Phase Space Methods for the Analysis and Simulation of CDM Dynamics

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Abel, Hahn, Kaehler (2012), MNRAS
Kaehler, Hahn, Abel (2012), IEEE TVCG
Hahn, Abel, Kaehler (2013), MNRAS
Angulo, Hahn, Abel (2013), MNRAS
Hahn, Angulo, Abel (2014), MNRAS subm.
Hahn & Angulo (2015), MNRAS subm.

What is Dark Matter?

microscopic → continuum limit

proton = 1GeV, WIMP 100GeV? -> 10^{21}/g

cold (or at most lukewarm) → $v_{thermal} \ll v_{bulk}$

e.g. thermally produced at very early times, cooled since then

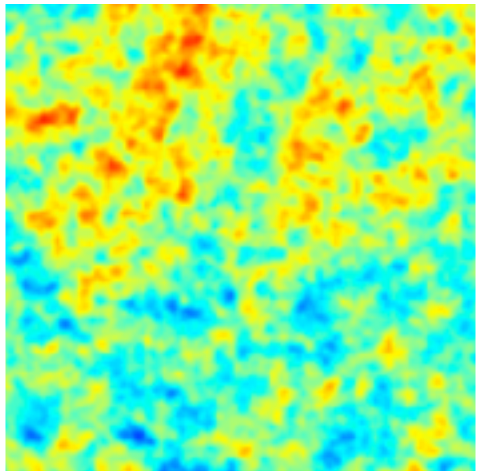
negligible cross-section → $\sigma_{DM} \ll \sigma_{em}$ collisionless

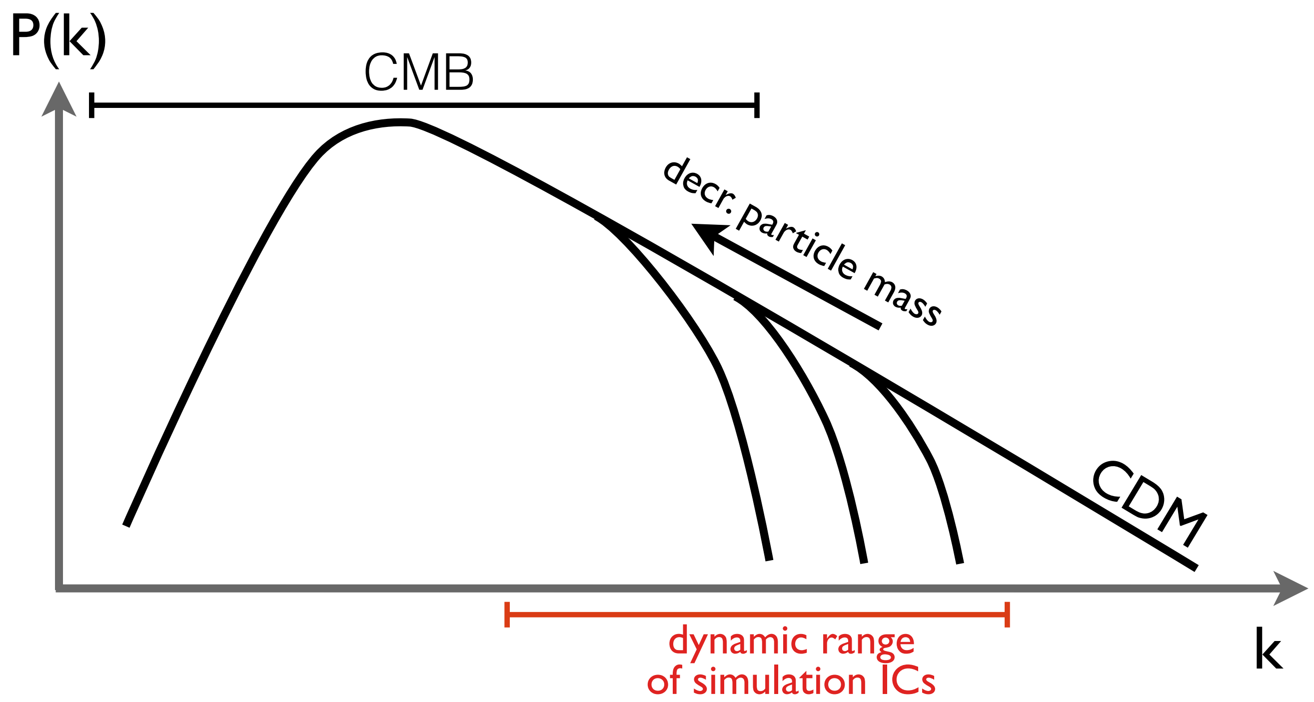
weak-scale or even weaker

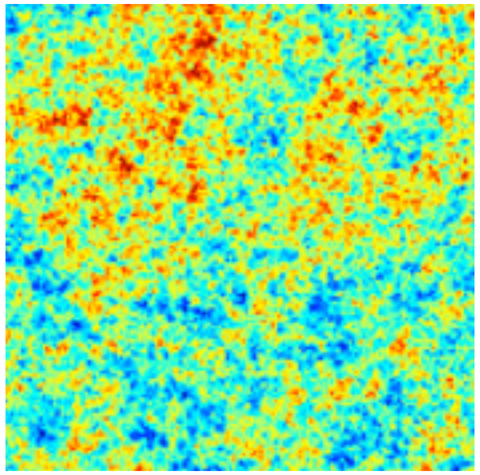
…and also the dominant gravitating component (~80%)

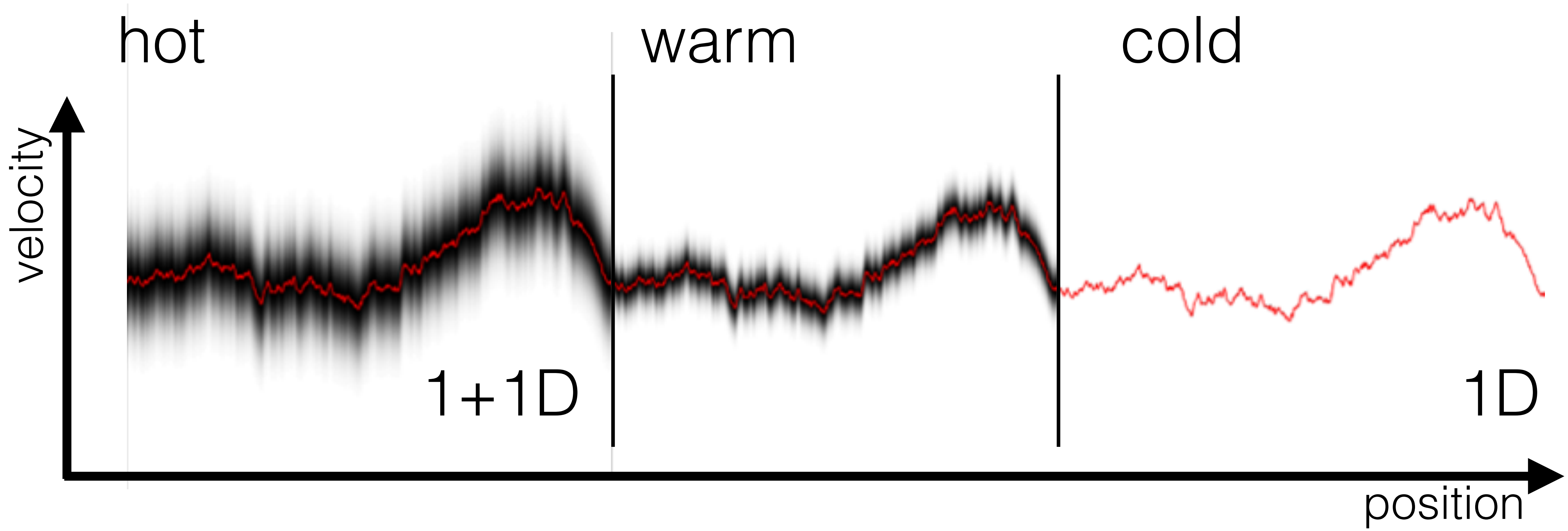
at first order, structure formation is well described by assuming all matter is dark matter

Dark Matter - properties on small scales

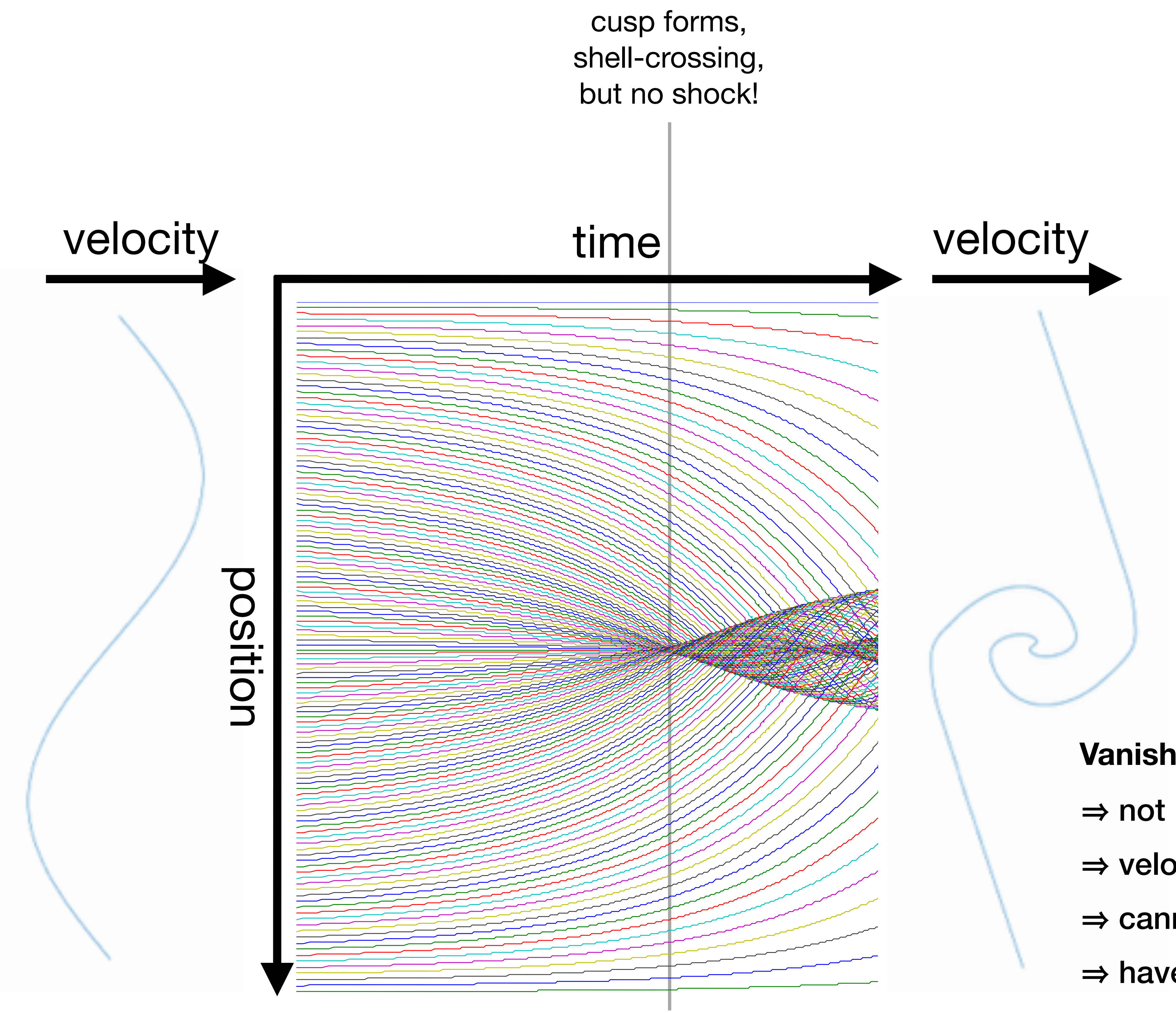








1D behaviour under self-gravity



Vanishing collision-term

⇒ not in hydro limit

⇒ velocity can be multi-valued

⇒ cannot stop at low order moments

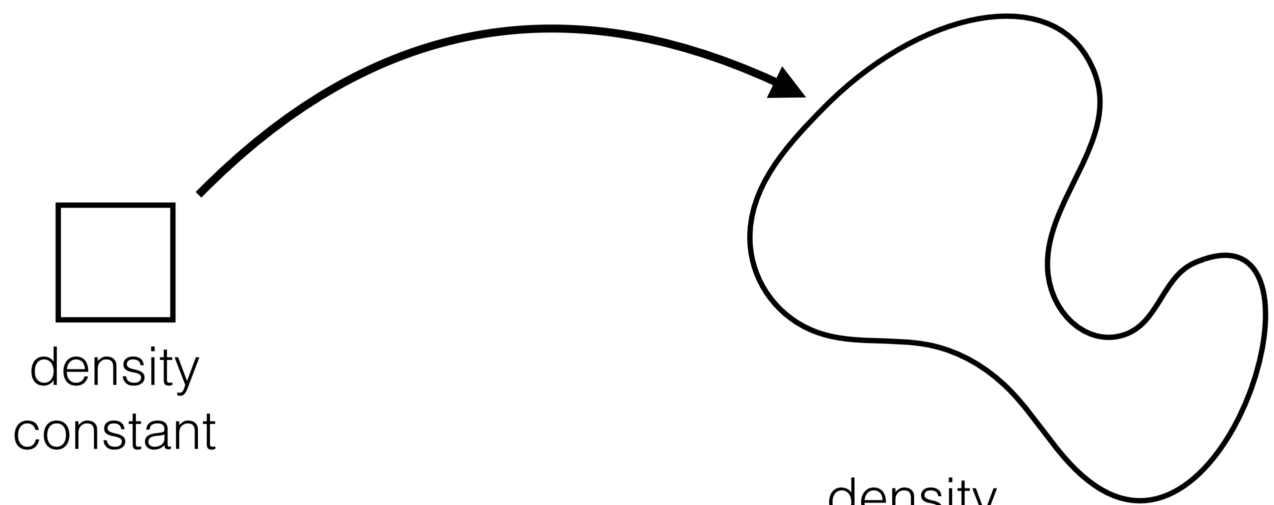
⇒ have to discretize distribution function

⇒ singular caustics emerge

Dark Matter - fluid flow

Lagrangian description, evolution of fluid element

$$\mathbb{Q} \subset \mathbb{R}^3 \to \mathbb{R}^6 \; : \; \mathbf{q} \mapsto (\mathbf{x}_\mathbf{q}(t), \mathbf{v}_\mathbf{q}(t))$$



$$\rho = m_{\rm DM} \left| \frac{\partial x_i}{\partial q_j} \right|^{-1}$$

For DM, motion of any point **q** depends only on gravity

$$(\dot{\mathbf{x}}_\mathbf{q}, \dot{\mathbf{v}}_\mathbf{q}) = (\mathbf{v}_\mathbf{q}, -\boldsymbol{\nabla}\phi)$$

unlike hydro, no internal temperature, entropy, pressure

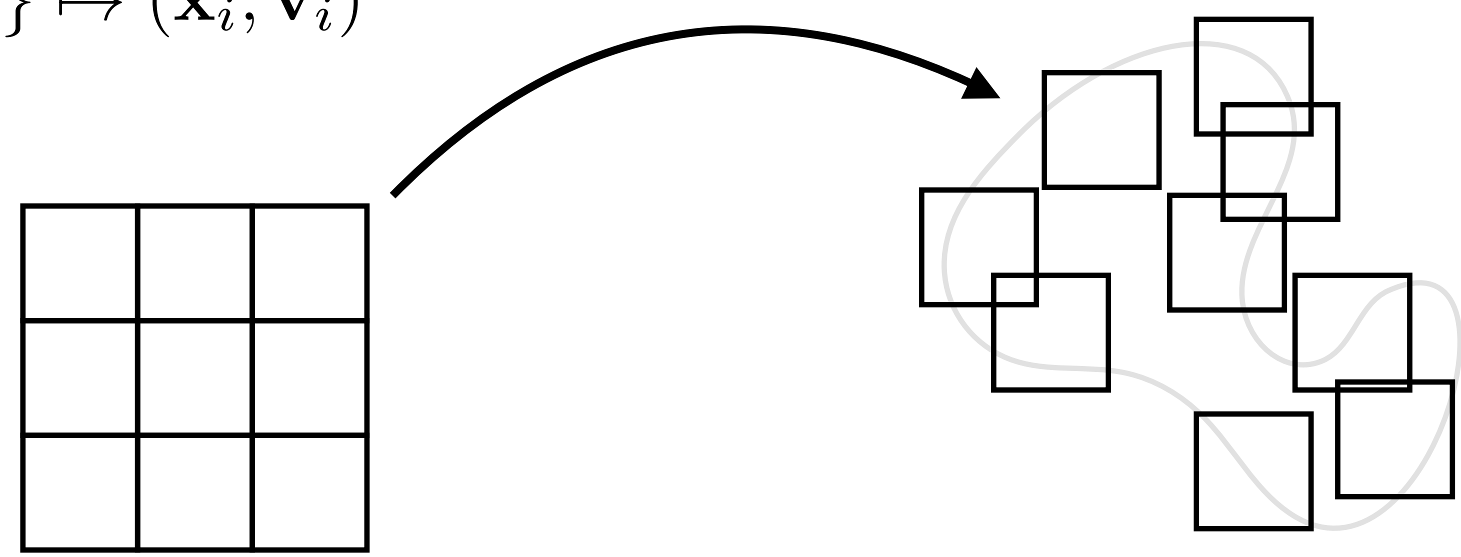
So the quest is to solve Poisson's equation

$$\Delta\phi = 4\pi G\rho$$

N-body vs. continuum approximation

The N-body approximation:

$$i \in \{1 \ldots N\} \mapsto (\mathbf{x}_i, \mathbf{v}_i)$$



⇒ EoM are just Hamiltonian N-body eq. (method of characteristics)

for small N, density field is poorly estimated,

$$\rho = m_p \sum \delta_D(x - x_i) \otimes W$$

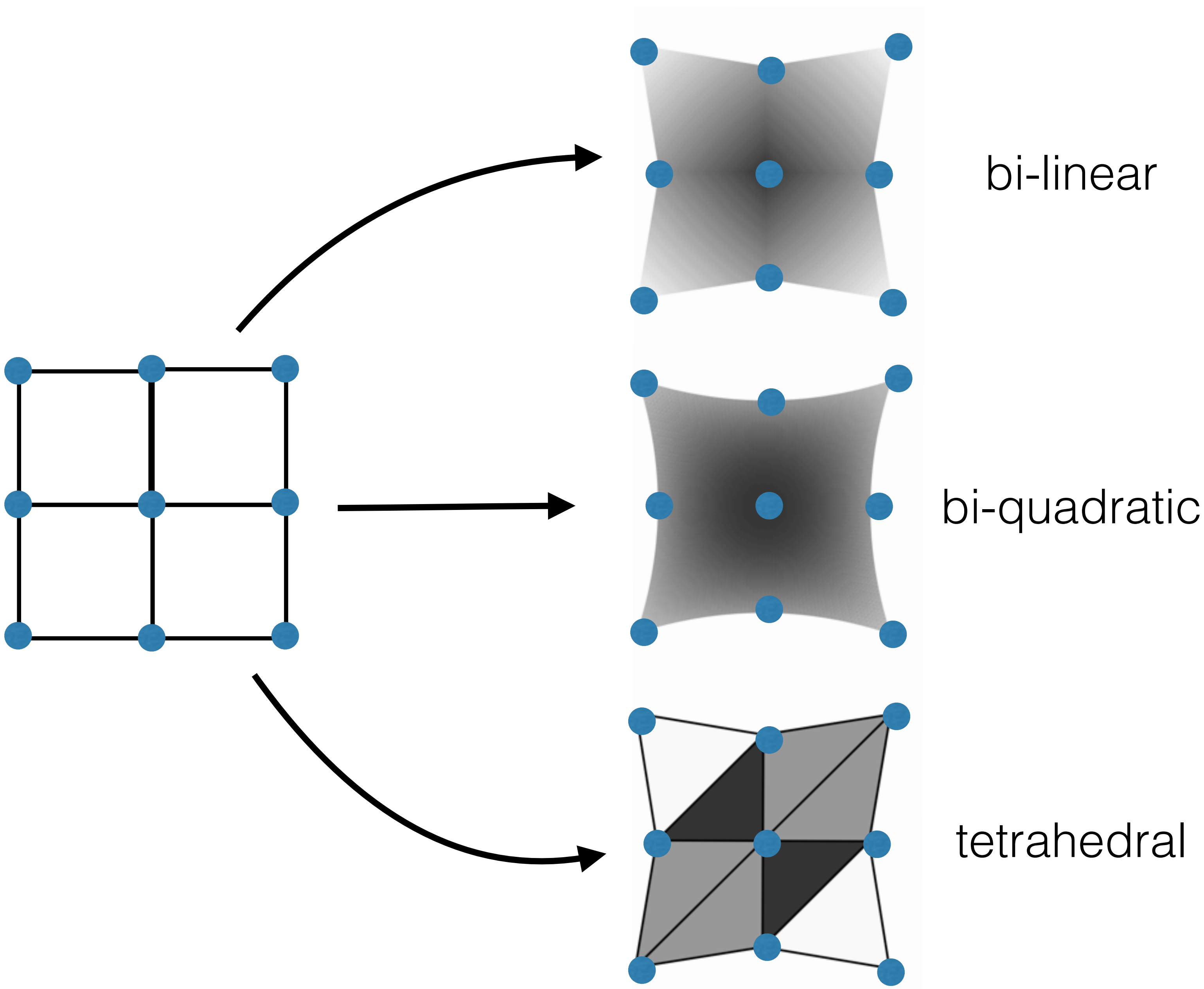
continuum structure is given up, but 'easy' to solve for forces

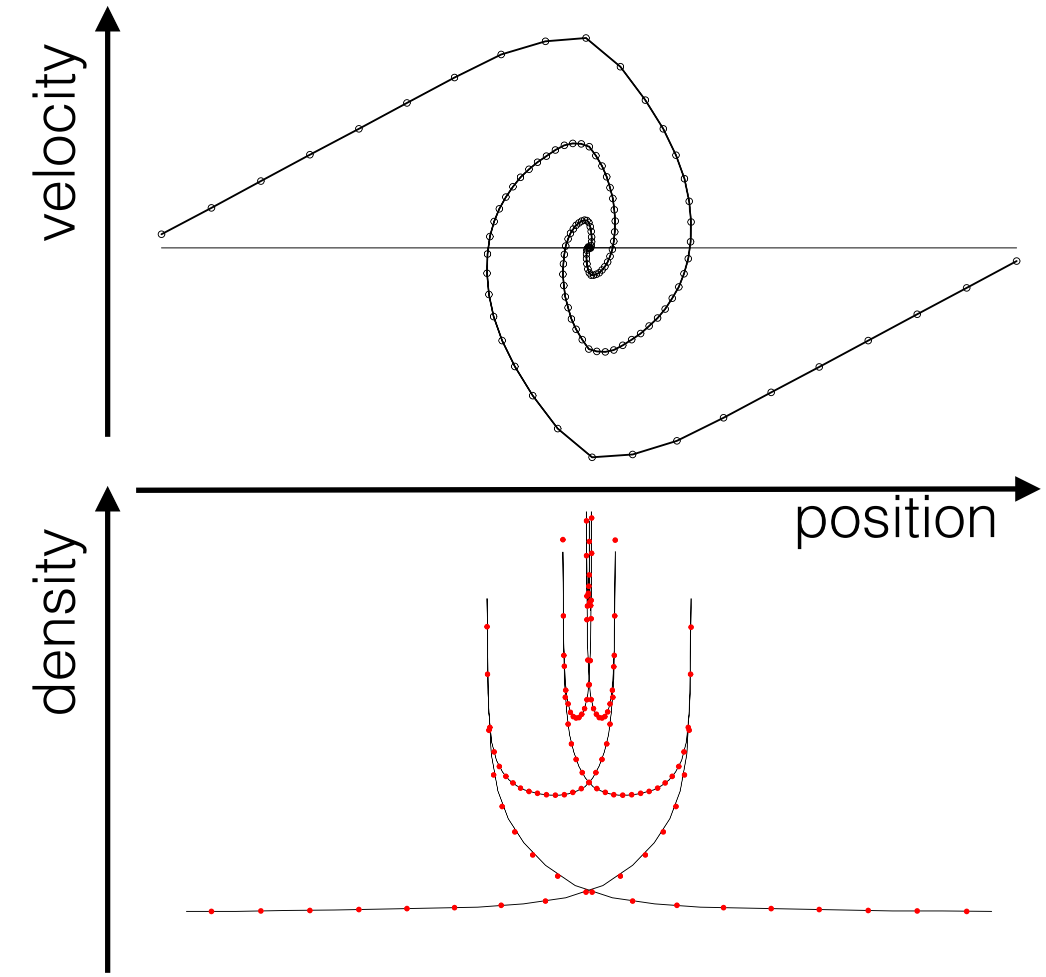
hope that as N->very large numbers, approach collisionless continuum

Lagrangian elements

Define little piecewise maps:

$$\mathbb{Q}_i \subset \mathbb{R}^3 \to \mathbb{R}^6 \; : \; \mathbf{q} \mapsto (\mathbf{x_q}(t), \mathbf{v_q}(t))$$

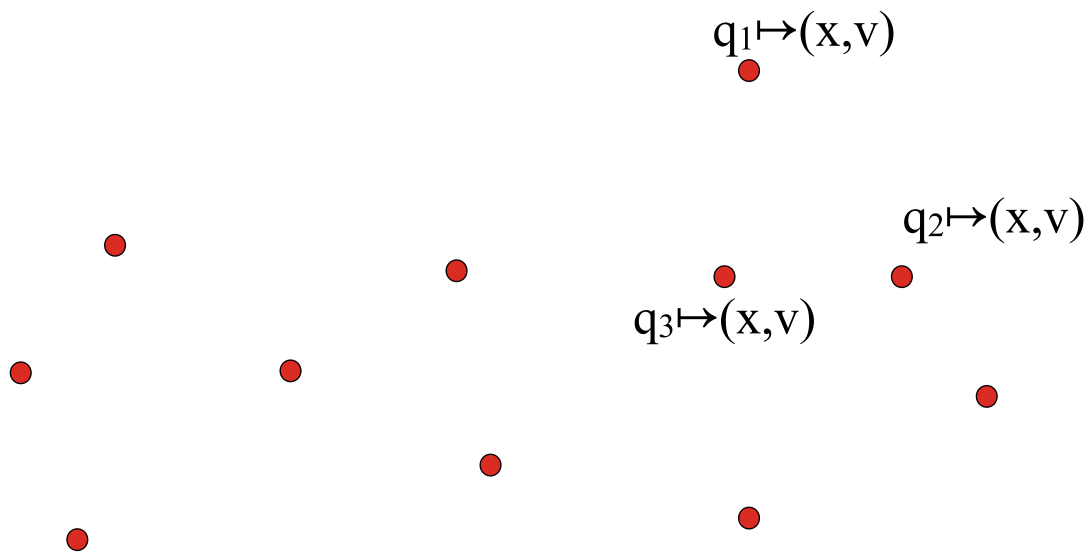


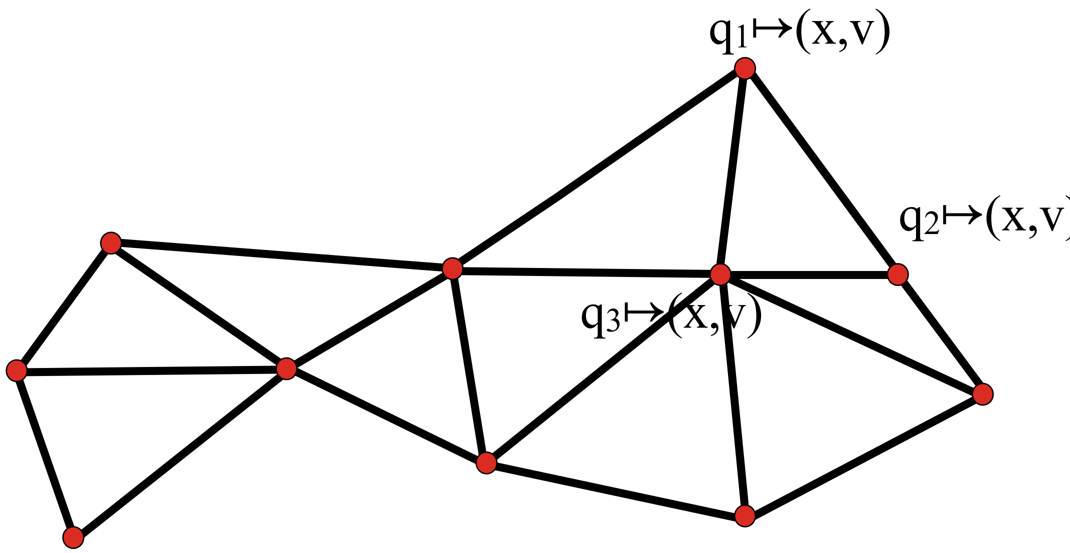


$$\rho = m_{\rm DM} \left| \frac{\partial x_i}{\partial q_j} \right|^{-1}$$

cost:
truncation error
in EoM!

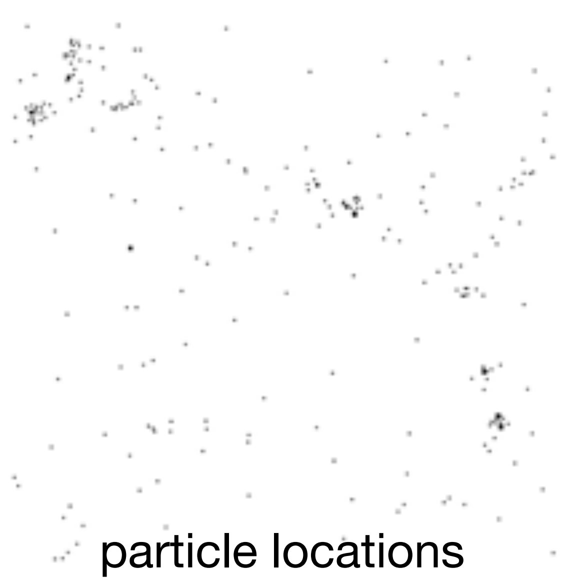
Describing the density field

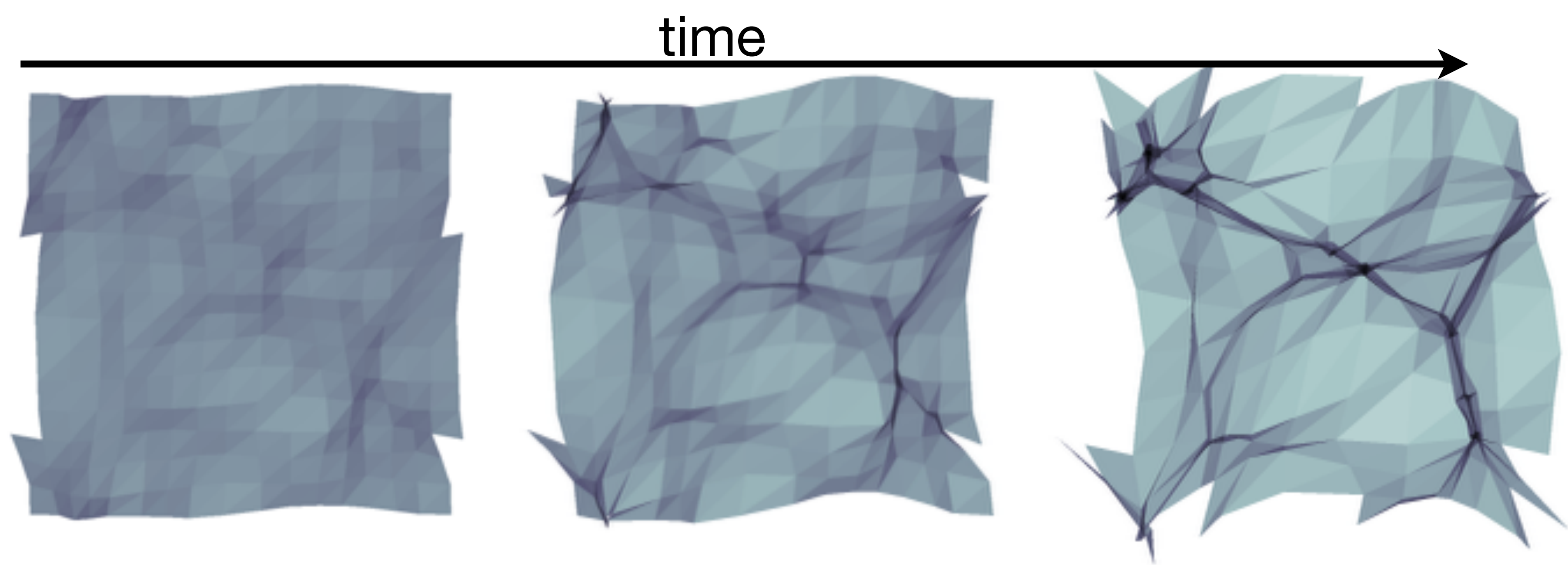




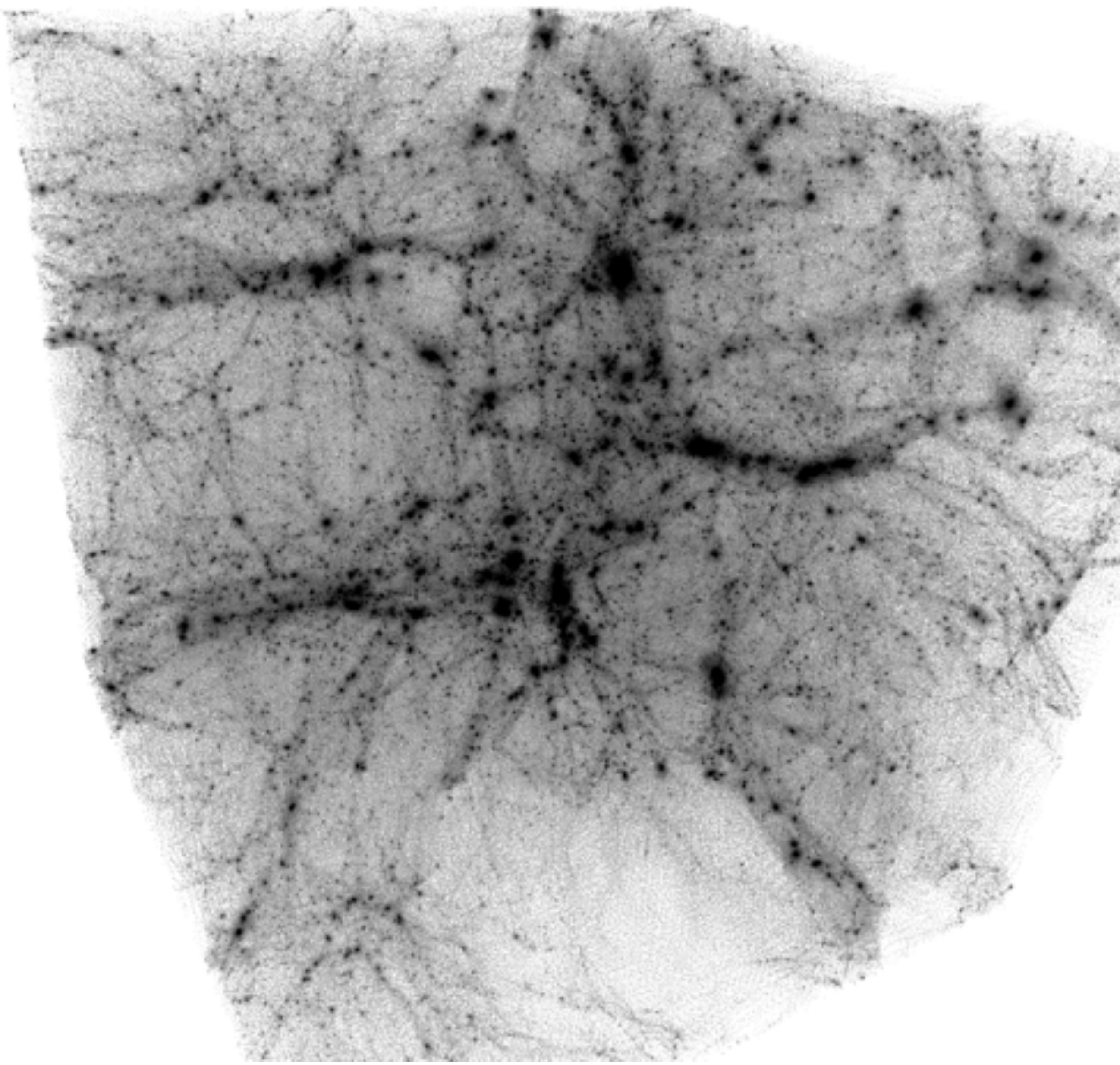
$$\rho = m_p \sum \delta_D(x - x_i) \otimes W$$

$$\rho = m_p \sum_{\text{streams}} \left| \det \frac{\partial x_i}{\partial q_j} \right|^{-1}$$

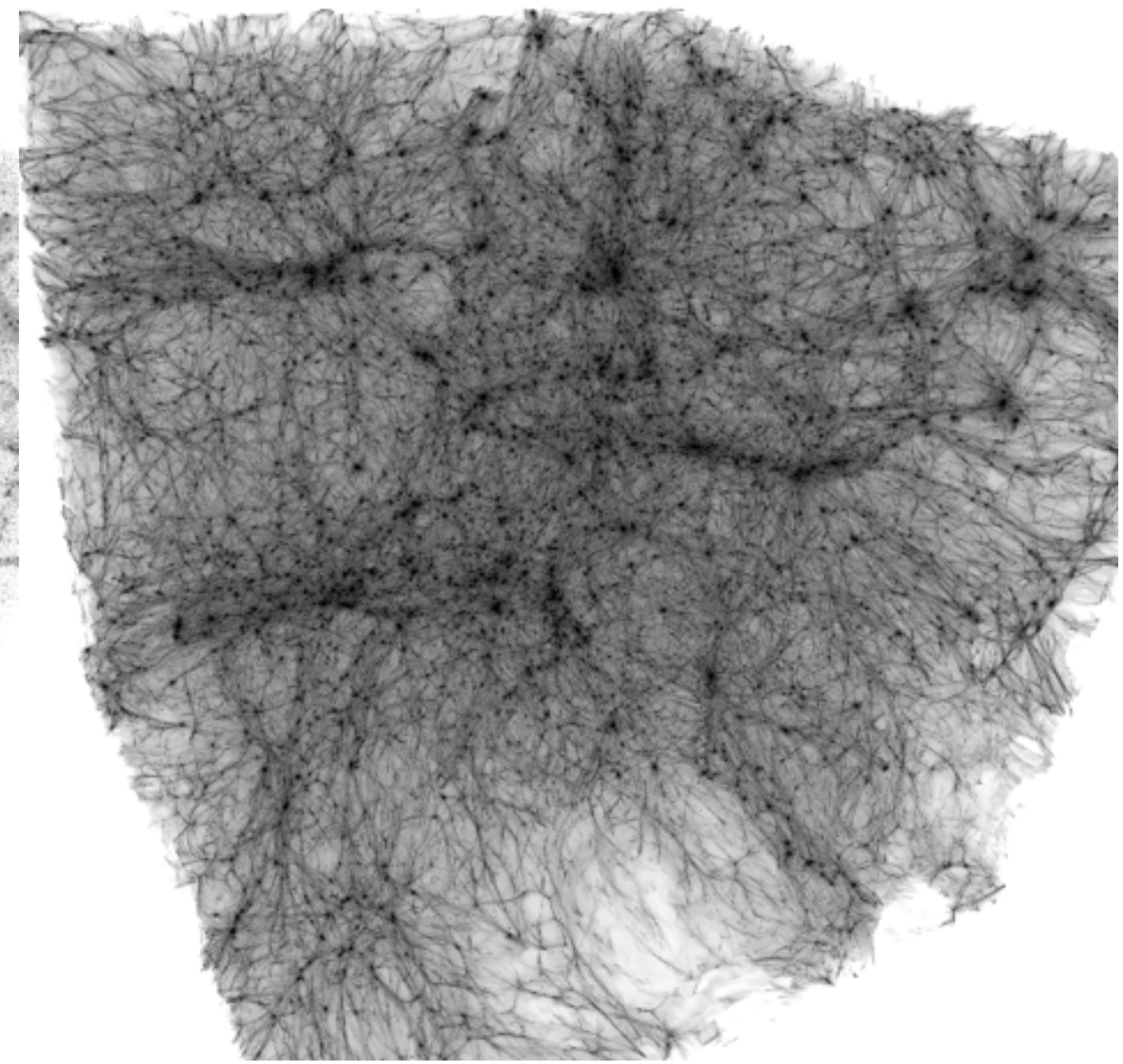




Three dimensions



rendering points for particles.

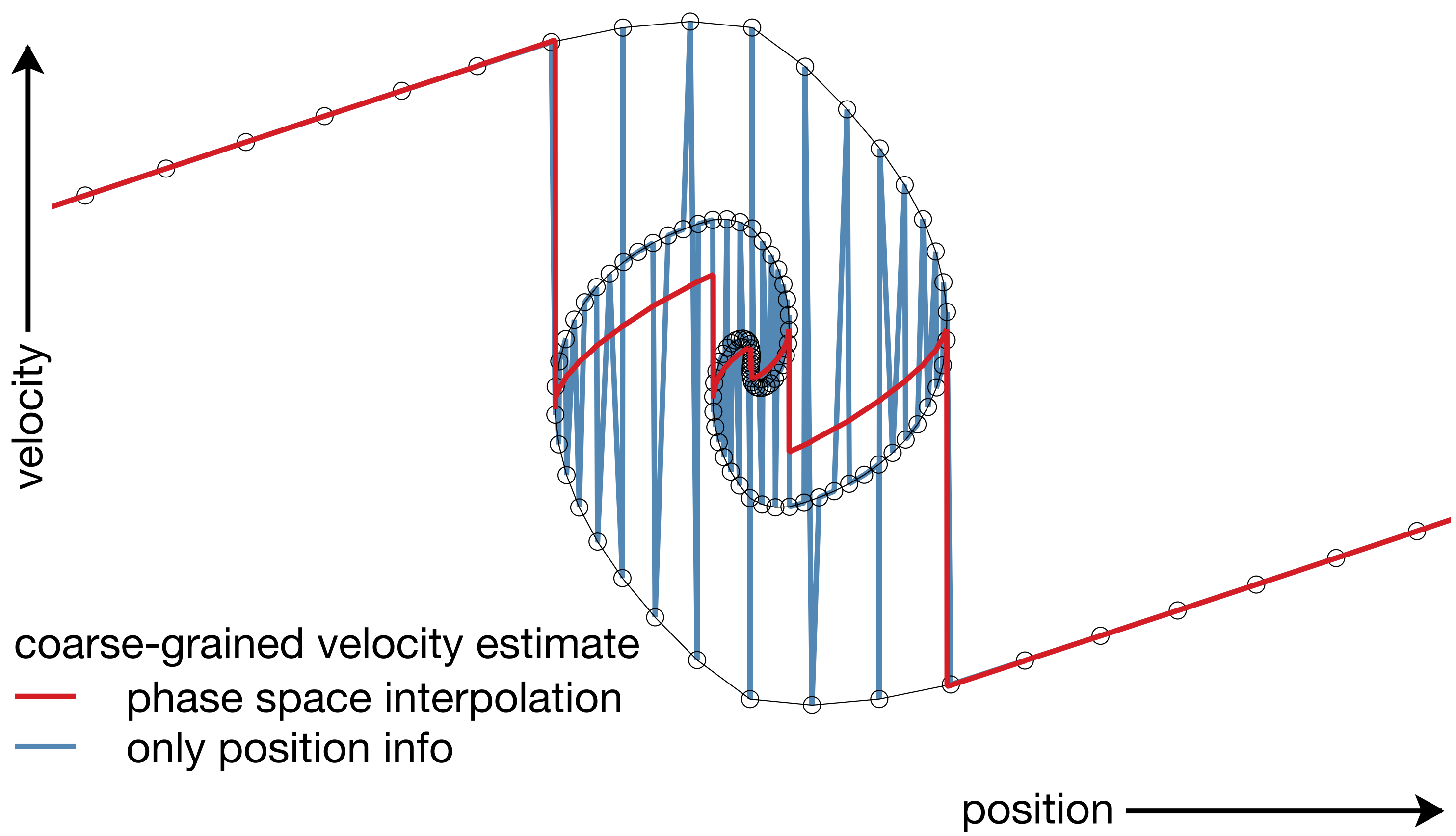


rendering tetrahedral phase space cells.

Same simulation data! (Abel, Hahn, Kaehler 2012)

Problem: How to measure the bulk velocity field?

- Interpolate between neighbouring N-body particles
- “neighbouring” in phase space, not configuration space
- account for averaging over streams (“coarse-graining”)



- Coarse-grained bulk velocity field:

$$\langle \mathbf{v} \rangle \equiv \frac{\int_{\mathbb{R}^3} \mathbf{v}\, f(\mathbf{x},\mathbf{v})\, \mathrm{d}^3 v}{\int_{\mathbb{R}^3} f(\mathbf{x},\mathbf{v})\, \mathrm{d}^3 v} = \frac{\sum_{s\in \mathrm{S}} \mathbf{v}_s(\mathbf{x})\, \rho_s(\mathbf{x})}{\sum_{s\in \mathrm{S}} \rho_s(\mathbf{x})}$$

- result is discontinuous across caustics

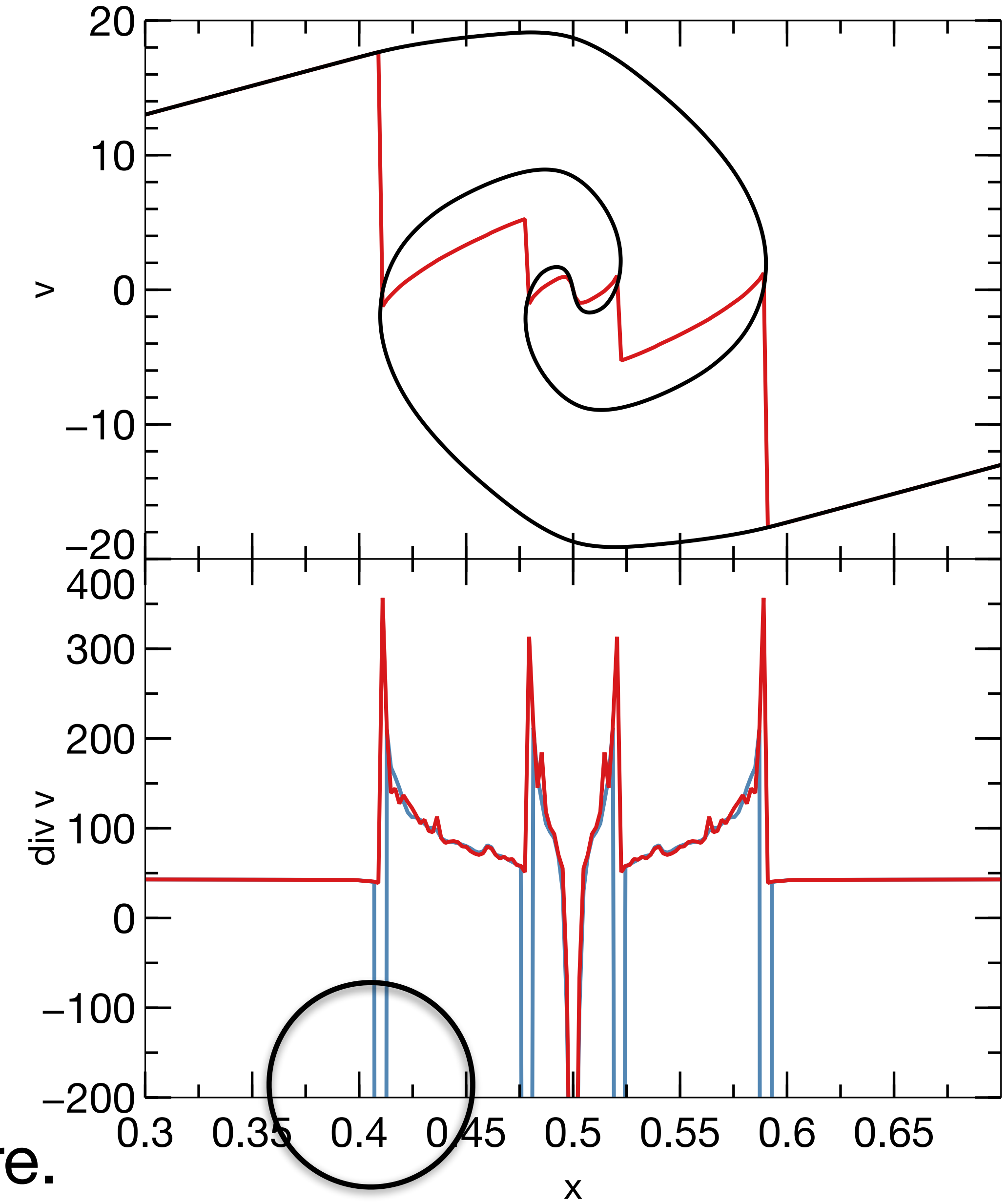
Derivatives of the bulk velocity field

- **Discontinuities make ordinary derivatives ill-defined without coarse-graining!**
- **Away from discontinuities:**
  Need to explicitly evaluate action of derivative on **projected** field:

$$\boldsymbol{\nabla} \cdot \langle \mathbf{v} \rangle = \big\langle (\boldsymbol{\nabla} \log \rho) \cdot (\mathbf{v} - \langle \mathbf{v} \rangle) \big\rangle + \langle \boldsymbol{\nabla} \cdot \mathbf{v} \rangle$$

$$\boldsymbol{\nabla} \times \langle \mathbf{v} \rangle = \big\langle (\boldsymbol{\nabla} \log \rho) \times (\mathbf{v} - \langle \mathbf{v} \rangle) \big\rangle + \langle \boldsymbol{\nabla} \times \mathbf{v} \rangle$$

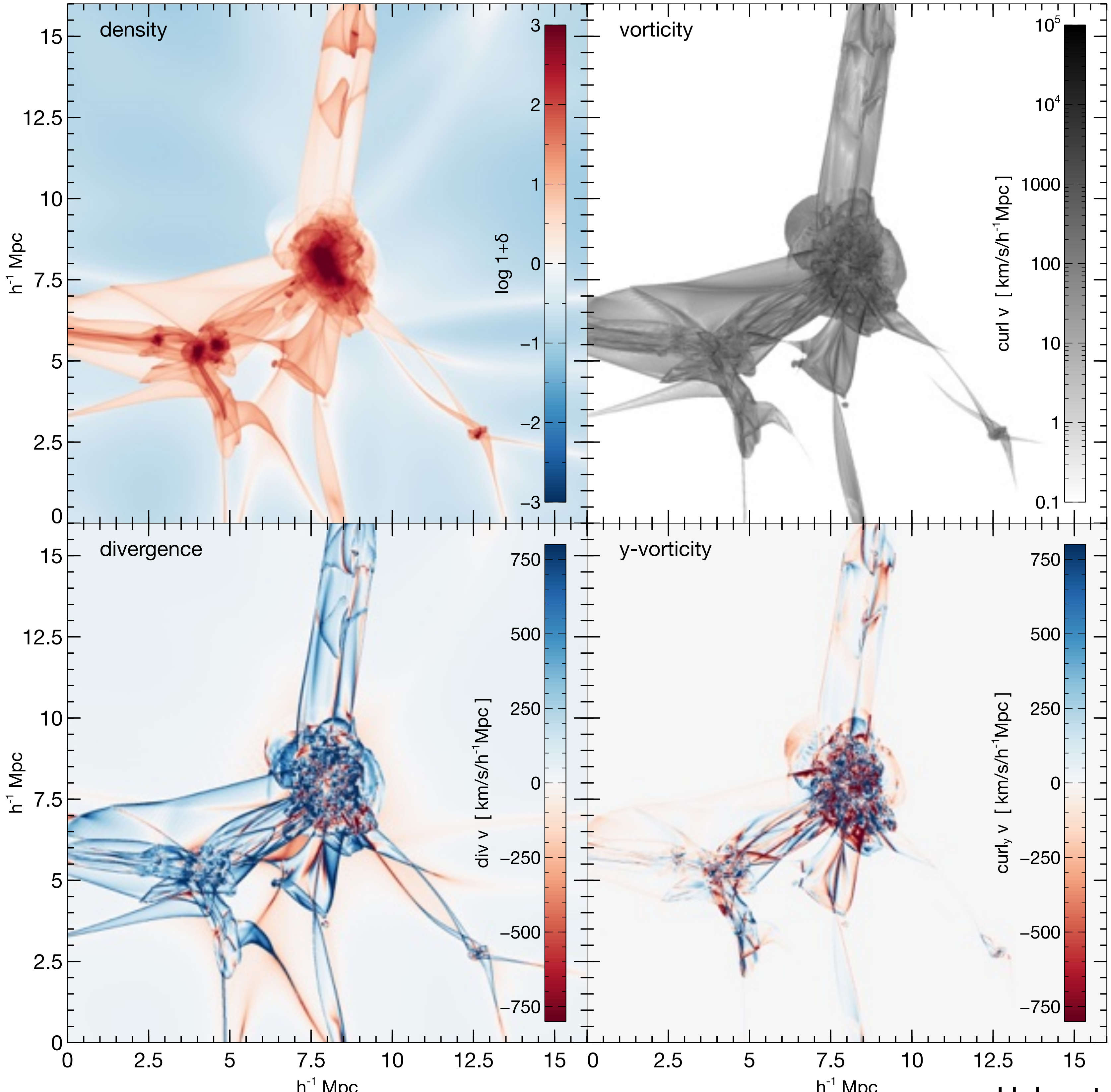
- **Vorticity for std. gravity pure multi-stream phenomenon!!**
- **At discontinuities:**
  Derivatives are singular, but have finite measure.



compressive singularities at caustics

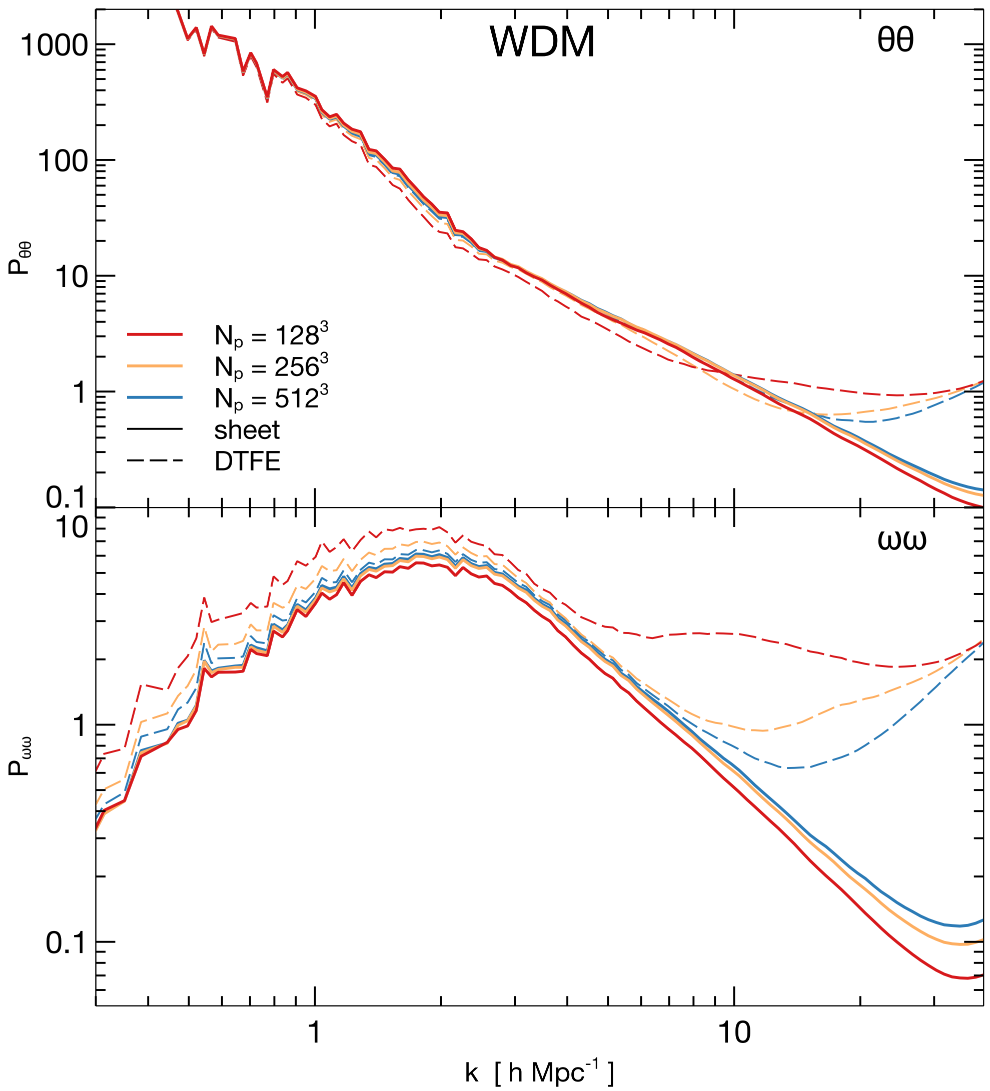
Properties of the cosmic velocity field II

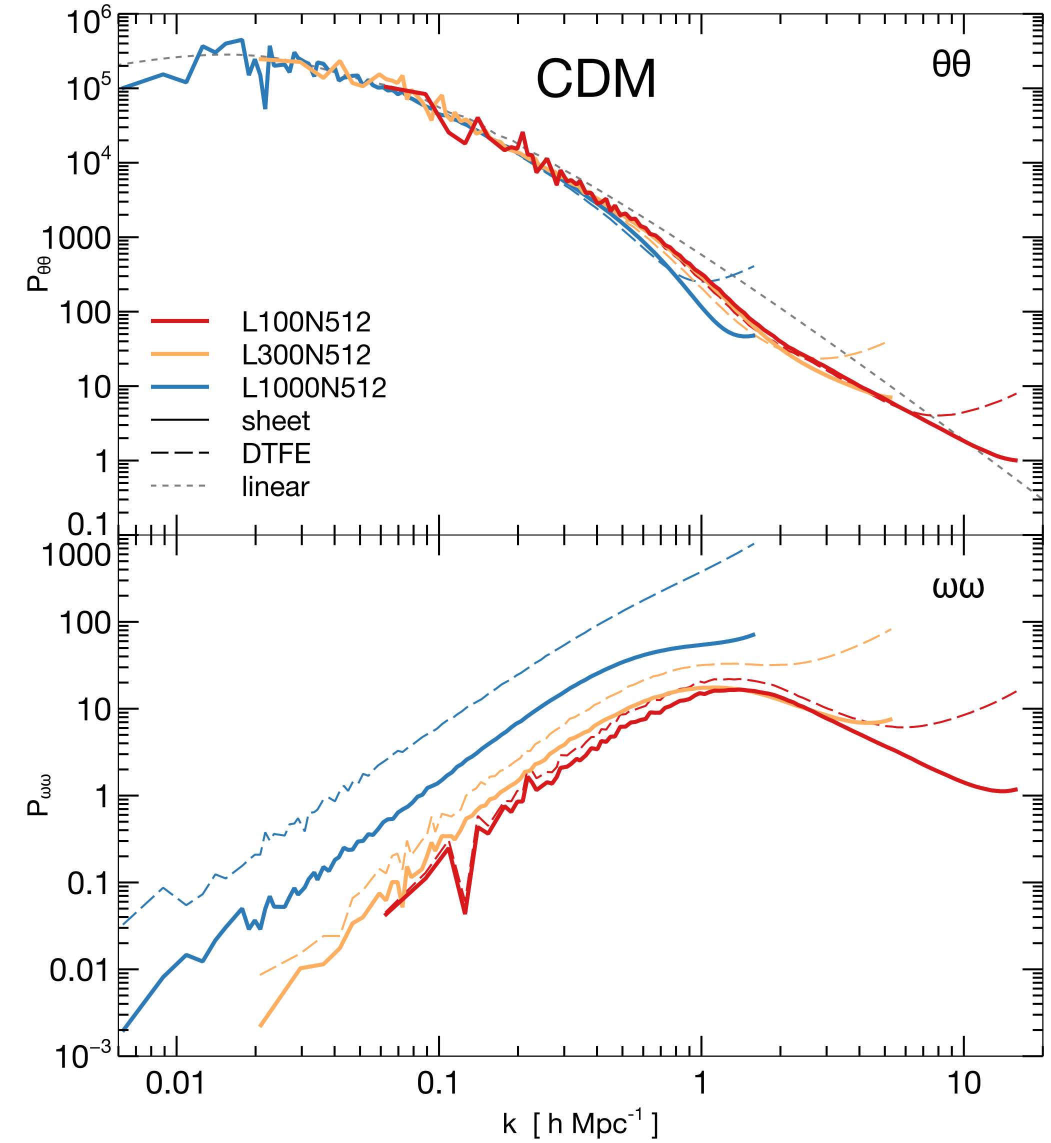
Dark matter
fluid mechanics!

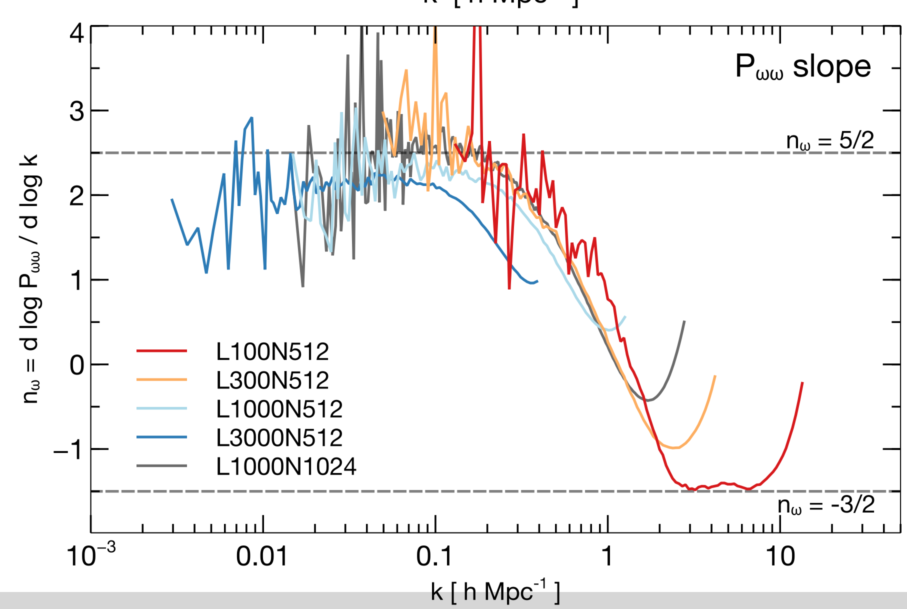


Hahn et al. 2014a

Spectral properties of the cosmic velocity field I





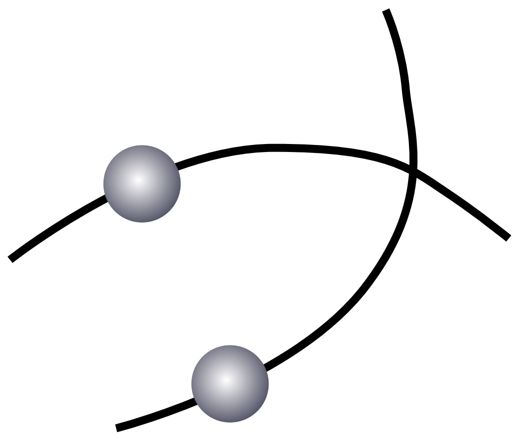


- Faster convergence (for WDM: convergence!)
- Better small scale properties

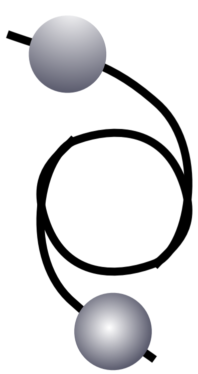
Problems of the N-body method: WDM

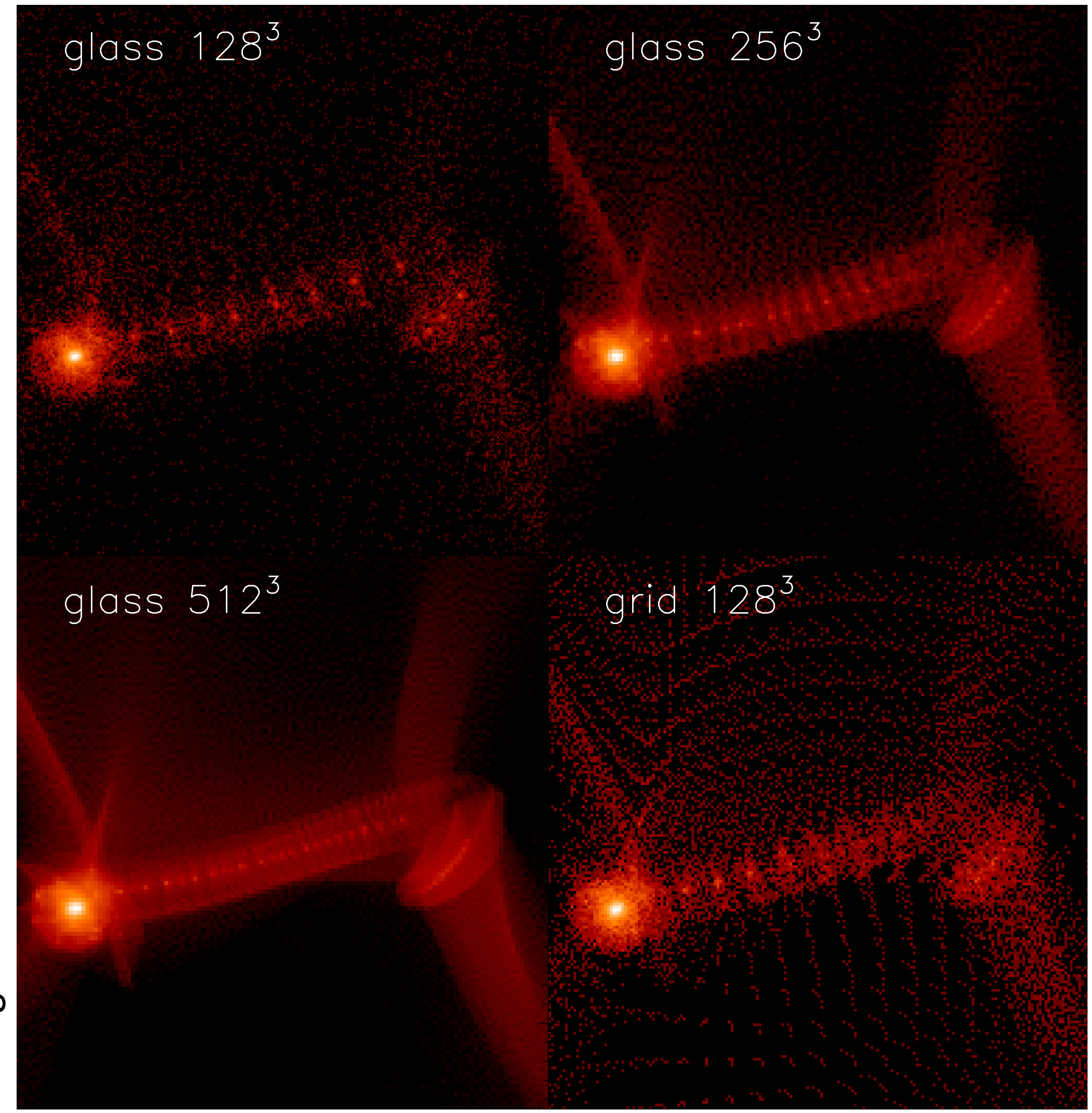
Main Problem: two-body effects, directly related to force softening

Scattering



Clumping/
Fragmentation





Wang&White 2007

Most obvious for non-CDM simulations!

(e.g. Centrella&Melott 1983, Melott&Shandarin 1989, Wang&White 2007)

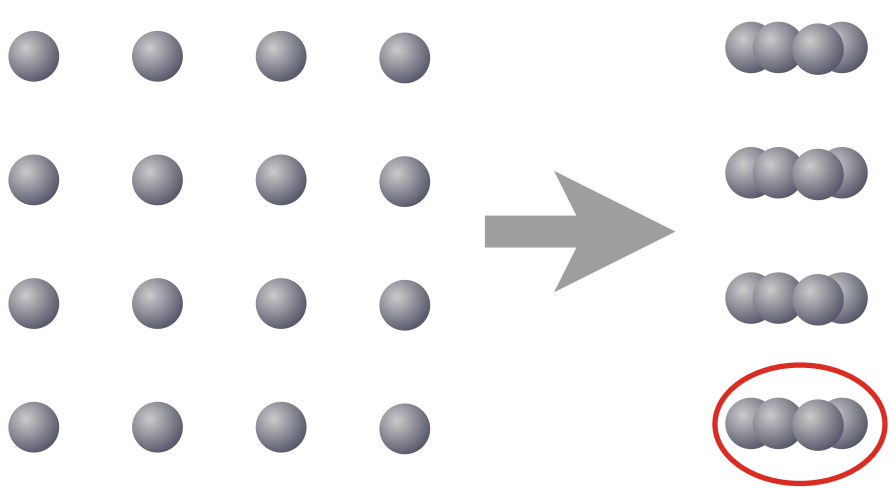
Improving on N-body….

N-body

$$\dot{\mathbf{x}}_i = \frac{1}{m\,a^2}\mathbf{p}_i \quad \text{and} \quad \dot{\mathbf{p}}_i = -m\,\boldsymbol{\nabla}_x\phi|_{\mathbf{x}_i}$$

point-wise and Hamiltonian

need softening,
no knowledge what it
should be (empirical)



Lagrangian phase-space element

$$\dot{\mathbf{x}}_\mathbf{q} = \mathbf{v}_\mathbf{q}, \quad \text{and} \quad \dot{\mathbf{v}}_\mathbf{q} = -\,\boldsymbol{\nabla}_x\phi|_{\mathbf{x}_\mathbf{q}}\,, \quad \text{with } \mathbf{q} \in \mathcal{Q}$$

continuum structure (diff w.r.t. q), approx by

$$P_k = \{\pi(\mathbf{q}) \mid \pi(\mathbf{q}) = \sum_{\alpha,\beta,\gamma=0}^{k} a_{\alpha\beta\gamma}\, q_0^\alpha\, q_1^\beta\, q_2^\gamma\}$$

-> EoM for polynomial coefficients

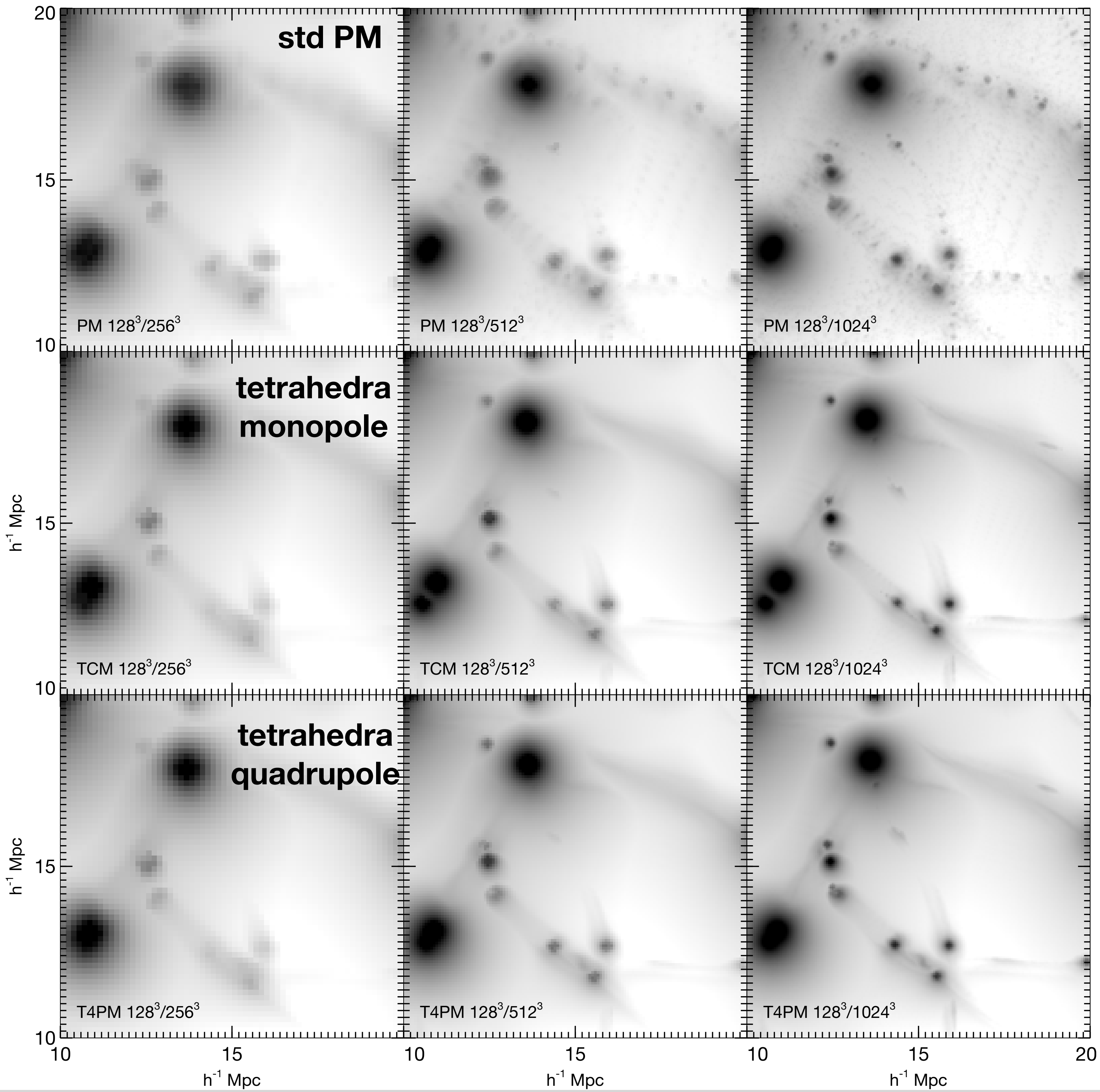
$$\dot{\mathbf{x}}_{\alpha\beta\gamma} = \mathbf{v}_{\alpha\beta\gamma}, \quad \dot{\mathbf{v}}_{\alpha\beta\gamma} = -J^{-1}\mathbf{f}_{\alpha\beta\gamma}$$

explicit truncation error:

$$\Delta\dot{\mathbf{v}} = -J^{-1}\sum_{\alpha,\beta,\gamma=k+1}^{\infty} \mathbf{f}_{\alpha\beta\gamma}\, q_0^\alpha\, q_1^\beta\, q_2^\gamma$$

Using tets for simulations: 300eV toy WDM problem

fixed mass resolution, varying force resolution:

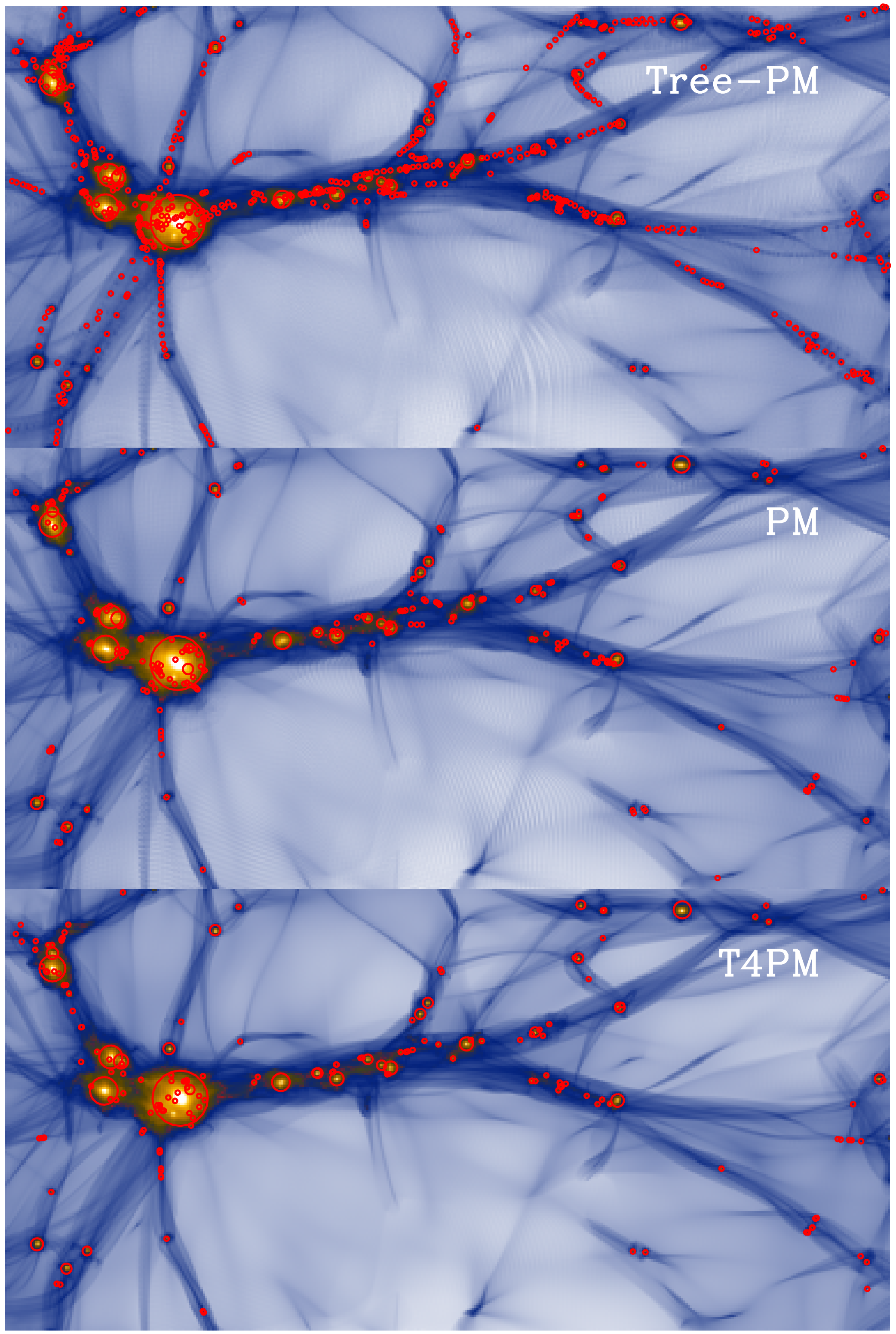


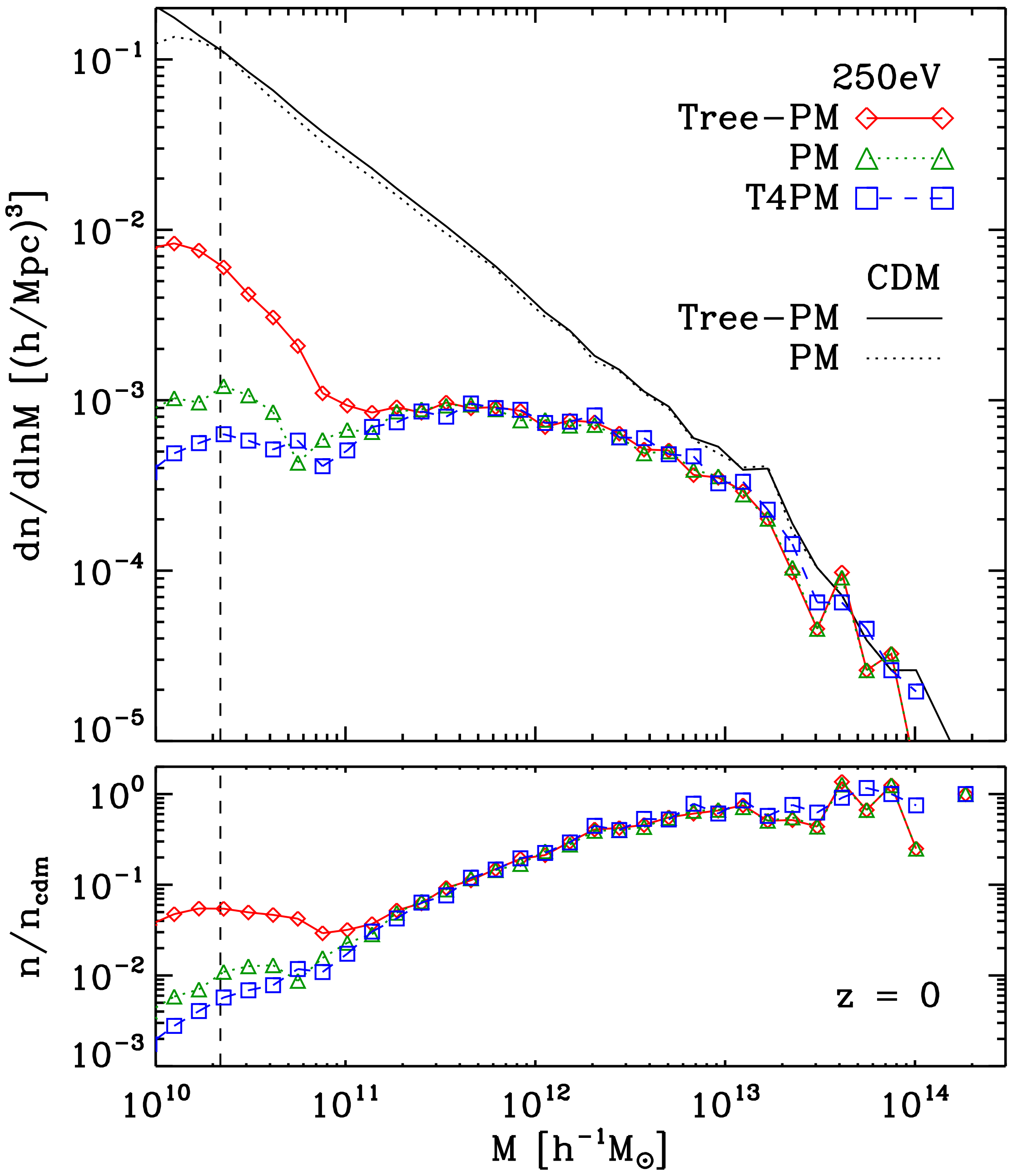
force res. →

features become sharper
fragmentation appears

sheet tesselation
based method cures
artificial fragmentation

First determination of WDM halo mass function!





Angulo, Hahn & Abel 2013

Limitations - diffusion/loss of energy cons.

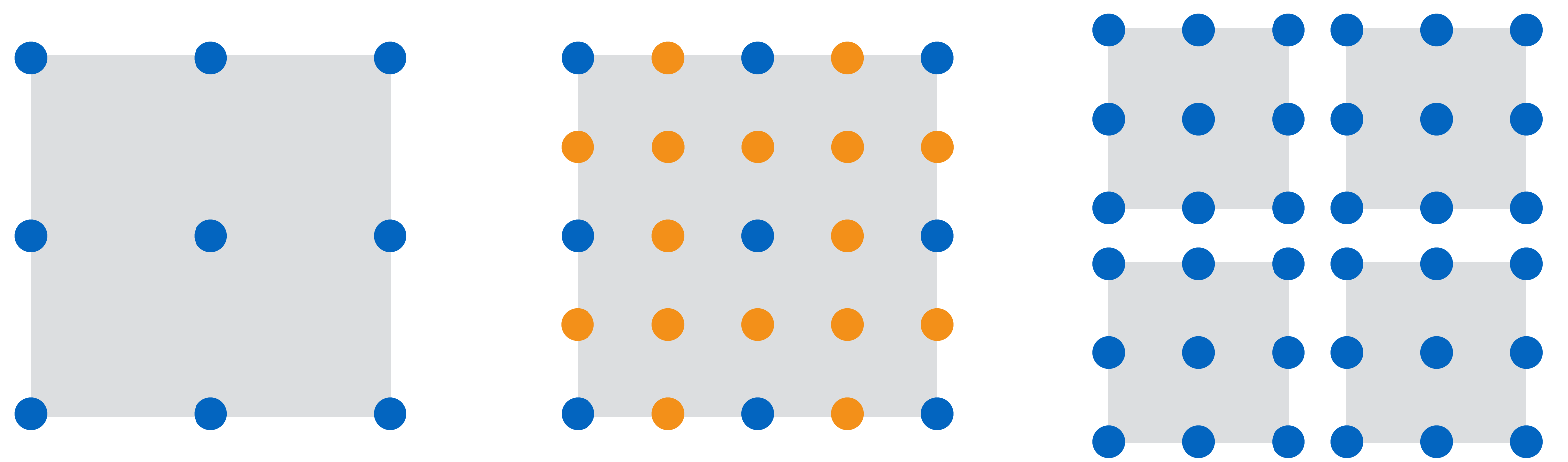
Mixing - (phase or chaotic)
need increasingly larger number of elements to trace the sheet surface

hi-res N-body

tesselated cube orbiting
in non-harmonic potential

Need adaptive refinement

adaptive refinement:

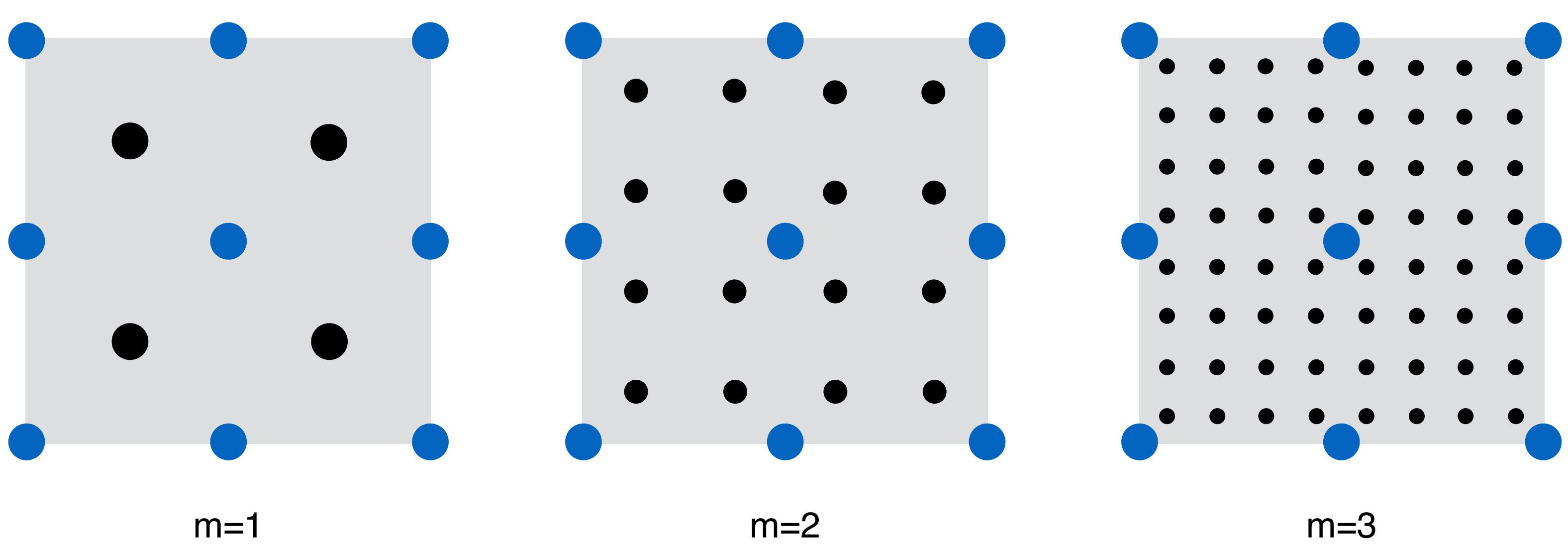


a. element is flagged for refinement

b. positions and velocities are determined at mid-points

c. new elements are created using the mid-point values

approximate element mass distribution by recursively deposited ‘mass carrier particles’ (these are not *active, i.e. no degrees of freedom*)



Hahn & Angulo 2015

refinement + higher order!

hi-res N-body

tesselated cube orbiting
in non-harmonic potential

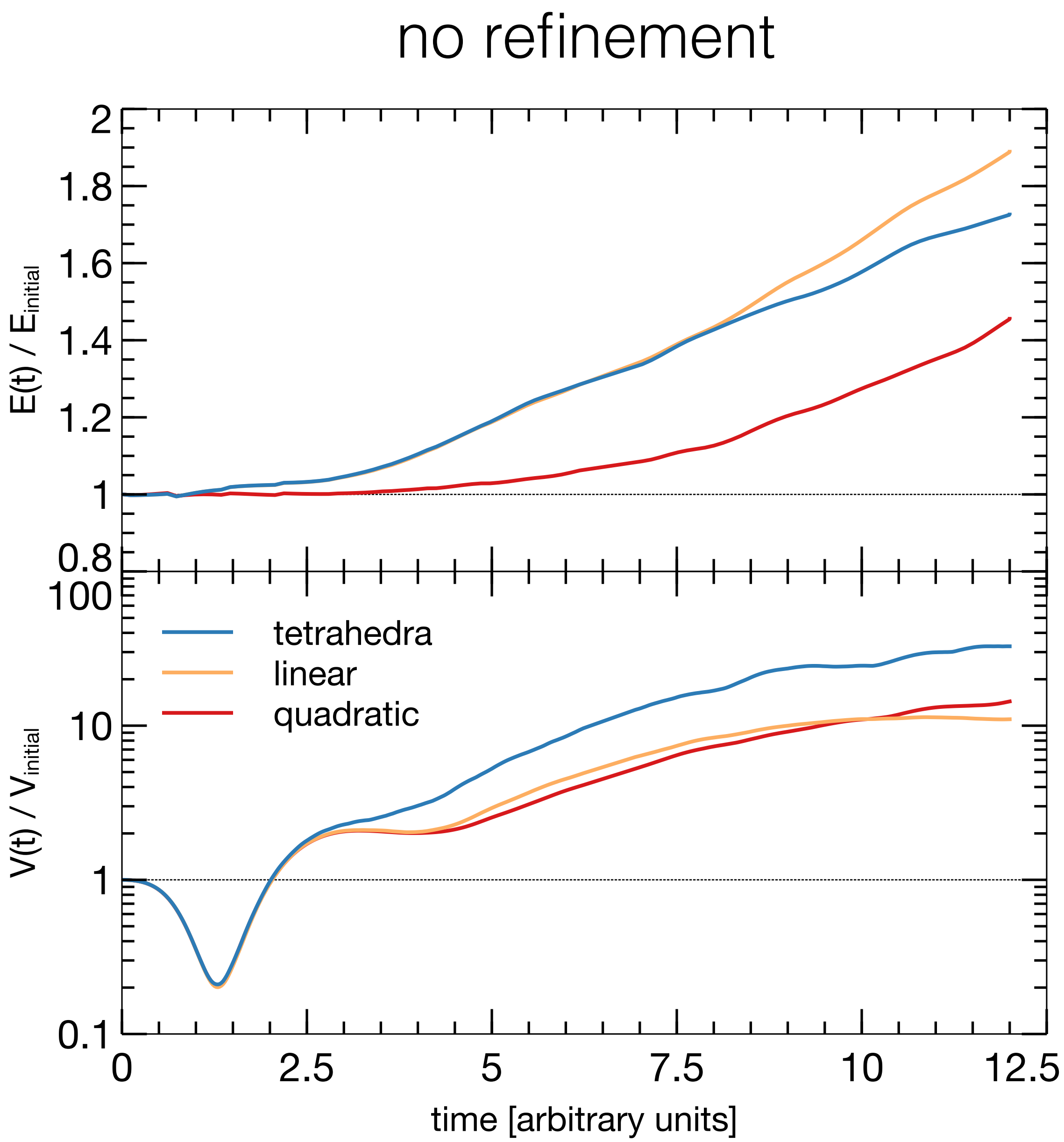
adaptively refined tri-quadratic
phase-space element

first alternative to N-body in highly non-linear regime!

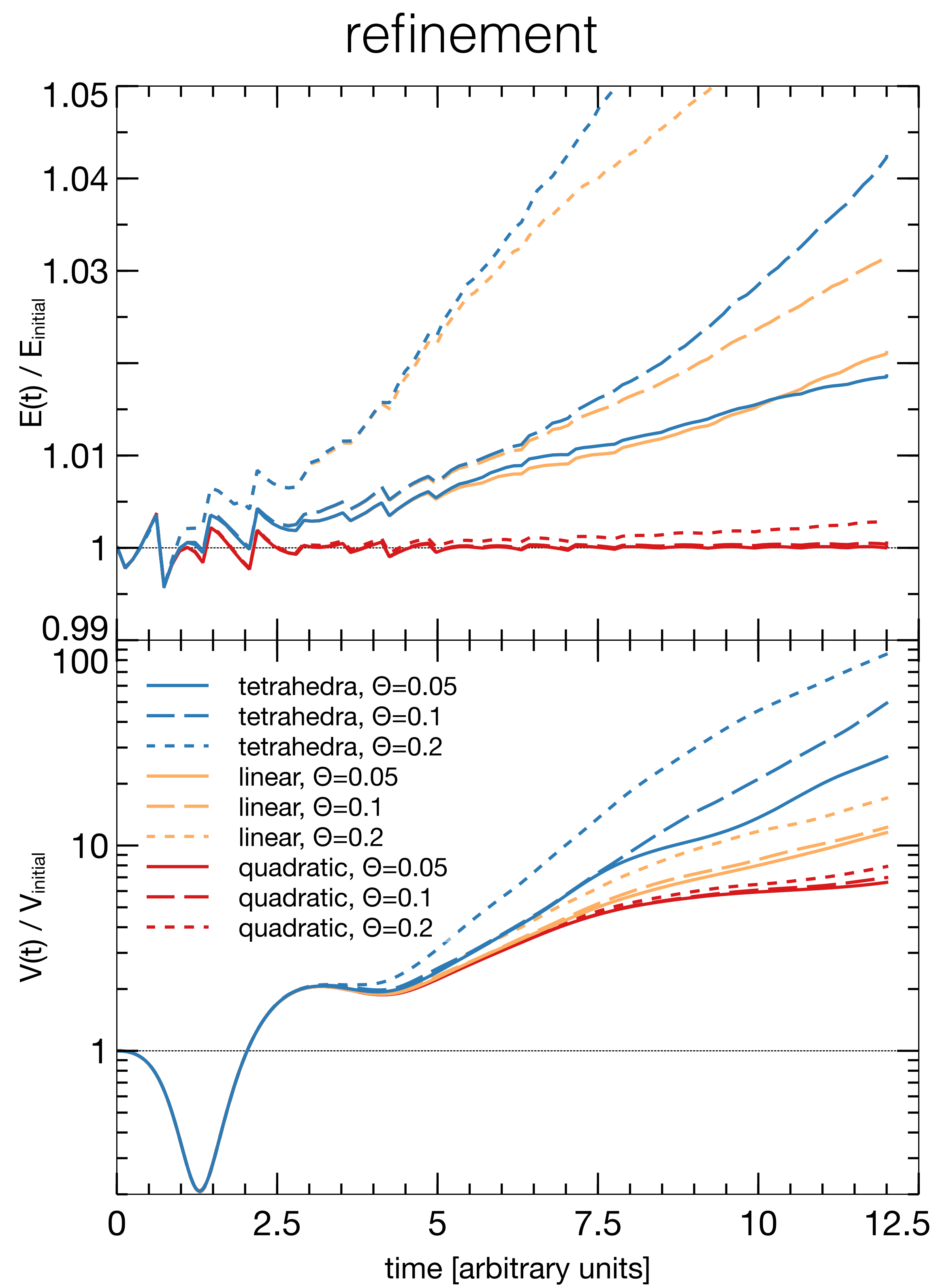
+ able to track fine-grained phase space

Hahn & Angulo 2015

Orbit test



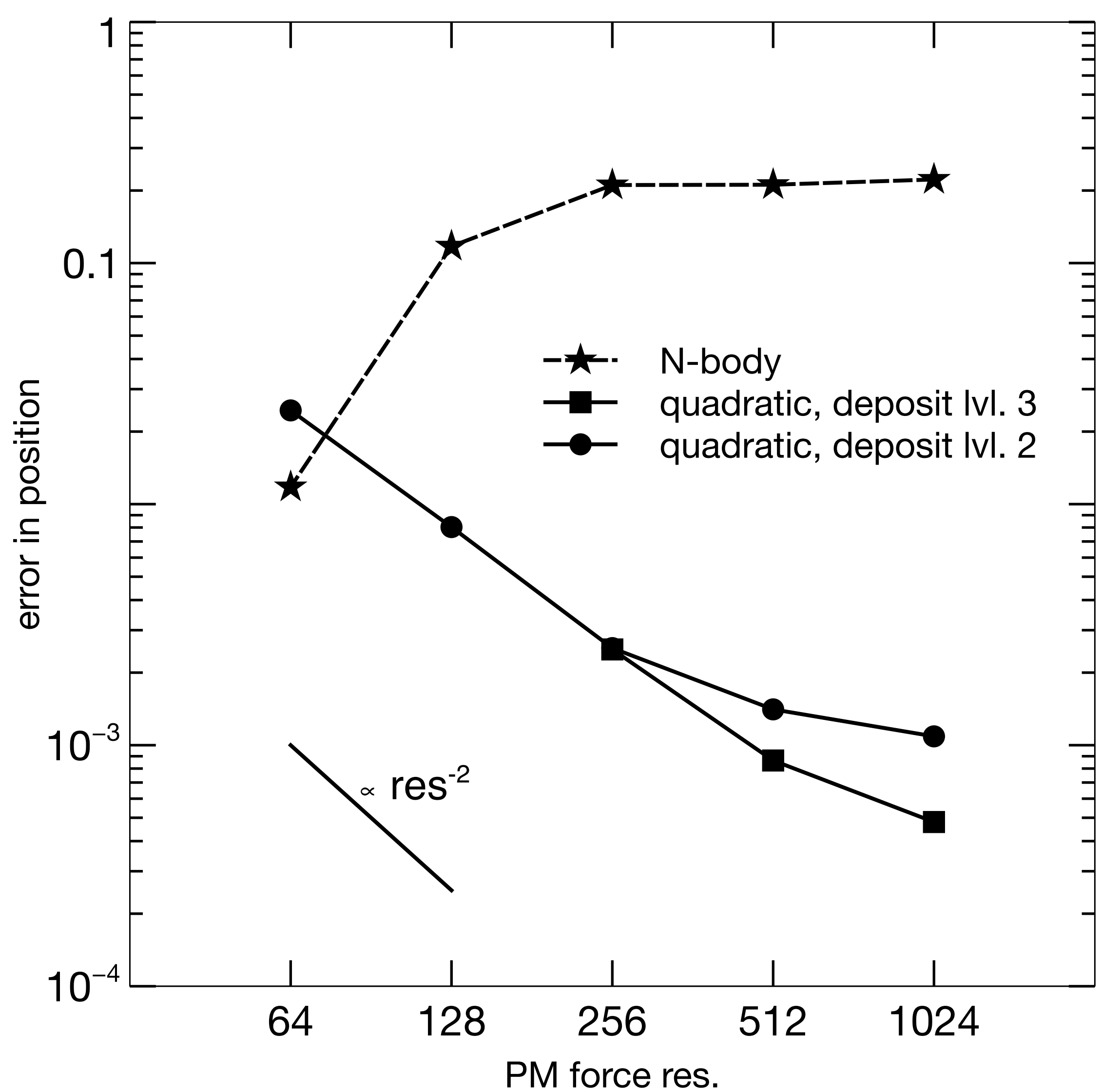
no refinement
E(t) / $E_{initial}$
V(t) / $V_{initial}$
tetrahedra
linear
quadratic
time [arbitrary units]



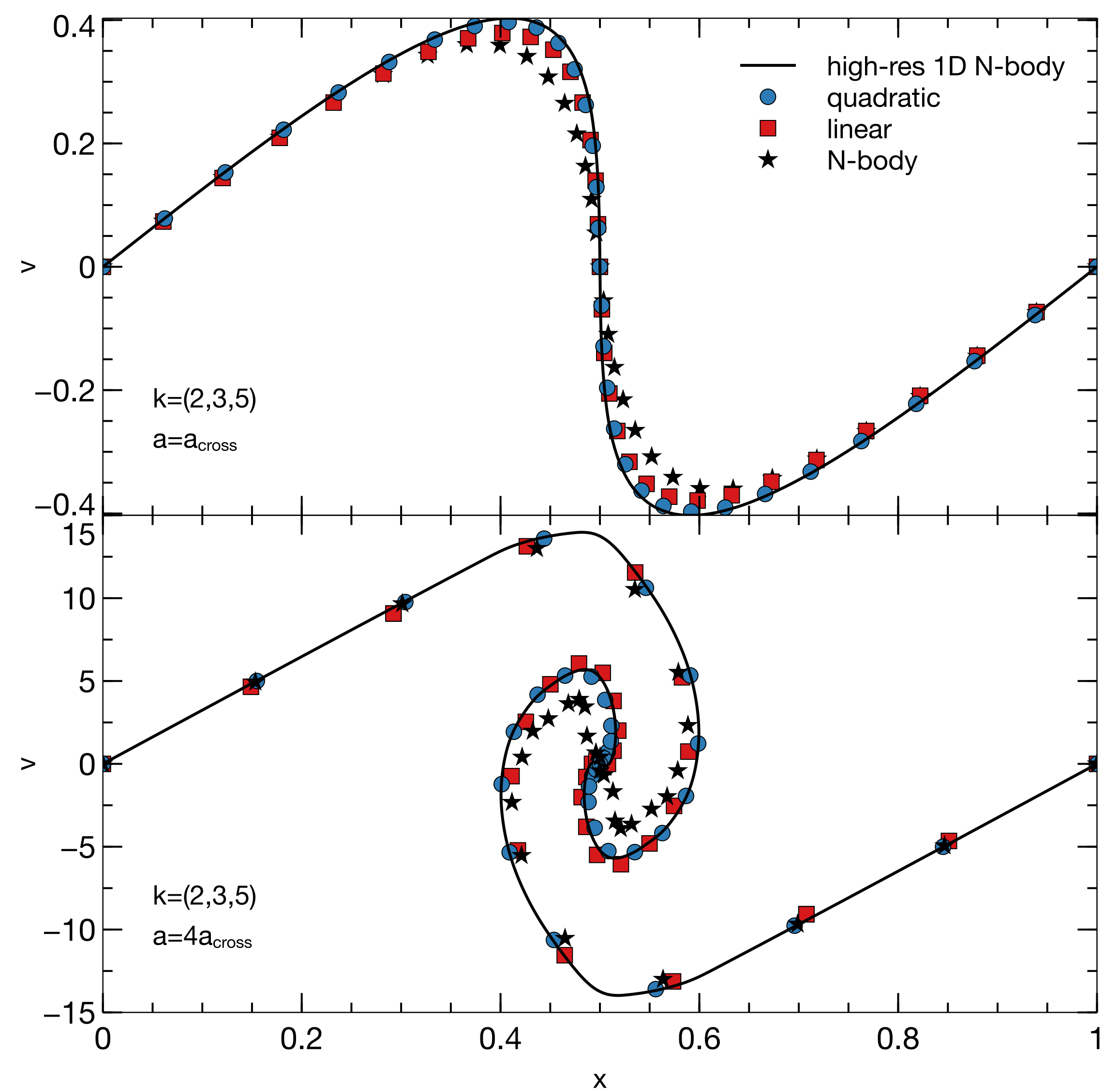
refinement
E(t) / $E_{initial}$
V(t) / $V_{initial}$
tetrahedra, Θ=0.05
tetrahedra, Θ=0.1
tetrahedra, Θ=0.2
linear, Θ=0.05
linear, Θ=0.1
linear, Θ=0.2
quadratic, Θ=0.05
quadratic, Θ=0.1
quadratic, Θ=0.2
time [arbitrary units]

Self-gravitating tests 1D

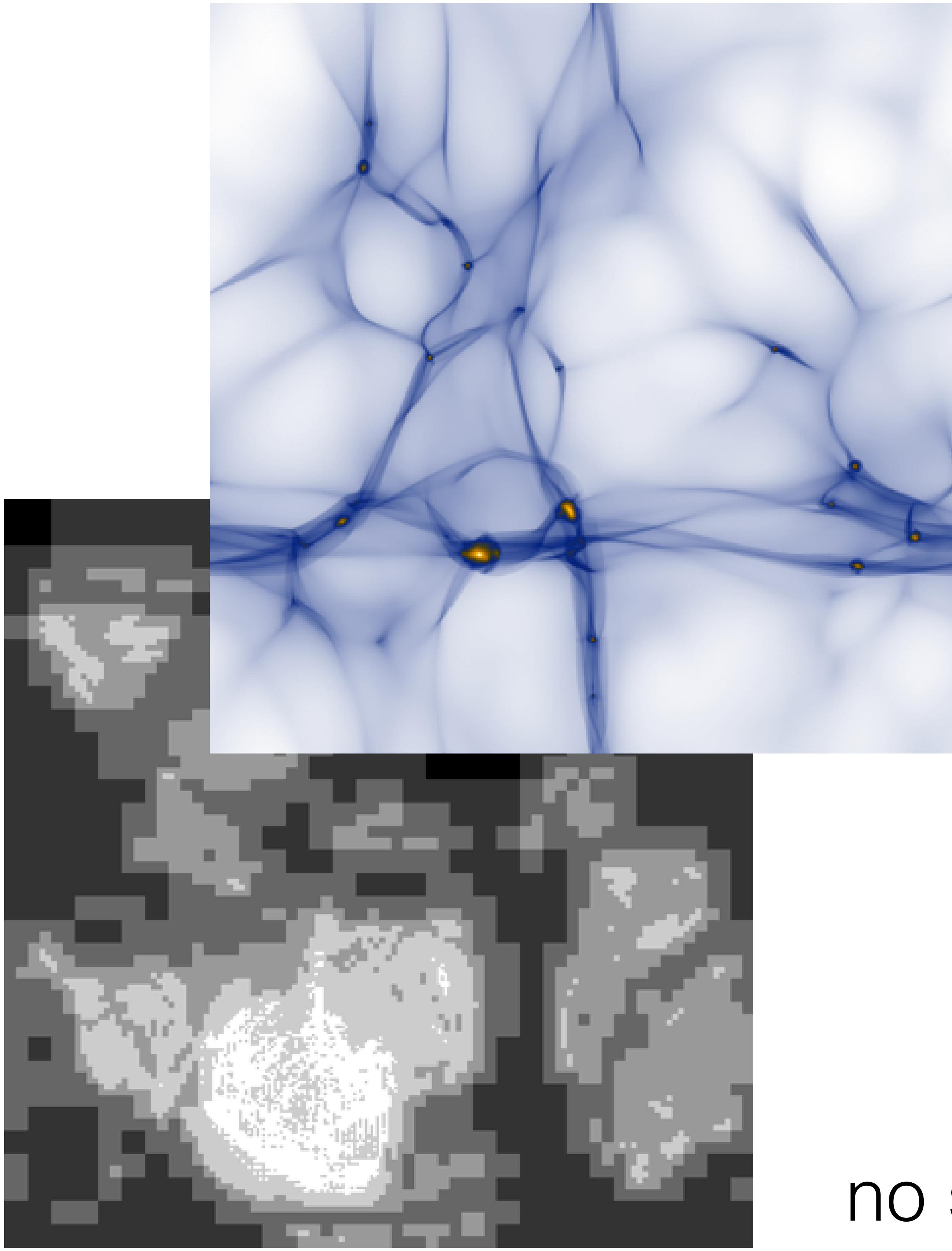
32^3 particle plane wave, axis aligned

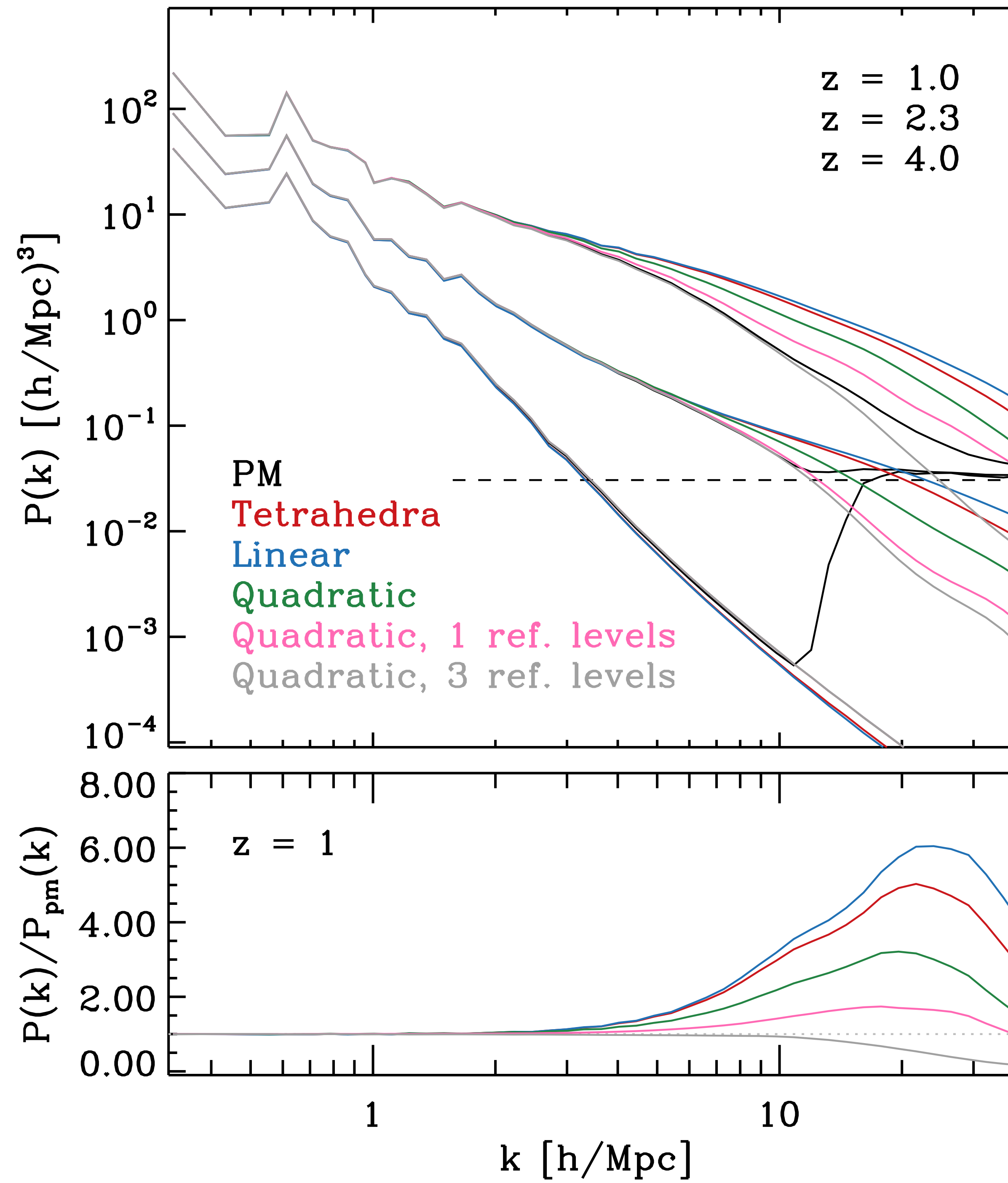


32^3 particle plane wave, oblique



let’s go cosmological





no shotnoise!!!

Hahn & Angulo 2015

Conclusions

- Lagrangian elements can give new insights into existing simulations (density/velocity fields, multi-stream analysis,…)
- Provide also self-consistent simulation technique. (functional when using high-order and adaptive refinement)
- Solves two-body and fragmentation problems of N-body
- First methodological test of N-body in deeply non-linear regime
- Stay tuned for halo properties…