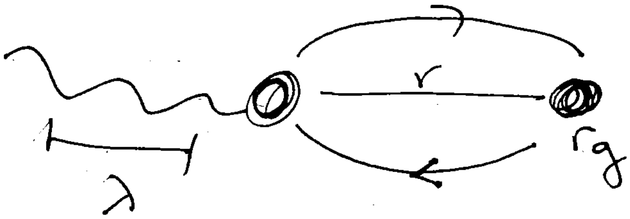
(16)

Lecture II –

The Binary Inspiral as an EFT

Recall that for $v \ll 1$, there is a hierarchy of scales



$$r_g \ll r \ll \lambda$$

$$(v^2 \sim r_g/r \ll 1)$$

$$(v \sim r/\lambda \ll 1)$$

Even though this is a classical problem, it turns out to be useful to think of it in ~~EFT~~ language. In fact three EFTs are necessary ($\hbar = 1$, still)

Full theory: $R_{\mu\nu}(g) = 0$

match

——————————————— $\mu = 1/r_s$

pt. particle + GR

exp. parameter
$\eta_0 = r_s/r$
$\sim r_g/r$

RG ↓

——————————————— $\mu = 1/r$

2-particle bound state
("NRGR")

exp. param
$\eta_1 = r_g/r$

RG ↓

——————————————— $\mu = \omega \sim \dfrac{v}{r}$

Radiation (Multipoles + GR)

exp. param

$\eta_2 = r/\lambda \sim v$
$\eta_3 = r_g/\lambda \sim v^3$ (NR)

⋮

In the remaining time, I'll discuss the construction of the EFTs as well as how to calculate observables.

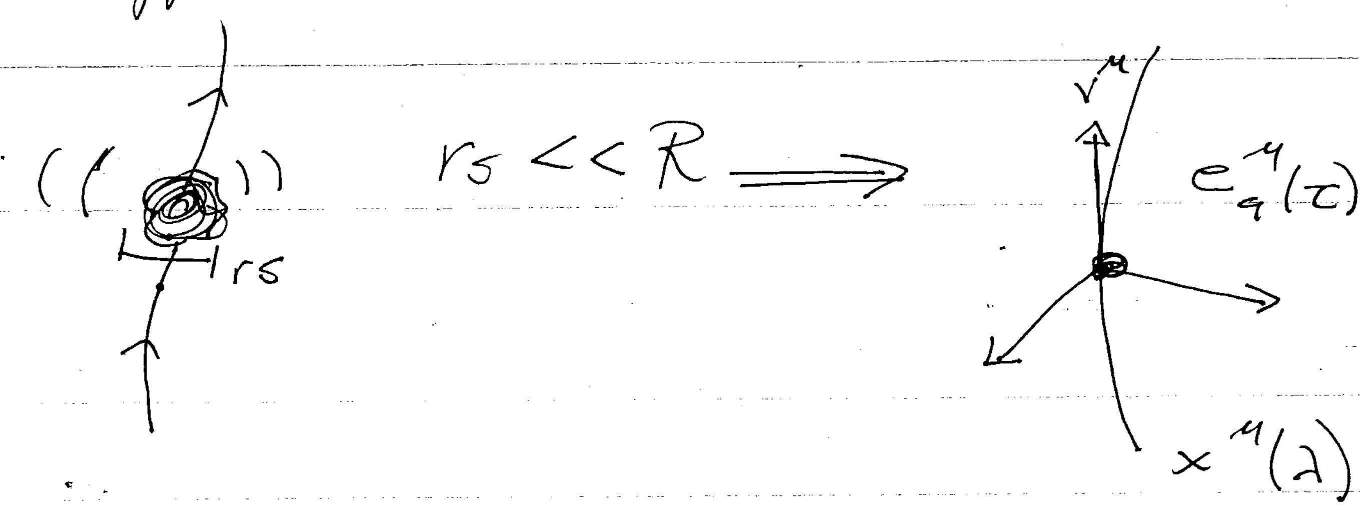
EFT For an Isolated Compact Object

Consider an isolated BH or NS in a background grav. field (e.g. an FRW universe) $g_{\mu\nu}(x)$. Let

$$R = \text{curvature scale of } g_{\mu\nu}(x)$$

$$r_s = \text{phys radius}$$

In regions of low curvature $r_s \ll R$ the object appears as a pt. particle (obviously)



i.e. when probed at long distances we assume that the relevant ("light") d.o.f's

- $g_{\mu\nu}(x)$ = spacetime metric
- $x^\mu(\lambda)$ = worldline ("CM coordinate")

18

w/ $v^\mu = dx^\mu/d\tau$ = 4-velocity

$d\tau^2 = g_{\mu\nu}(x)\,dx^\mu dx^\nu$ = proper time

– $e^\mu_{\ a=1,2,3}(\tau)$ = Spatial frame, localized on worldline $x^\mu(\lambda)$

$(v \cdot e_a = 0)$

This object $e^\mu_{\ a}$ describes the spatial orientation of the object in space: In the rest frame of the particle, $v^\mu = (1, \vec{0})$, it is given by a 3x3 orthogonal matrix

$$e^\mu_{\ a} = \begin{pmatrix} 0 & 0 & 0 \\ \hline & R_{3\times 3} \in SO(3) & \\ & & \end{pmatrix}$$

These modes can be parametrized, eg as Euler angles $R(\alpha, \beta, \gamma) \in SO(3)$ etc. Physically this corresponds to the particle's spin, which can be relativistic.

The EFT formalism (Lagrangian + power counting) of spin was 1st done by R. Porto, 2005. An elegant formulation using non-linear realizations of Poincare ISO(3,1) is in 1405.7384 (EPFL group). See refs. for details

(19)

Neglecting dissipation (see Rothstein + WG '05) the result of integrating out UV dof's of the compact object should result, by decoupling, in a local Lagrangian for the light d.o.f's

We do not need to carry out this matching procedure to know the form of the Lagrangian. It should be consistent w/ the symmetries, namely

– Diffs: $x^\mu \longrightarrow x'^\mu(x)$

– RPI: $\lambda =$ wordline param $\longrightarrow \tilde{\lambda}(\lambda)$

Thus the effective pt. particle theory is, in full generality

$$S = S_{EH} + S_{pp}$$

$$S_{EH} = -2 m_{pl}^2 \int d^4x \sqrt{g}\, R(x)$$

w/

$$S_{pp} = -\int d\tau\, m - \int dx^\mu\, \omega_\mu{}^{ab}(x(\lambda))\, L_{ab}^{(\lambda)}$$

$$+ c_E \int d\tau\, E_{ab} E^{ab} + c_B \int d\tau\, B_{ab} B^{ab} + \cdots$$

20

In this eqn:

$$d\tau^2 = g_{\mu\nu}(x(\lambda))\, dx^\mu dx^\nu = $$ proper time along worldline

$m(\lambda) =$ BH mass (actually a dynamical variable, see WG, Ross, Rothstein 2012, although we'll ignore this here)

Ignore for now
$\omega_\mu{}^{ab}$ = spin-connection for frame $e^\mu{}_a$
L_{ab} = particle spin

The leading term, $S = -m\int d\tau + \cdots$ describes geodesic motion, while the couplings to curvature

$$c_E \int d\tau\, E_{ab} E^{ab}, \quad c_B \int d\tau\, B_{ab} B^{ab}$$

w/

(even parity) $E_{ab}(x)$ = "electric Weyl"
$$= R_{\mu\alpha\nu\beta}\, e_a{}^\alpha e_b{}^\beta v^\mu v^\nu$$

B_{ab} = "magnetic Weyl"

(21)

$$= \frac{1}{2} e^{\alpha}_{a} e^{\beta}_{b} V^{\mu} V^{\lambda} \epsilon_{\alpha\mu\rho\sigma} R_{\beta}{}^{\lambda\rho\sigma}$$

describe deviations away from geodesic motion, i.e. the tidal response of the object to ext. grav. fields. For a classical object the Lagrangian $S = \int dt\, L$ has units of energy $= M$ so

$$M = [\, c_E \, E_{ab} E_{ab} \,] = [c_E] \cdot L^{-4} \quad \text{so}$$

one would expect (M = mass of object)

$$c_{E,B} \sim M r_{phys}^{4} \propto r_g r_{phys}^{4}$$

which is in fact ~~what~~ what one obtains for newtonian stars, ~~so~~ one might guess that it plays a small role in the orbital mechanics at a relative order

$$\epsilon \sim \frac{r_g r_{phys}^{4}}{r^{5}} \sim v^{10} \left(\frac{r_{phys}}{r_g} \right)^{4}$$

For NS $r_{phys} \sim 10\, r_g$, this need not be so small (Tinclerer + Flanagan), but safe to ignore for BH.

(In writing S_{pp}, I have ignored the contribution of terms involving the Ricci curvature, of the ~~form~~

$$\int d\tau \, R(x), \quad \int d\tau \, R_{\mu\nu}(x) v^\mu v^\nu$$

Such operators are called "redundant" because they do not give rise to physical effects. This turns out to be a consequence (see Georgi, NPB 1991) in general EFTs of the fact that terms such as the above, which vanish by the leading order eqn. of motion ("on-shell")

$$R_{\mu\nu}(g(x)) = 0$$

i.e. operators that "vanish on-shell" may be ignored, since they can be removed by field redefinitions.)

Feynman Diagram Expansions for Observables

For a system of widely separated objects interacting gravitationally, we have

$$S = S_{EH} + \sum_{a=1}^{N=2} S_{pp}^{(a)} [x^a, g_{\mu\nu}(x)]$$

we assume that the orbital separation is $r >> r_g$, so we are necessarily in the NR, Newtonian limit

$$v^2 \sim \frac{r_s}{r} \sim \frac{m}{m_{Pl}^2} \frac{1}{r} << 1$$

The orbital angular momentum of the system is thus of order

$$L \sim mvr = \frac{1}{v} mv^2 r \sim \frac{m^2}{m_{Pl}^2} \cdot \frac{1}{v} >> 1$$

in our units w/ $\hbar = 1$. Since $L/\hbar >> 1$ we might suspect that we can recover the classical observables from the $\hbar \to 0$ limit of a perturbative quantum calculation, done in terms of Feynman diagrams. This is useful, since there is a whole set ~~for~~ of ~~computing Feynman diagrams~~ sophisticated methods for computing Feynman diagrams in the QFT literature, which we can transplant to the classical GR problem. In particular, we can use <u>dimensional regularization</u> to define IR and UV divergent integrals.

To set up the Feynman diagram expansion, write the metric as

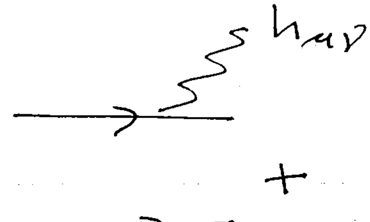
$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \bar{h}_{\mu\nu}(x) + H_{\mu\nu}(x)/m_{pl}$$

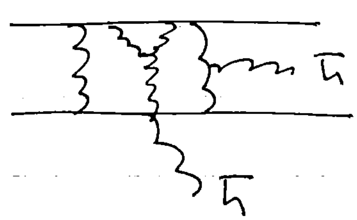
where $\bar{h}_{\mu\nu}(x)$ = some background field, and define

$$W[\bar{h}, x_{a=1,2}] = e^{i\Gamma[x_a, \bar{h}]}$$

$$= \int [DH_{\mu\nu}(x)]_{b.cs}\, e^{iS[\eta + \bar{h} + \frac{H}{m_{pl}}, x_a]}$$

This quantity has an expansion in terms of diagrams, of the form

$W = 1 +$  $+$  $+$ 

$+ \cdots +$  $+ \cdots$

Before trying to understand what these diagrams mean, I will note that $W[\bar{h}, x_a]$ is a generating fn for all the observables in the binary. In particular

$$\Gamma[\bar{h}=0, x_a] = \int dt\, L(\vec{x}_a, \dot{\vec{x}}_a|t))$$

defines the 2-body gravitational Lagrangian of the system. It yields the time evolution of the two-particle system.

On the other hand, the linear term

$$T[\bar{h}, x_a] = \frac{1}{2}\int d^4x\, \bar{h}_{\mu\nu}(x)\, T^{\mu\nu}(x)$$

defines an effective energy momentum tensor for the system, including gravitational energy + momentum. It is invariant under gauge transformations of the background, i.e

$$\bar{h}_{\mu\nu} \longrightarrow \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

gauge inv $\Rightarrow$ $T[\bar{h}, x_a] = T[\bar{h} + \partial\xi + \partial\xi, x_a]$

$$\Longrightarrow \quad \partial_\mu T^{\mu\nu}(x) = 0 \quad \text{(Ward id.)}$$

The quantity $T^{\mu\nu}(x)$ can be regarded as an energy momentum "pseudotensor" ~~for gravity~~, like the Landau-Lifshitz pseudotensor or the analogous quantity defined in Weinberg's GR book

The b.c's in the path integral define ~~a set~~ the choice of free graviton propagator. Using standard Feynman in/out b.c's, $T^{\mu\nu}(x)$ can be regarded as the S-matrix to emit one-graviton from the system

$$A_{h=\pm 2}(k) = \underset{\text{emission}}{\text{graviton}} \text{ amplitude}; \quad \text{"momentum"} \; k^\mu,\ k^2=0, \; \text{helicity} = h$$

$$= \int d^4x \, \epsilon^*_{\mu\nu}(k) T^{\mu\nu}(x) e^{-ik\cdot x}$$

The graviton emission rate is then

$$d\Gamma_h(\vec{k}) = \frac{1}{T} \frac{d^3\vec{k}}{(2\pi)^3 2|\vec{k}|} |A_h(k)|^2$$

w/ $T \Rightarrow \infty$ = detector integration time. From the differential rate, we obtain time averaged radiation observables e.g.

$$\langle \dot{p}^\mu \rangle_{h=\pm 2} = \text{time-averaged rate of energy-momentum emission}$$

$$= \int k^\mu \, d\Gamma_h(\bar{k})$$

and $\langle \dot{\vec{J}} \rangle =$ angular momentum loss

$$= 2 \int \frac{\bar{k}}{|\bar{k}|} d\Gamma_{h=2}(\bar{k}) - 2 \int \frac{\bar{k}}{|\bar{k}|} d\Gamma_{h=-2}(\bar{k})$$

On the other hand, we can take for b.c's "In/In" conditions, as in cosmology. This can be used to compute time dependent quantities, such as the waveform at $\bar{x} \to \infty$:

$$h_{\mu\nu}(\vec{x} \to \infty, t) = \int d^4y \, D^{ret}_{\mu\nu,\alpha\beta}(x-y) \, T^{\alpha\beta}(y)$$

= retarded graviton propagator.

We can now understand how to calculate the diagrams. To do so, it is easier to 1st consider a toy model w/ a scalar graviton $\phi(x)$:

$$S = S_{grav} + S_{pp}$$

$$S_{grav} = \frac{1}{2}\int d^4x\, \partial_\mu\phi(x)\partial^\mu\phi(x)$$

$$S_{pp} = -\sum_{a=1}^{2} m_a \int d\tau_a \left(1 + \frac{\phi(x_a)}{2\sqrt{2}\, m_{pl}}\right)$$

or $$S = -\sum_a m_a \int d\tau_a + \int d^4x\, \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + J(x)\phi(x)$$

w/ $$J(x) = -\sum_a \frac{m_a}{2\sqrt{2}\, m_{pl}} \int d\tau_a\, \delta^4(x - x_a)$$

Then, expanding about a classical background $\phi = \bar{\Phi} + \Phi$

$$W[J, \bar{\Phi}] = \int D\Phi\, e^{iS[\bar{\Phi}+\Phi,\, x_a]}$$

$$= e^{-i\sum_a m_a \int d\tau_a}\, e^{i\int dx\, J(x)\bar{\Phi}(x)}$$

$$\cdot \int D\Phi\, e^{iS[\Phi, J]}$$

The integral

$$Z[J] = \int D\Phi\, e^{i\int dx\, \frac{1}{2}(\partial\Phi)^2 + J(x)\Phi(x)}$$

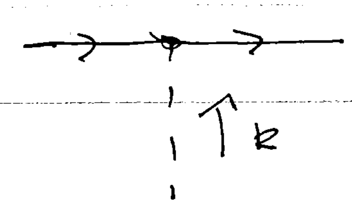
is Gaussian, so it can be explicitly computed. Up to an irrelevant normalization

$$\ln Z[J] = -\frac{1}{2}\int d^4x\, d^4y\, J(x) D_F(x-y) J(y)$$

where recall the Feynman propagator

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2+i\epsilon} e^{-ik\cdot(x-y)}$$

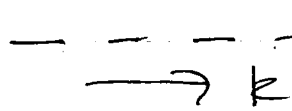
We can interpret this result diagrammatically, by introducing a Feynman vertex



$$= i\int d^4x\, J(x) e^{-ik\cdot x}$$

$$= -i\sum_a \frac{m_a}{2\sqrt{2 m_{pl}}}\int d\tau_a\, e^{-ik\cdot x_a}$$

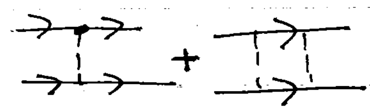
and a propagator rule

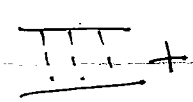


$$= \frac{i}{k^2+i\epsilon}$$

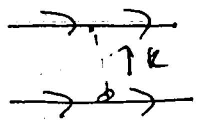
so that

$$Z[J] = e^{\text{(diagram 3)}} = 1 + \text{(diagram 4)} + \cdots$$





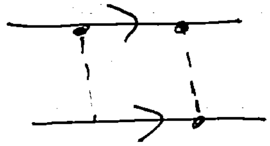
Note that the factor of 1/2 in the diagram



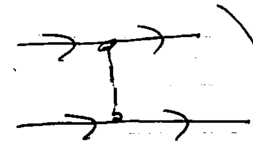
$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2+i\epsilon}\left[-i\sum \frac{m_a}{2\sqrt{2}\,m_{pl}}\int d\tau_a\, e^{-ik\cdot x_a}\right]$$

$$\cdot\left[-i\sum \frac{m_b}{2\sqrt{2}\,m_{pl}}\int d\tau_b\, e^{-ik\cdot x_b}\right]$$

is interpreted as a symmetry factor. Note also that the particle line $\longrightarrow$ represents a classical source, so it has no propagator associated to it. Thus diagrams factorize, e.g



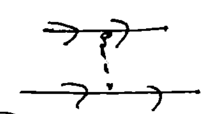
$$= \frac{1}{2!}\Big(\ \ \ \ \ \Big)^2$$



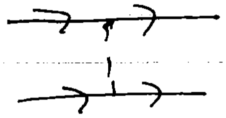
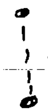
↑ symmetry factor

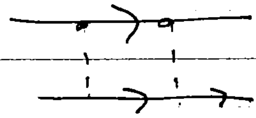
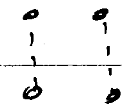
and thus

$$1 + \ \ \ + \ \ \ + \ \ \ + \cdots = e^{\ \ }$$



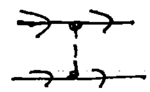
This exponentiation is a general property of the perturbative expansion. In general

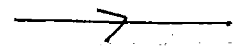
$\Gamma[\Phi, x_a]$ = sum of diagrams that remain connected after the particle worldlines are removed ("irreducible") eg  $\xrightarrow[\text{particle}]{\text{remove}}$  = connected ✓, while

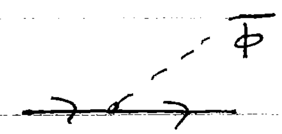
 $\xrightarrow[\text{particle}]{\text{remove}}$ =  = disconnected

So we have

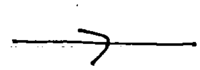
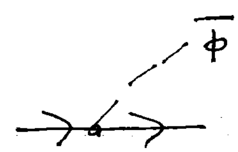
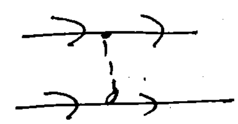
$$W[x_a, \bar{\Phi}] = e^{-i \sum_a m_a \int d\tau_a \left(1 + \bar{\Phi}(x_a)/2\sqrt{2} m_{Pl}\right)}$$

$\times\ e^{\text{diagram}}$ 

or by representing $-i \sum m_a \int d\tau_a \equiv$ 

$-i \sum m_a \int d\tau_a \dfrac{\bar{\Phi}(x_a)}{2\sqrt{2} m_{Pl}} \equiv$ 

we get the diagramatic expansion

$\Gamma[x_a, \bar{\Phi}] =$  +  + 

which is the exact background field effective action in our toy model of scalar gravity.

(32)

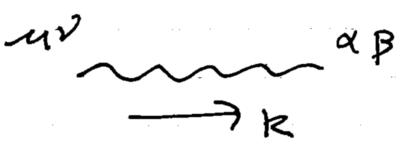
For the full GR case, the diagrammatic expansion is constructed in a similar manner. Expand

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}/m_{Pl}$$

Then schematically

$$-2m_{Pl}^2 \int d^4x \sqrt{g}\, R \rightarrow \int d^4x \left[(\partial h)^2 + \frac{h(\partial h)^2}{m_{Pl}} + \frac{h^2(\partial h)^2}{m_{Pl}^2} + \cdots \right]$$

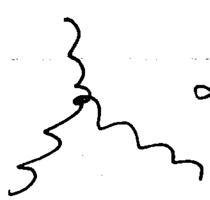
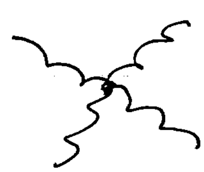
Leading to a propagator (after adding a suitable gauge fixing term)



$$= \frac{i}{k^2} P_{\mu\nu;\alpha\beta}$$

w/ $$P_{\mu\nu,\alpha\beta} = \frac{1}{2}\left[\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\alpha\beta} \right]$$

as well as interaction vertices

 $\propto \frac{1}{m_{Pl}}$  $\propto \frac{1}{m_{Pl}^2}$

etc (see Donoghue '95 for explicit formulas + derivations

The couplings to the point particles also generate interaction terms. Let $d\bar{\tau}^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$.

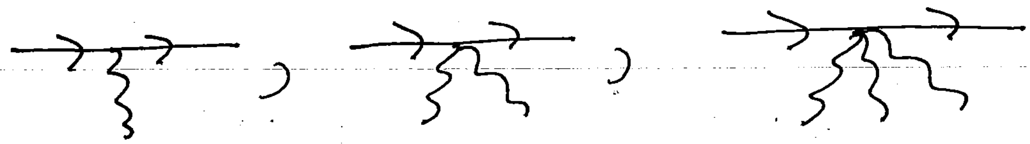
$$-m\int d\tau = -m\int \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$$

$$= -m\int \sqrt{d\bar{\tau}^2 + \frac{h_{\mu\nu}}{m_{Pl}} \frac{dx^\mu}{d\bar{\tau}} \frac{dx^\nu}{d\bar{\tau}}}$$

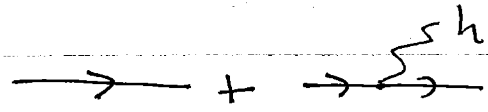
$$= -m\int d\bar{\tau} - \frac{m}{2m_{Pl}} \int d\bar{\tau}\; h_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu$$

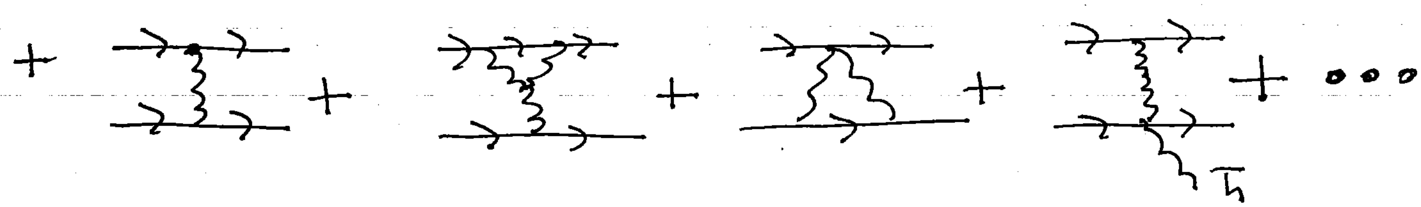
$$- \frac{m}{8 m_{Pl}^2} \int d\bar{\tau} \left(h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu\right)^2 + \cdots$$

so we have a series of vertices



If we further decompose $h_{\mu\nu} = \bar{h}_{\mu\nu}(x) + H_{\mu\nu}(x)$ w/ $\bar{h}_{\mu\nu}(x)$ = background field, we have

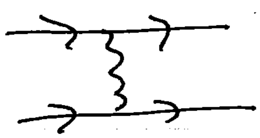
$\ln W[x^a, \bar{h}] = i\Gamma[x^a, \bar{h}] =$ 



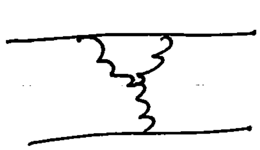
This provides a systematic way of computing observables, as an expansion in the gravitational coupling $\sim 1/m_{Pl}$.

Unfortunately, this form of the perturbative expansion is not well suited to the kinematic limit $v \ll 1$.

The problem is that the diagrams do not have definite scaling properties w.r.t the expansion parameter $v \ll 1$. For instance

 $\sim v^2 + v^4 + v^6 + \cdots$

contains an infinite # of terms in $v \ll 1$. It is also not clear how other diagrams scale, e.g

 $= v^{?}$

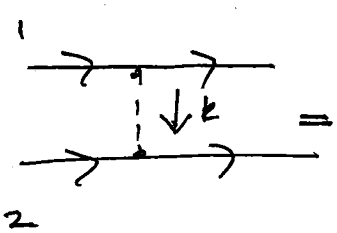
so if we want to work at fixed order in v, we dont know when to stop.

The problem w/ this Lorentz covariant perturbative expansion is that the diagrams involve momentum integrals receiving contributions over all the scales in the problem. We need to explicitly integrate out the short distance scale $\sim r$ = orbital radius in order to construct an EFT w/ manifest power counting in v.

Integrating out the Orbital Scale –

To solve the power counting problem, we borrow from EFT's for heavy quarks constructed in the 1990's – early 2000's.

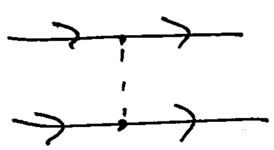
To explain the trick, consider, in the toy gravity theory, the diagram.



$$= \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2+i\epsilon} \left[\int d\tau_1 \left(\frac{-im_1}{2\sqrt{2}m_{pl}} \right) e^{-ik\cdot x_1(\tau)} \right]$$

$$\cdot \left[\int d\tau_2 \left(\frac{-im_2}{2\sqrt{2}m_{pl}} \right) e^{ik\cdot x_2(\tau)} \right]$$

In the NR limit $|\vec{v}_i| \ll 1$, this becomes



$$= \frac{-im_1 m_2}{8 m_{pl}^2} \int dx_1^0 dx_2^0 \, e^{-ik^0(x_1^0 - x_2^0)}$$

$$\cdot \, e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)} \quad \frac{1}{k_0^2 - \vec{k}^2} \quad \frac{d^4k}{(2\pi)^4}$$

Non relativistic kinematics implies that the ntegral d^4k receives dominant contributions from two regions:

(1) "Potential": $k^\mu \sim \left(\frac{v}{r}, \frac{1}{r}\right)$; $r = |\bar{x}_1 - \bar{x}_2|$

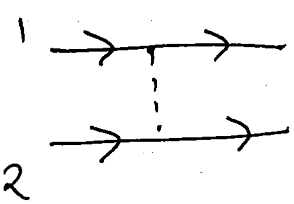
Potential graviton exchange mediates instantaneous nteractions bet. the point particles. Such modes are never on-shell, since $k^2 \neq 0$

For momenta in this region

$$\frac{1}{k_0^2 - \bar{k}^2} = -\frac{1}{\bar{k}^2}\left(1 + \frac{k_0^2}{\bar{k}^2} + \cdots\right)$$

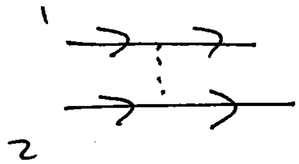
$$= -\frac{1}{\bar{k}^2}\left(1 + \mathcal{O}(v^2) + \cdots\right)$$

and the diagram becomes



$$= \frac{i m_1 m_2}{8 m_{Pl}^2} \int dx_1^0\, dx_2^0\, \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x_1 - x_2)} \cdot \left[\frac{1}{\bar{k}^2} + \mathcal{O}(v^2)\right]$$

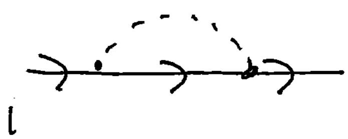
Using $\displaystyle\int \frac{d^4k}{(2\pi)^4} \frac{1}{\vec{k}^2} e^{-ik^0(x_1^0 - x_2^0)} e^{i\vec{k}\cdot\vec{x}} = \frac{1}{4\pi|\vec{x}|}\delta(x^0)$

we got  $= \int dt \, \frac{i m_1 m_2}{32\pi m_{Pl}^2} \frac{1}{|\vec{x}_1(t) - \vec{x}_2(t)|} + \cdots$

so the potential region of momentum space gives rise to a term in the ~~#~~two-particle Lagrangian

$$L = \cdots + \frac{G m_1 m_2}{|\vec{x}_1 - \vec{x}_2|} + \cdots$$

(In addition, there are self-energy corrections, of the form

 $\propto \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\vec{k}^2}$ = linearly divergent

However, if we work in dimensional regularization, this diagram vanishes, by the formula

$$\int \frac{d^d\vec{k}}{(2\pi)^d} \left(\frac{1}{\vec{k}^2}\right)^{\alpha} e^{i\vec{k}\cdot\vec{x}} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)} \left(\frac{\vec{x}^2}{4}\right)^{\frac{\alpha-d}{2}}$$

so in d-dimensions:

$$\lim_{d\to 3} \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{\vec{k}^2} = \frac{1}{(4\pi)^{d/2}} \Gamma(d/2-1) \left(\frac{0}{4}\right)^{1-d/2}$$

$$= 0 \,)$$

(2) Radiation : NR systems emit massless radiation w/ freq. $\omega \sim v/r$. So "radiation gravitons" have dispersion relation

$$k^\mu \sim (v/r, v/r) \quad ; \quad k^2 = 0 \ (\text{on-shell})$$

Because the potential modes are off-shell, they should be integrated out of the theory at long distances. To do this, we decompose the graviton into modes

$$h_{\mu\nu}(x) = \underbrace{\frac{\bar{h}_{\mu\nu}(x)}{m_{Pl}}}_{\text{rad. field}} + \sum_{\vec{k}} \underbrace{\frac{H_{\vec{k}\mu\nu}(x^0)}{m_{Pl}} e^{i\vec{k}\cdot\vec{x}}}_{\text{potential}}$$

w/ $\vec{k} \sim 1/r$. Thus all spacetime derivatives on fields scale in the same way

$$\partial_\alpha \bar{h}_{\mu\nu}(x) \sim \frac{v}{r} \bar{h}_{\mu\nu}(x)$$

$$\partial_\alpha H_{\vec{k},\mu\nu}(x^0) \sim \frac{v}{r} H_{\vec{k},\mu\nu}(x^0)$$

The propagator for the potential can be read off the quadratic terms in the grav. action

$$S_{H^2} = -\frac{1}{2}\int dx^0 \sum_{\vec{k}} \left(\vec{k}^2 H_{\vec{k}\mu\nu} H_{-\vec{k}\mu\nu} - \frac{\vec{k}^2}{2} H_{\vec{k}} H_{-\vec{k}} \right)$$

in a suitable gauge. Thus there is a potential propagator

$$\underset{\mu\nu}{x^0 \bullet} \overset{\vec{k}}{\text{- - - -}} \underset{\alpha\beta}{\bullet\, 0} = -\frac{i}{\vec{k}^2} P_{\mu\nu;\alpha\beta}\, \delta^3(\vec{k}+\vec{q})\, \delta(x^0)$$

Given that $x^0 \sim 1/\omega \sim r/v$; $\vec{k} \sim 1/r$ we can assign a scaling to the potential modes

$$\langle T\, H_{\vec{k}\mu\nu}(x^0)\, H_{\vec{q}\alpha\beta}|0\rangle \sim \left(\frac{1}{r}\right)^{-2} \left(\frac{1}{r}\right)^{-3} \left(\frac{r}{v}\right)^{-1}$$

$$\Rightarrow \quad H_{\vec{k}\mu\nu}^{\;2} \sim r^4 \cdot v$$

$$\Rightarrow \quad H_{\vec{k}\mu\nu} \sim r^2 \cdot \sqrt{v}$$

For the radiation mode, the propagator scales like that of a massless particle

$$\langle T\, h_{\mu\nu}(x)\, h_{\alpha\beta}|0\rangle \sim \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} e^{-ik\cdot x}$$

$$\sim k^2 \sim (v/r)^2$$

$$\Rightarrow \quad h_{\mu\nu} \sim v/r$$

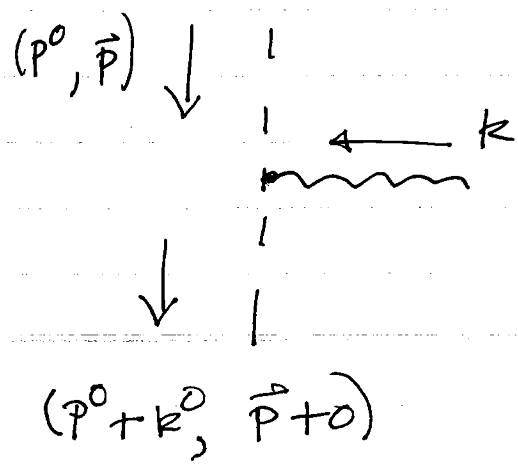
Finally, it is necessary to multipole expand the couplings of radiation to the pot. particles and to the potential modes about some point inside the binary bound state

$$\bar{h}_{\mu\nu}(x^0, \vec{x}) = \bar{h}_{\mu\nu}(x^0, \vec{0}) + \vec{x}\cdot\nabla \bar{h}_{\mu\nu}(x^0, \vec{0}) + \frac{1}{2} x^i x^j \partial_i \partial_j \bar{h}_{\mu\nu}(x^0, \vec{0}) + \cdots$$

or in momentum space

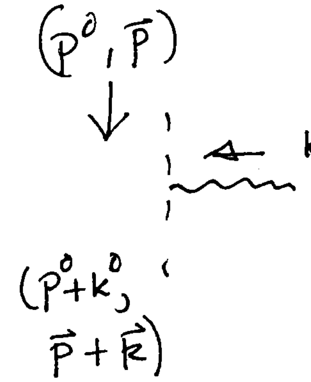
$$\bar{h}_{\mu\nu}(k) = (2\pi)^3 \delta^3(\vec{k})\, \bar{h}_{\mu\nu}(k) + (2\pi)^3 \partial_i \delta^3(\vec{k})\, \partial_i \bar{h}_{\mu\nu}(k) + \cdots$$

This ensures that radiation graviton absorption does not change the spatial momentum of a potential mode, eg in a subdiagram of the form



$$\propto \frac{1}{\vec{p}^{\,2}} \times \frac{1}{(\vec{p}+\vec{0})^2} \cdot \frac{1}{k^2}$$

as opposed to



$$\propto \frac{1}{(\vec{p}+\vec{k})^2}\,\frac{1}{\vec{p}^2}\,\frac{1}{k^2}$$

$$= \frac{1}{\vec{p}^2 k^2}\cdot\frac{1}{\vec{p}^2+2\vec{p}\cdot\vec{k}+\vec{k}^2}$$

$$= \frac{1}{k^2}\,\frac{1}{\vec{p}^4}\left[1+\mathcal{O}(v) + \mathcal{O}(v^2)+\cdots\right]$$

which does not scale as a definite power of the velocity.

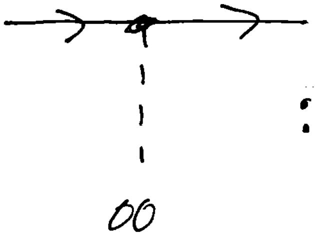
Given these steps, we now have a well defined set of rules that allows us to count powers of velocity in any diagrammatic contribution to $\Gamma[x_a, \bar{h}]$:

~~Another Example~~

Every term in the action consists of powers of m, m_{Pl}, $H_{\vec{k}}$, $\bar{h}$, $\vec{k}$, ∂_μ w/ scaling

m/m_{Pl}	$\vec{k}$	∂_μ	$H_{\vec{k}}$	$\bar{h}$
$\sqrt{Lv}$	$1/r$	$\frac{v}{r}$	$r^2\sqrt{v}$	v/r

For example, the coupling of a potential mode to a particle:



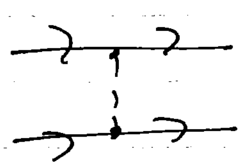
$$: \; -m\int d\tau \simeq -m\int dx^0 \sqrt{1 + h_{00}/m_{pl}}$$

$$= \cdots - \frac{m}{2m_{pl}} \int dx^0 \, h_{00} + \cdots$$

$$= -\frac{m}{2m_{pl}} \int dx^0 \int \frac{d^3\vec{q}}{(2\pi)^3} \, H_{\vec{q}}(x^0)_{00} \, e^{i\vec{q}\cdot\vec{x}} + \cdots$$

$$\sim \left(\sqrt{Lv}\right)\left(\frac{r}{v}\right)\left(\frac{1}{r}\right)^3\left(r^2\sqrt{v}\right) \sim \sqrt{L} \gg 1$$

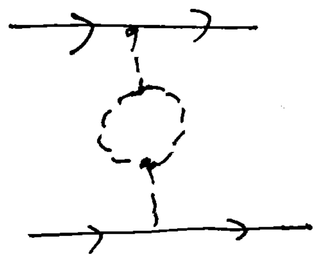
Thus the leading order exchange interaction term in $T[x^q, h]$ is

$$T = $$  $$+ \cdots \sim \left(\sqrt{L}\right)^2 = L$$

In general diagrams scale as

$$L^n v^m$$

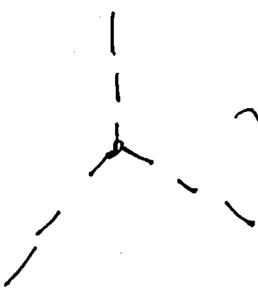
w/ $n \lesssim 0$ and $m \geq 0$. In the classical limit we can ignore diagrams w/ graviton loops, such as



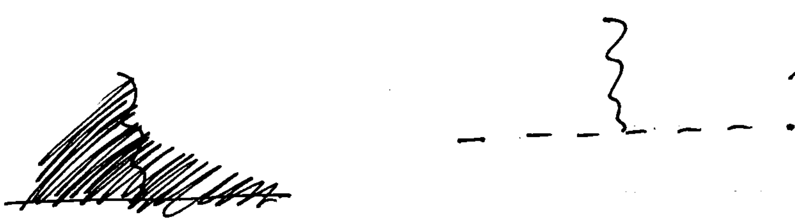
$$\sim \frac{\hbar}{L} \cdot v^4 \ll 1$$

although in principle it gives rise to a quantum gravity correction to the potential (see Donoghue, PRD 1995)

Other examples



$$\sim \frac{v^2}{\sqrt{L}} =$$ potential mode self-interaction



$$\sim \frac{v^{5/2}}{\sqrt{L}}$$

= rad + potential coupling

etc.