A Comment on Multi-Gluon Amplitudes and DPIs

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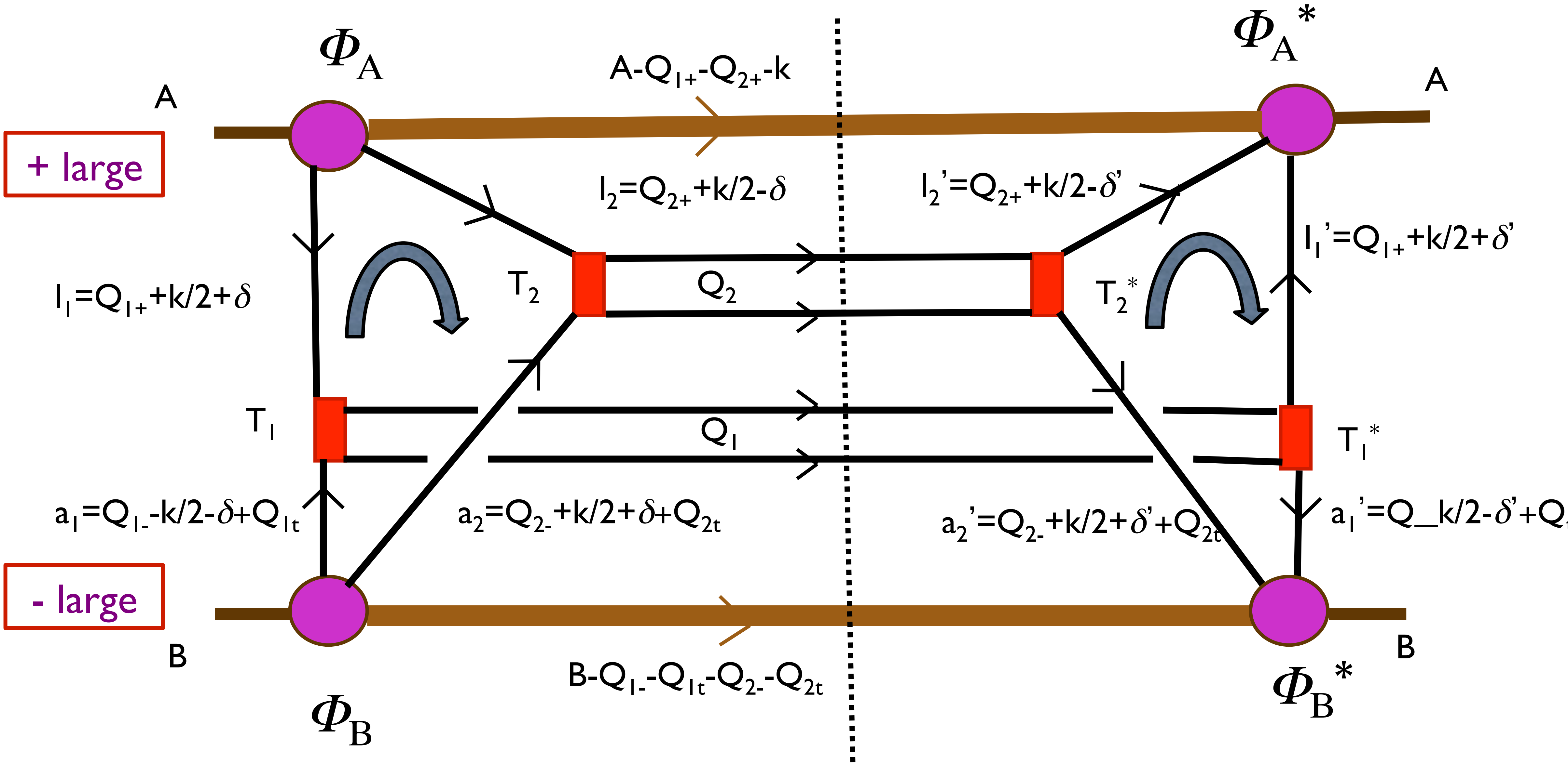
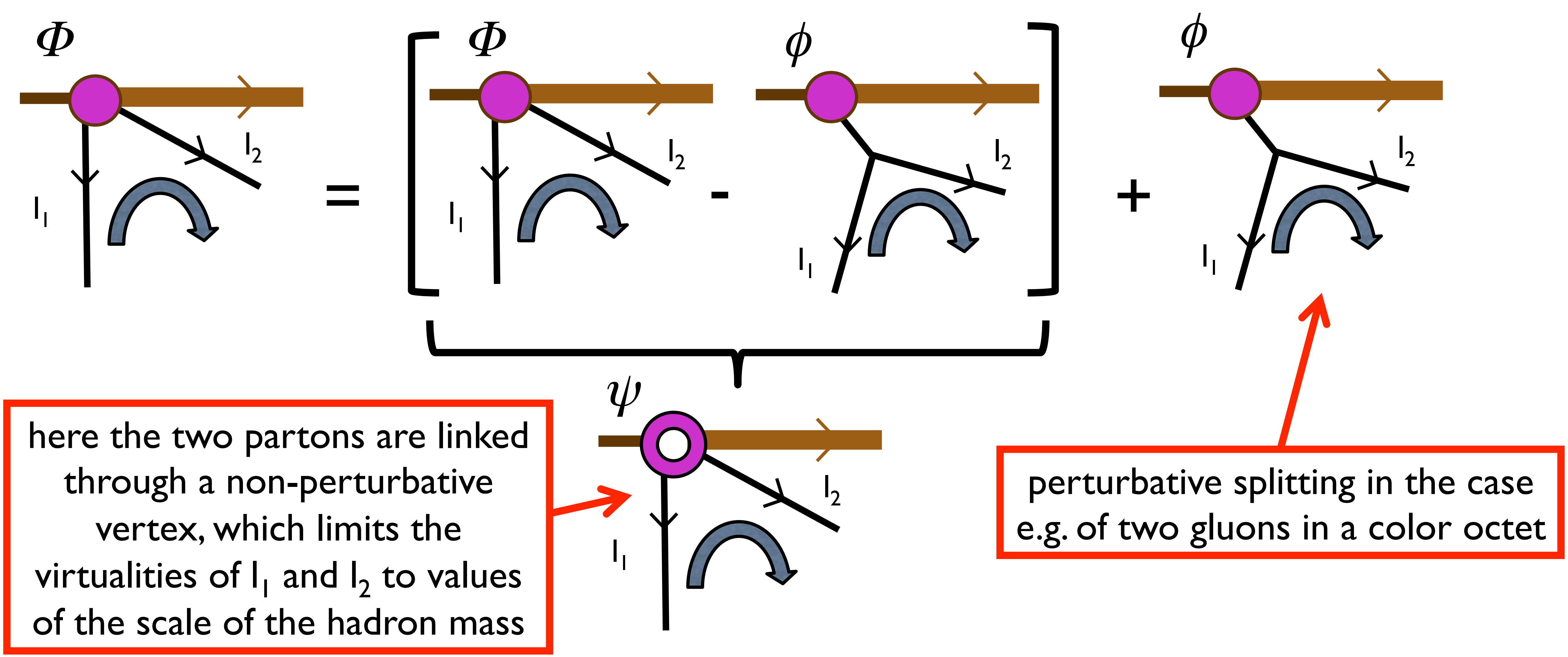
In a Double Parton Interaction two different pairs of partons interact with large momentum transfer exchange in the same inelastic event.



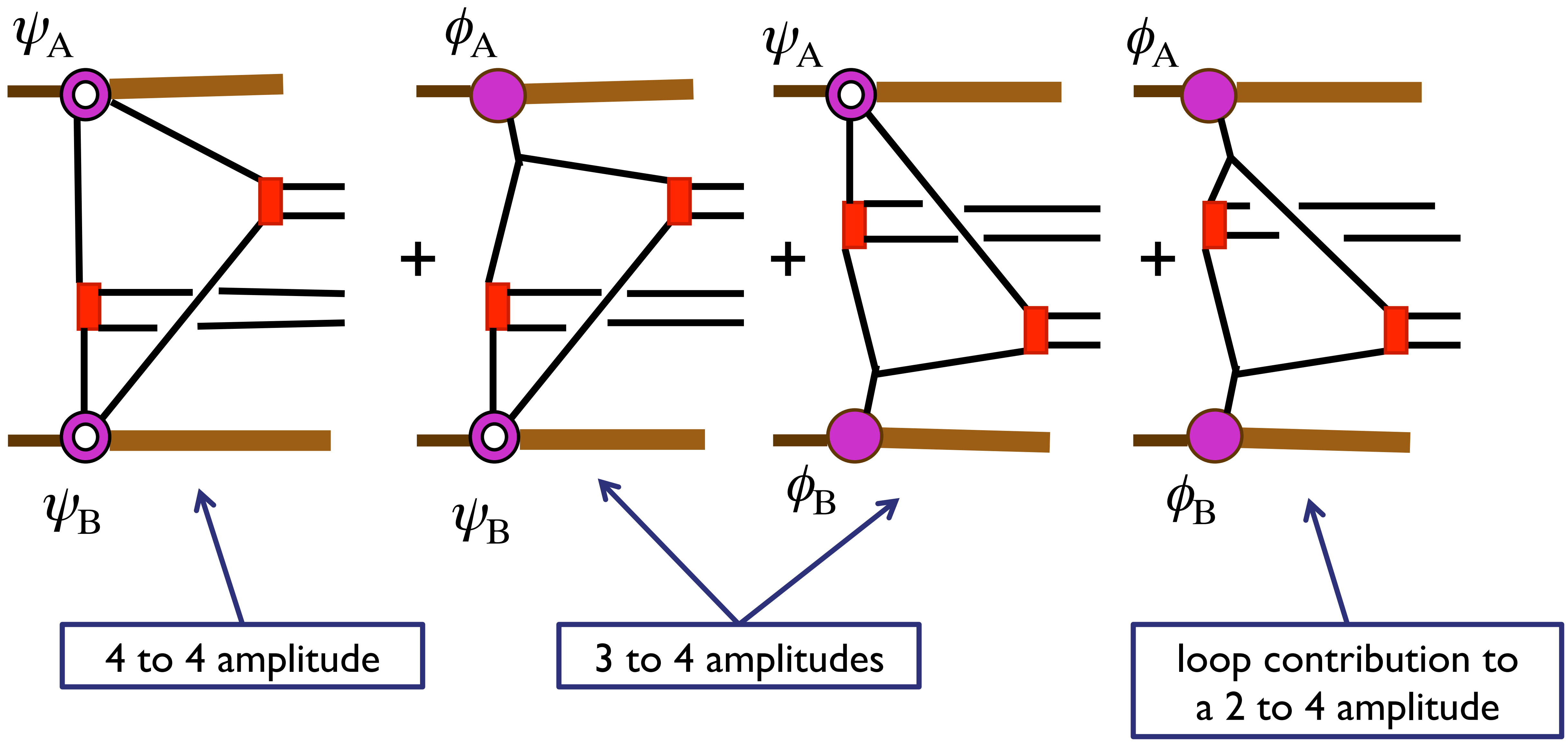
The cross section can be expressed as a contribution to the forward elastic scattering amplitude, which is characterized by two independent loops where initial state momenta of the partonic interactions are integrated independently in the amplitude of the process and in its complex conjugate.

A Double Parton Interaction cross section therefore is not diagonal as a function of initial state momenta.

It is convenient to distinguish two different contributions in the non-perturbative vertices :



The amplitude thus splits into 4 different pieces, the first 3 introduce a non perturbative dimensional factor in the cross section and are thus of interest to DPIs, the 4th tem is a higher order correction to the 2 to 4 parton scattering amplitude :



Due to the loop integration, the Double Parton Interaction cross section is not diagonal as a function of initial state partons momenta.

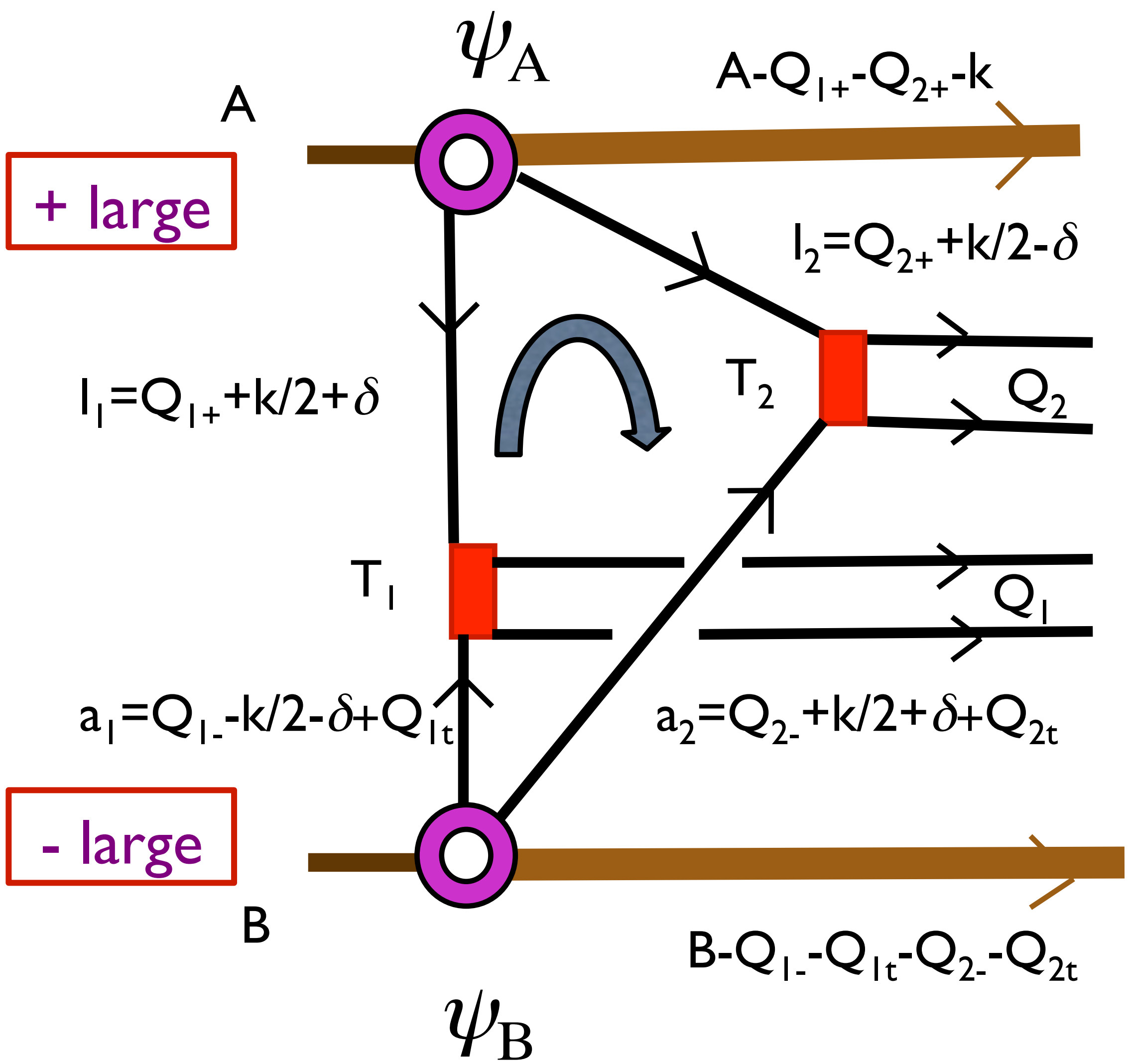
The non perturbative vertex in the loop introduces a non-perturbative scale, which limits the virtualities in the loop to values of the order of the hadron mass. The upper lines in the loop are thus characterized by large light cone '+' components, while the lines in the lower part of the loop are characterized by large light cone '-' components.

The loop integrations, on the light cone '+' and '-' integration variables are thus kinematically decoupled in the upper and in the lower part of the forward scattering amplitude diagram.
In the case of the 4 to 4 scattering process the integrations on δ_+ and δ_- are thus separated and define the functions

$$\Psi_A = \int \frac{\psi_A}{l_1^2 l_2^2} d\delta_-, \qquad \Psi_B = \int \frac{\psi_B}{a_1^2 a_2^2} d\delta_+$$

which allow to introduce the double parton distributions in the DPI cross section. The loop integrations on the transverse variables are on the contrary all linked. The amplitude is however diagonalized when going to the transverse coordinates representation, due to overall conservation of transverse momenta.

Loop integration on δ_+ and δ_-:



$$l_{1+} = Q_{1+} + (k/2+\delta)_+ \approx Q_{1+}$$
$$a_{1-} = Q_{1-} - (k/2+\delta)_- \approx Q_{1-}$$

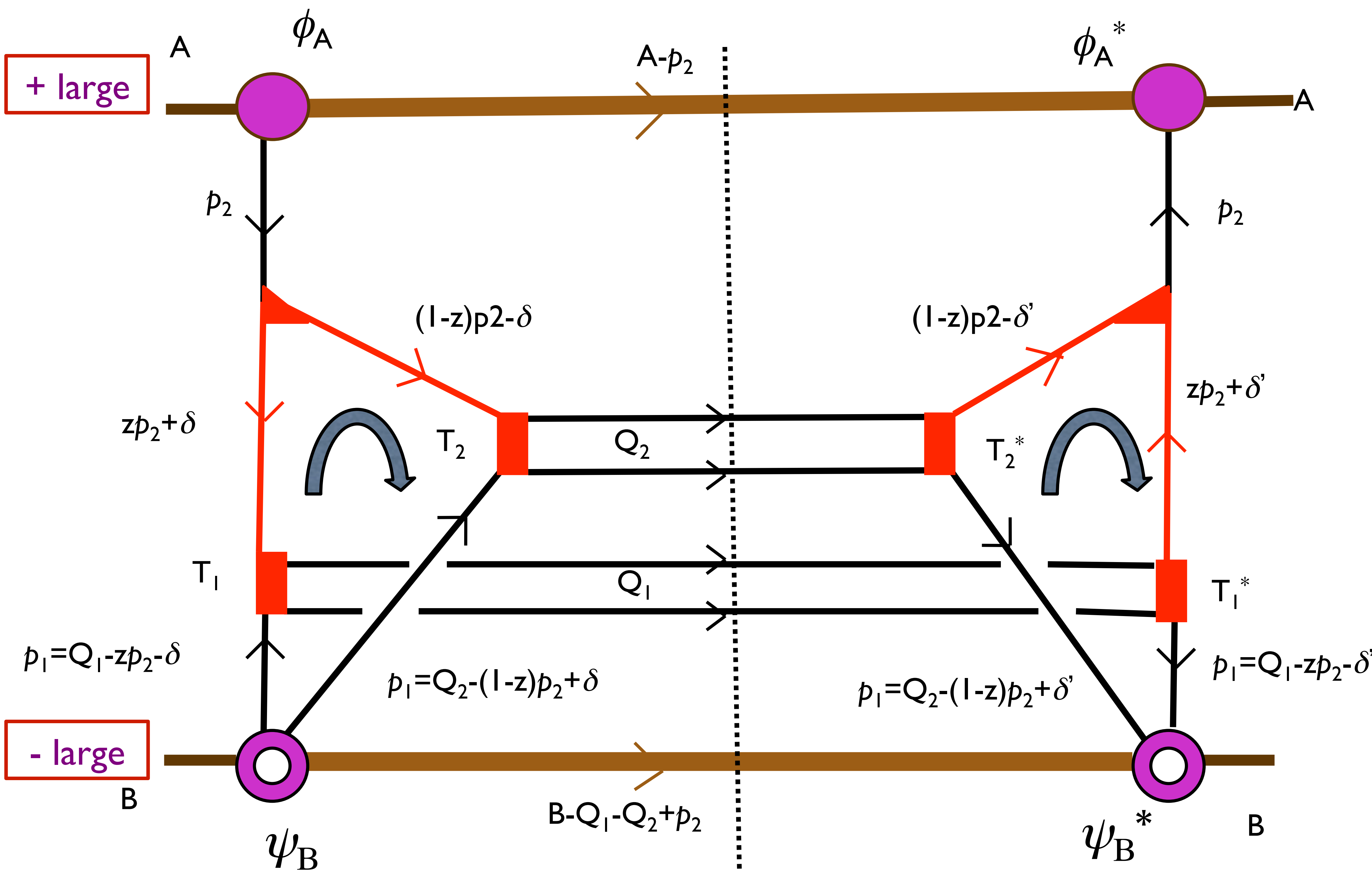
$$\int \frac{\psi_A}{l_1^2 l_2^2} T_1 T_2 \frac{\psi_B}{a_1^2 a_2^2} d\delta_+ \approx \left\{ \frac{\psi_A}{l_1^2 l_2^2} T_1 T_2 \right\}\Big|_{\delta_+=0} \times \int \frac{\psi_B}{a_1^2 a_2^2} d\delta_+$$

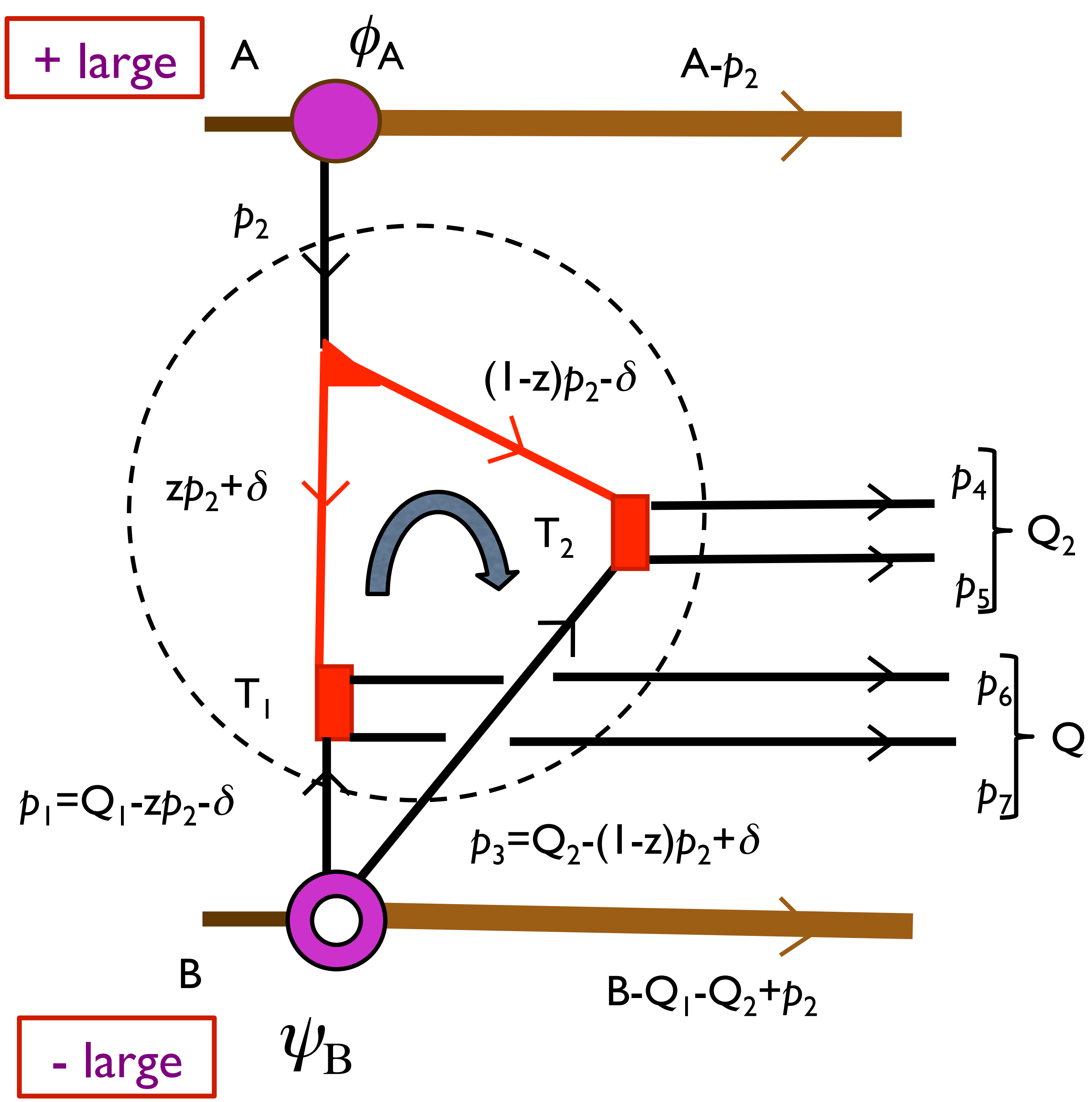
$$l_{2+} = Q_{2+} + (k/2-\delta)_+ \approx Q_{1+}$$
$$a_{2-} = Q_{2-} - (k/2-\delta)_- \approx Q_{1-}$$

$$\int \frac{\psi_A}{l_1^2 l_2^2} T_1 T_2 \frac{\psi_B}{a_1^2 a_2^2} d\delta_- \approx \int d\delta_- \frac{\psi_A}{l_1^2 l_2^2} \times \left\{ T_1 T_2 \frac{\psi_B}{a_1^2 a_2^2} \right\}\Big|_{\delta_-=0}$$

$$\int \frac{\psi_A}{l_1^2 l_2^2} T_1 T_2 \frac{\psi_B}{a_1^2 a_2^2} d\delta_- d\delta_+ \approx \overbrace{\left(\int d\delta_- \frac{\psi_A}{l_1^2 l_2^2} \right)}^{\Psi_A} \times \left\{ T_1 T_2 \right\}\Big|_{\delta_-=0} \times \overbrace{\left(\int d\delta_+ \frac{\psi_B}{a_1^2 a_2^2} \right)}^{\Psi_B}$$

A contribution to the forward scattering amplitude due to the 3 to 4 process:





The partonic process (in red in the figure) in the interaction amplitude represents a particular case of a 7 parton interaction (the 3 incoming and 4 outgoing black lines in the figure).

When considering all gluons amplitudes, the number of tree level diagrams, which contribute to the 7-gluon amplitude, is huge: 2485.

A standard approach to the study of the 7-gluon amplidude looks therefore impractical. In the zero mass case, tree level amplitudes are nevertheless successfully worked out with the Spinor-Helicity formalism. In the all gluons case, tree level amplitudes have in fact been worked out explicitly for any number of external gluons.

In the spinor-helicity formalism, the tree level amplitude of n gluons, with colors $a_1 \ldots a_n$, helicities $\varepsilon_1 \ldots \varepsilon_n$ and momenta $p_1 \ldots p_n$, is expressed as

$$\mathcal{M}_n = \sum_{perms'} \mathrm{Tr}(T^{a_1} T^{a_2} \ldots T^{a_n}) A(p_1, \epsilon_1; p_1, \epsilon_2; \ldots; p_n, \epsilon_n)$$

where the sum runs over all non-cyclic permutations of1,2,...,n and the Ts are the SU(3) generators in the fundamental representation of the group.

The partial amplitudes *A*(1,2,...,n) are called *color ordered amplitudes* and satisfy various important properties. In particular each *A*(1,2,...,n) is a *gauge invariant* quantity and the different terms are incoherent in M_n to leading order in the number of colors.

The partial amplitudes are expressed in terms of spinor products:

$$\langle ij \rangle = \sqrt{s_{ij}}\, e^{i\phi}, \qquad [ij] = \sqrt{s_{ij}}\, e^{-i\phi}, \qquad s_{ij} = (p_i + p_j)^2$$

where ϕ is a phase factor.

All external momenta are conventionally defined as outgoing. Pure-gluon amplitudes, for which all the gluon helicities are equal (corresponding to the case where all final state helicities are opposite to all initial ones), or at most one is different, are all zero when the number of external gluons is 4 or larger (it may be different from zero in the case of 3 gluons). The simplest case, where color ordered amplitudes are different from zero, is the case of the Maximally Helicity Violating (MHV) amplitudes, where all external gluons have the same helicity with the exception of two. The remarkably simple expression of a MHV amplitude is due to Parke and Taylor. In the case of a n-gluon scattering amplitude with all helicities positive, except for the gluons labeled with *i* and *j*, the expression is:

$$A(1^+2^+\dots i^-\dots j^-\dots n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle (n-1)n\rangle\langle n1\rangle}$$

Notice that a contribution to a DPI is characterized by multi-particle singularities in the amplitude. In the case of MHV the only singularities are of the kind $1/(S_{k(k+1)})^{1/2}$

***No multi-particle singularities are thus possible in the case of* MHV**.

n-gluons amplitude

Amplitude near a collinear singularity (e.g. gluons 1 and 2 are parallel):

(n-1) gluons amplitude

$$A(1^+2^+\dots i^-\dots j^-\dots n^+) \simeq \frac{1}{\sqrt{z(1-z)}\langle 12\rangle} \times \frac{\langle ij\rangle^4}{\langle P3\rangle\langle 34\rangle\dots\langle (n-1)n\rangle\langle nP\rangle}$$

Notice that if gluons 1 and 2 were incoming, in the parallel limit the amplitude would describe the fusion of two initial state gluons into the gluon labelled *P*, followed by the interaction of *P* with the remaining n-2 gluons

To contribute to a DPI, a multi-parton amplitude has to be characterized by at least two multi-particle singularities.

All contributions to the 7-gluon NMHV amplitude can be obtained from 4 different color ordered amplitudes. The simplest one is

$$A(1^-2^-3^-4^+5^+6^+7^+) \\ = A(1^-2^-3^-4^+5^+6^+7^+)|_a + A(1^-2^-3^-4^+5^+6^+7^+)|_b + A(1^-2^-3^-4^+5^+6^+7^+)|_c$$

$$A(1^-2^-3^-4^+5^+6^+7^+)|_a = \frac{\langle 1|2+3|4]^3}{P_{234}^2\langle 56\rangle\langle 67\rangle\langle 71\rangle[23][34]\langle 5|4+3|2]}$$

$$A(1^-2^-3^-4^+5^+6^+7^+)|_b = -\frac{\langle 3|2+1|7]^3}{P_{712}^2\langle 34\rangle\langle 45\rangle\langle 6|7+1|2]\langle 65\rangle[71][12]}$$

$$A(1^-2^-3^-4^+5^+6^+7^+)|_c = -\frac{\langle 3|(4+5)(6+7)|1\rangle^3}{P_{345}^2P_{671}^2\langle 34\rangle\langle 45\rangle\langle 6|7+1|2]\langle 67\rangle\langle 71\rangle\langle 5|4+3|2]}$$

two multi-particle singularities

where

$$P_{ijk} = (p_i+p_j+p_k), \qquad \langle i|j+k|l] = \langle ij\rangle[jl] + \langle ik\rangle[kl]$$

$$\langle i|j+k|l] = \langle i|i+j+k|l] = \langle iP_{ijk}\rangle[P_{ijk}l]$$

$$\langle i|(j+k)(l+m)|n\rangle = \langle i|(i+j+k)(l+m+n)|n\rangle = \langle iP_{ijk}\rangle[P_{ijk}P_{lmn}]\langle P_{lmn}n\rangle$$

Notice that the color ordered amplitude (which is gauge invariant) is given by the sum of all three terms: $A = A|_a + A|_b + A|_c$, while one has two multi-particle singularities only in $A|_c$.

By using

$$P_{ijk} = (p_i + p_j + p_k), \qquad \langle i|j+k|l] = \langle i|i+j+k|l] = \langle iP_{ijk}\rangle[P_{ijk}l]$$
$$\langle i|(j+k)(l+m)|n\rangle = \langle i|(i+j+k)(l+m+n)|n\rangle = \langle iP_{ijk}\rangle[P_{ijk}P_{lmn}]\langle P_{lmn}n\rangle$$

one can express $A(1^-2^-3^-4^+5^+6^+7^+)|_c$ as follows:

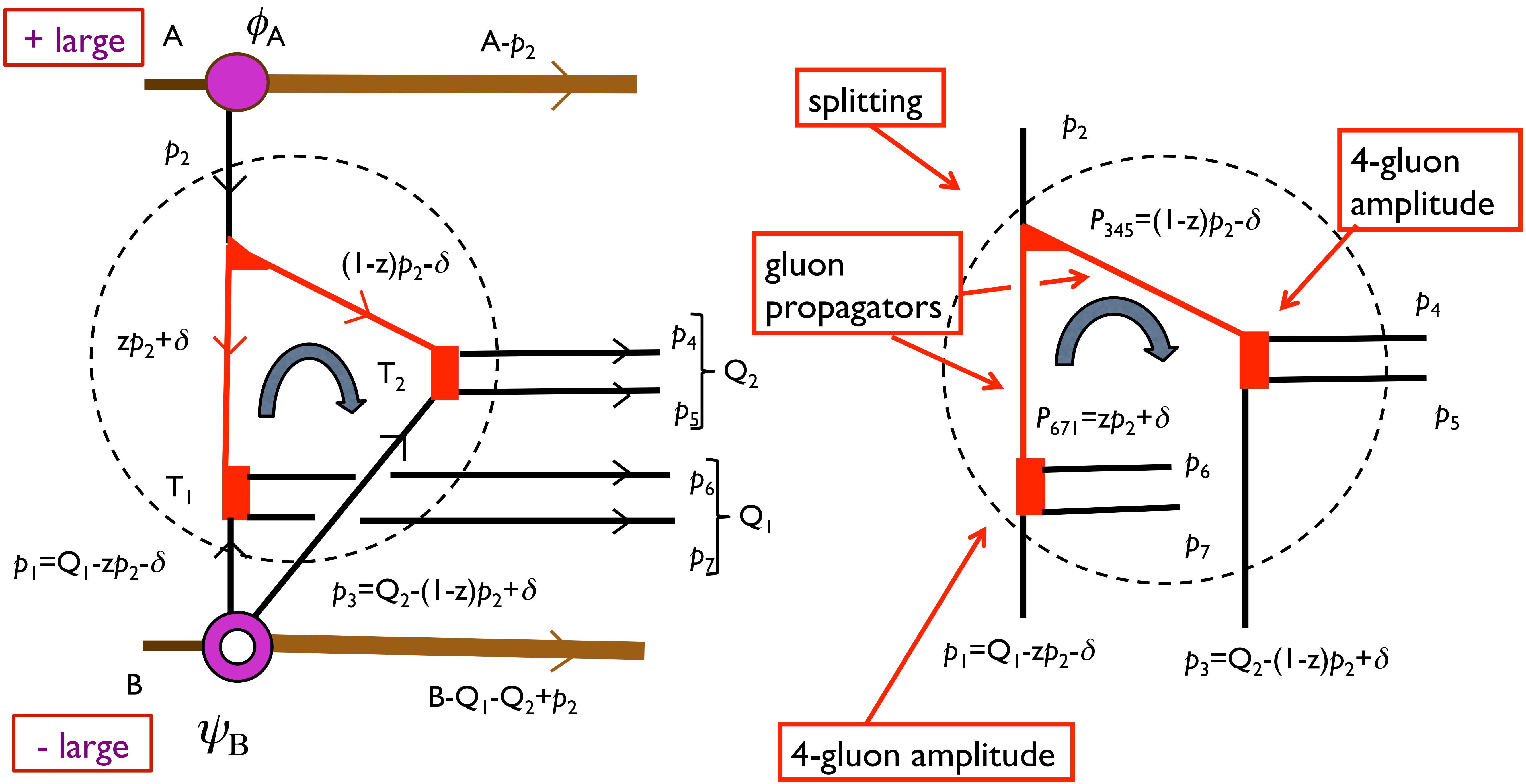
4-gluon Parke-Taylor scattering amplitudes

$$A(1^-2^-3^-4^+5^+6^+7^+)|_c = -\frac{\langle 1P_{671}\rangle^3}{\langle 67\rangle\langle 71\rangle\langle P_{671}6\rangle} \times \frac{1}{P_{671}^2}$$
$$\times \frac{[P_{671}P_{345}]^3}{[P_{671}2][2P_{345}]} \times \frac{1}{P_{345}^2} \times \frac{\langle P_{345}3\rangle^3}{\langle 34\rangle\langle 45\rangle\langle 5P_{345}\rangle}$$

splitting amplitude

gluon propagators

$A(1^-2^-3^-4^+5^+6^+7^+)|_c$ can thus contribute to a DPI :



Notice that the $A|_c$ is integrated in the loop with the non-perturbative vertex ψ_B

Loop integration:

virtualities cannot be large, because the loop lines enter in the non perturbative vertex ψ_B.
One can thus integrate on the loop integration variables δ_+ and δ_- (c.f.r. the figure in the previous slide) by keeping into account of the dependence on δ_+, only in the upper part of the loop, and of the dependence on δ_-, only in the lower part of the loop:

$$\Psi_B = \int d\delta_+ \frac{\psi_B(p_1, p_3)}{p_1^2\, p_3^2}$$

$$A|_c = \mathcal{A}|_c \times \frac{1}{P_{345}^2 P_{671}^2}, \qquad \mathbf{A}|_c = \int d\delta_- \frac{\mathcal{A}|_c}{P_{345}^2 P_{671}^2}$$

The integration on δ_- is done with the singularities of $P_{345}{}^2$ and $P_{671}{}^2$:

$$\int d\delta_- \frac{1}{P_{345}^2 P_{671}^2} = \int \frac{d\delta_-}{\left((zp_2+\delta)^2+i\epsilon\right)\left((1-z)p_2-\delta)^2+i\epsilon\right)} = \frac{2\pi i}{\delta_t^2\, p_{2+}}$$

To obtain the final expression of the amplitude, one has to integrate on δ_t.
When $\delta = 0$ the two intermediate gluons are on mass shell and the process factorizes into the product of a splitting function and two on shell scattering amplitudes.
One is thus interested in the behavior of the integrand in the limit of small δ. At small δ one has:

$$\mathbf{A}|_c = \frac{\big(z\langle 32\rangle\langle\delta 1\rangle + (1-z)\langle 3\delta\rangle\langle 21\rangle\big)^3[\delta 2]}{\langle 34\rangle\langle 45\rangle\langle 67\rangle\langle 71\rangle\langle\delta 6\rangle\langle\delta 5\rangle} \times \frac{2\pi i}{\delta_t^2 p_{2+}} \to \frac{\text{const.}}{\delta_t} \quad \text{for} \quad \delta \to 0$$

The integration of $A|_c$ in $d^2\delta_t$ thus washes out the singularity at $\delta_t=0$.
As a consequence he amplitude is not enhanced in the configuration where the transverse momenta of the two pairs of large p_t partons are compensated pairwise.

Notice that $A|_c$ *has to be summed coherently with* $A|_a$ *and* $A|_b$ *to obtain the gauge invariant color orderd ampltitude* A. As $A|_c$ is not enhanced and interferes with $A|_a$ and $A|_b$, its contribution cannot be isolated from the other two, in the color ordered contribution to the cross section.

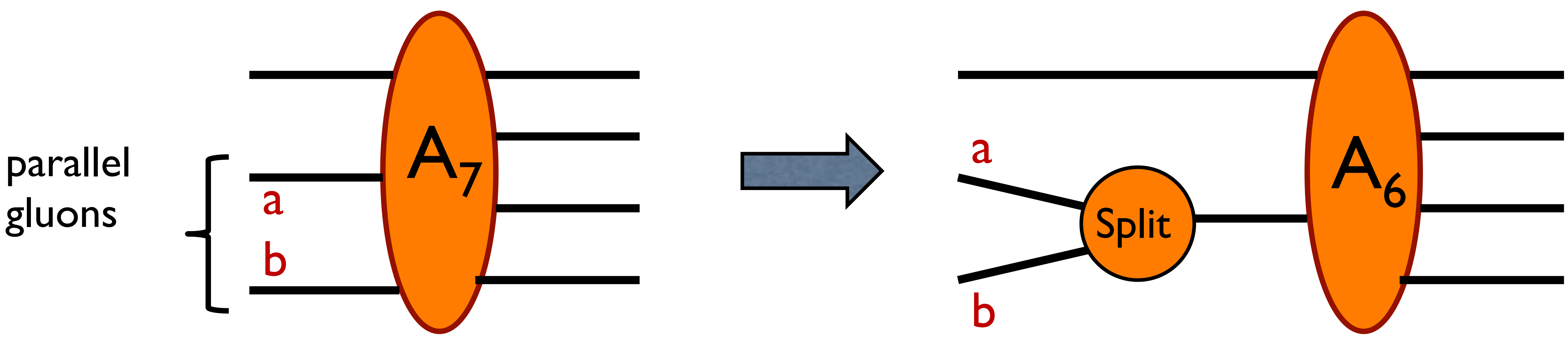
This property holds not only for the particular case of color ordered amplitude discussed here. It holds also for all other contributions, to the tree level 7-gluons amplitude, characterized by multi-particle singularities.

A further feature is that the complete 3 to 4 scattering amplitude results from the sum of all color ordered contributions. In the actual case of interest one has two gluons in the initial state, which are originated by the same hadron and which thus have a rather small relative transverse momentum.

The color ordered terms, where the almost parallel initial state gluons are cyclicaly-adjacent in the amplitude, are singular, in the invariant obtained by the sum of the two almost parallel momenta, and therefore give a leading contribution to the amplitude.

A main contribution to the 7-gluon amplitude, in the kinematics considered here, is therefore factorized into a fusion amplitude and a 6-gluon scattering amplitude, the latter with only two gluons in the initial state.

While initiated by three partons, a main contribution to the cross section is thus effectively given by a 2 to 4, rather than by a 3 to 4 parton process.



CONCLUSIONS

In the kinematics of DPIs, a main contribution to the 3 to 4 parton processes is given by terms corresponding to the fusion of the two initial state partons, which originate from the same hadron.

In the 7 gluons tree level amplitude one finds terms, that contribute to the color ordered amplitudes, which can be factorized into a splitting amplitude and two almost on shell four partons (namely 2 to 2) scattering amplitudes.

However these contributions to the amplitude

- are not enhanced in the configuration where the final state partons have transverse momenta balanced pair-wise because of the loop integration

- are not gauge invariant and thus interfere with the other contributions to the same (gauge invariant) color ordered amplitude,

As a consequence their contribution cannot be isolated in the final cross section.

Backup Slides

$$
\begin{aligned}
&A(1^-,2^-,3^-,4^+,5^+,6^+,7^+) = \\
&\frac{\langle 1|2+3|4]^3}{t_2^{[3]}\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 7\ 1\rangle[2\ 3][3\ 4]\langle 5|4+3|2]} \\
&-\frac{1}{\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 6|7+1|2]}\left(\frac{\langle 3|(4+5)(6+7)|1\rangle^3}{t_3^{[3]}t_6^{[3]}\langle 6\ 7\rangle\langle 7\ 1\rangle\langle 5|4+3|2]}+\frac{\langle 3|2+1|7]^3}{t_7^{[3]}\langle 6\ 5\rangle[7\ 1][1\ 2]}\right).
\end{aligned}
$$

$$
\begin{aligned}
&A(1^-,2^-,3^+,4^-,5^+,6^+,7^+) = \\
&\frac{\langle 1\ 2\rangle^3[3\ 5]^4}{t_3^{[3]}[3\ 4][4\ 5]\langle 6\ 7\rangle\langle 7\ 1\rangle\langle 2|3+4|5]\langle 6|4+5|3]} \\
&+\frac{\langle 2\ 4\rangle^4\langle 1|7+6|5]^3}{t_2^{[3]}t_6^{[3]}\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 6\ 7\rangle\langle 7\ 1\rangle\langle 2|3+4|5]\langle 6|(7+1)(2+3)|4\rangle} \\
&+\frac{\langle 1\ 2\rangle^3\langle 4|5+6|3]^4}{t_4^{[3]}t_7^{[3]}\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 7\ 1\rangle\langle 6|4+5|3]\langle 7|1+2|3]\langle 4|(5+6)(7+1)|2\rangle} \\
&+\frac{\langle 4|1+2|3]^4}{t_1^{[3]}[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 4|3+2|1]\langle 7|1+2|3]} \\
&+\frac{\langle 2\ 4\rangle^4\langle 4|5+6|7]^3}{\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 6\rangle[7\ 1]\langle 4|3+2|1]\langle 4|(5+6)(7+1)|2\rangle\langle 6|(7+1)(2+3)|4\rangle}.
\end{aligned}
$$

$$
\begin{aligned}
&A(1^-,2^-,3^+,4^+,5^-,6^+,7^+) = \\
&\frac{\langle 5|1+2|3]^4}{t_1^{[3]}[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 7|1+2|3]\langle 4|3+2|1]} \\
&+\frac{\langle 1\ 2\rangle^3\langle 5|4+6|3]^4}{t_4^{[3]}t_7^{[3]}\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 7\ 1\rangle\langle 6|5+4|3]\langle 7|1+2|3]\langle 4|(5+6)(7+1)|2\rangle} \\
&+\frac{\langle 1\ 2\rangle^3[3\ 4]^3}{t_3^{[3]}[4\ 5]\langle 6\ 7\rangle\langle 7\ 1\rangle\langle 2|3+4|5]\langle 6|4+5|3]} \\
&+\frac{\langle 1\ 2\rangle^3\langle 2|3+4|6]^4}{\langle 7\ 1\rangle\langle 2\ 3\rangle\langle 3\ 4\rangle[5\ 6]\langle 2|1+7|6]\langle 2|3+4|5]\langle 2|(3+4)(5+6)|7\rangle\langle 4|(5+6)(7+1)|2\rangle} \\
&+\frac{\langle 2|(3+4)(7+6)|5\rangle^4}{t_2^{[3]}t_5^{[3]}\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 5|6+7|1]\langle 4|2+3|1]\langle 2|(3+4)(5+6)|7\rangle} \\
&+\frac{\langle 2\ 5\rangle^4[6\ 7]^3}{t_6^{[3]}\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle[7\ 1]\langle 5|6+7|1]\langle 2|7+1|6]}.
\end{aligned}
$$

$$
\begin{aligned}
&A(1^-,2^+,3^-,4^+,5^-,6^+,7^+) = \\
&\frac{\langle 1\ 5\rangle^4[2\ 4]^4}{t_2^{[3]}[2\ 3][3\ 4]\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 7\ 1\rangle\langle 1|2+3|4]\langle 5|3+4|2]} \\
&-\frac{\langle 1\ 3\rangle^4\langle 5|6+7|4]^4}{t_1^{[3]}t_5^{[3]}\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 1|2+3|4]\langle 7|5+6|4]\langle 5|(6+7)(1+2)|3\rangle} \\
&+\frac{\langle 1\ 3\rangle^4[4\ 6]^4}{t_4^{[3]}\langle 7\ 1\rangle\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]\langle 3|4+5|6]\langle 7|5+6|4]} \\
&-\frac{\langle 3\ 5\rangle^4[2\ 7]^4}{t_7^{[3]}\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 6\rangle[7\ 1][1\ 2]\langle 6|7+1|2]\langle 3|2+1|7]} \\
&+\frac{\langle 3\ 5\rangle^4\langle 1|6+7|2]^4}{t_3^{[3]}t_6^{[3]}\langle 6\ 7\rangle\langle 7\ 1\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 6|7+1|2]\langle 5|3+4|2]\langle 3|(4+5)(6+7)|1\rangle} \\
&-\frac{\langle 1\ 3\rangle^4\langle 3\ 5\rangle^4[6\ 7]^3}{\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 3|4+5|6]\langle 3|2+1|7]\langle 3|(4+5)(6+7)|1\rangle\langle 5|(6+7)(1+2)|3\rangle}.
\end{aligned}
$$