

match $\longrightarrow$ $\quad \mu = \Lambda_2$

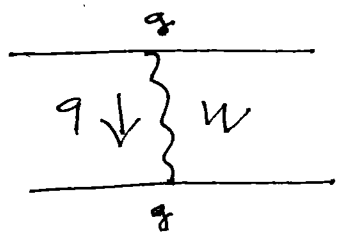
RG $\downarrow$ $\quad$ EFT_2 : $S_{eff}[\phi]$

$\mu = \omega$

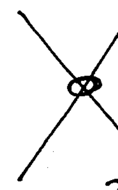
Typical Example: $SU(2)\times U(1)$ EW ints.

$\longrightarrow$ Fermi theory of weak interactions

For $E \ll m_W \sim 80$ GeV



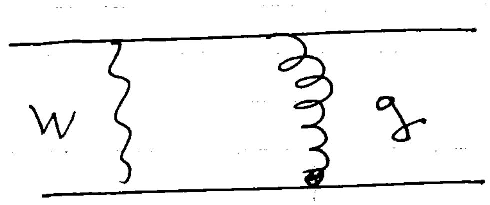
$$\propto \frac{g^2}{q^2 - m_W^2} \xrightarrow{q^2 \ll m_W^2}$$



$$+\ \mathcal{O}\left(\frac{q^2}{m_W^2}\right)$$

$$G_F \sim \frac{g^2}{m_W^2}, \quad \Delta = 5$$

For $E \ll m_W$, it is a lot easier to compute radiative corrections (eg due to QCD) in the EFT rather than the full theory



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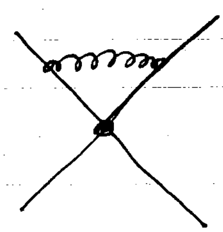

Other examples: NRGR, EFT of LSS ...

(2) Full Theory $S[\Phi, \phi]$ not known: Even if the UV physics at scales $\sim \Lambda$ is not known, decoupling guarantees that at energies $\omega \ll \Lambda$ the physics is described by an effective Lagrangian of the form

$$\mathcal{L}_{eff} = \sum c_i(\mu)\, \mathcal{O}_i(x)$$

Thus by writing the most general $\mathcal{L}_{eff}$

consistent w/ <u>low energy symmetries</u>, we are necessarily capturing UV effects in a <u>model independent way</u>. Predictive power then follows from finite exp. resolution $\epsilon \ll 1 \Rightarrow$ finite # of unknown coefs $c_i(\mu)$ that must be fit to data

<u>Examples</u>:

(1) <u>Classical GR</u> is an EFT for quantum gravity at energies

$$E < m_{Pl} \sim 10^{19}\ \text{GeV}$$

(see review by Donoghue, 1995)

(2) <u>The Standard Model</u> is $SU(3)\times SU(2)\times U(1)$ invariant EFT at energy scales

$$\Lambda \gtrsim \text{few TeV } (?)\ (\text{see LHC})$$

or so

$$\mathcal{L}_{EFT} = \underset{\text{"SM"}}{\mathcal{L}(\Delta \leq 4)} + \sum \frac{d_i}{\Lambda^{\Delta_i - 4}} \mathcal{O}_i$$

w/ $\mathcal{O}_i(x)$ = gauge inv. SM operator.

(3) <u>QCD Chiral Lagrangian</u>: Non-linear $SU(3)\times SU(3)$ Goldstone Lagrangian below

$$E < \Lambda_{QCD} \sim 1\ \text{GeV}$$

(4) <u>EFT's in CM</u>: Eg EFT for <u>fluids</u> (Nicolis et al hep-th/0512260)