Introduction to neutron-induced reactions and the R-matrix formalism

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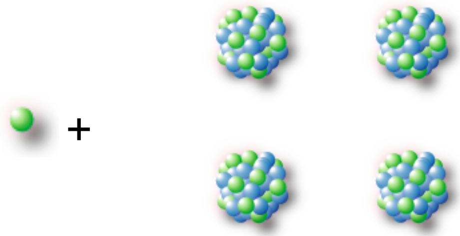
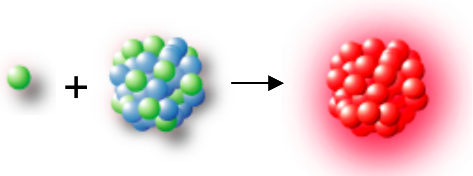
Neutron induced reactions

solid state



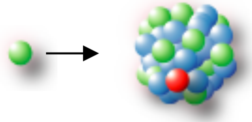
compound nucleus reactions



$\tau \sim 10^{-16}$

$E_n < 10\ \mathrm{MeV}$

direct reactions



$\tau \sim 10^{-22}$

$E_n > 10\ \mathrm{MeV}$

1 nm — **neutron wave length** — 10 fm

1 meV — **neutron kinetic energy** — 10 MeV

Neutron-nucleus reactions

Reaction:
- $X + a \rightarrow Y + b$
  - $X(a,b)Y$
  - $X(a,b)$

Examples of equivalent notations:

$^{10}B + {}^{1}n \rightarrow {}^{7}Li + {}^{4}He$
$^{10}B + n \rightarrow {}^{7}Li + \alpha$
$^{10}B(n,\alpha)$

$^{238}U + n \rightarrow {}^{239}U^*$
$^{238}U + n \rightarrow {}^{239}U + \gamma$
$^{238}U(n,\gamma)$

Reaction cross section σ, expressed in barns, $1\ b = 10^{-28}\ m^2$

Neutron induced nuclear reactions:
- elastic scattering (n,n)
- inelastic scattering (n,n')
- capture (n,γ)
- fission (n,f)
- particle emission (n,α), (n,p), (n,xn)

Total cross section σ_{tot}: sum of all reactions

Neutron-nucleus reactions

Reaction:

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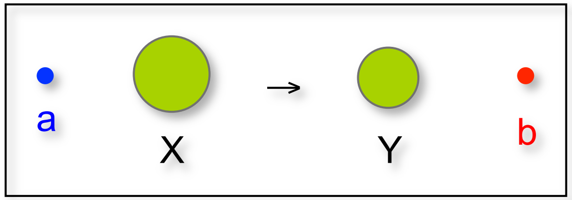
- X(a,b)Y



Cross section:
function of the kinetic energy of the particle a

$$\sigma(E_a) = \int\int \frac{d^2\sigma(E_a, E_b, \Omega)}{dE_b d\Omega} dE_b d\Omega$$

Differential cross section:

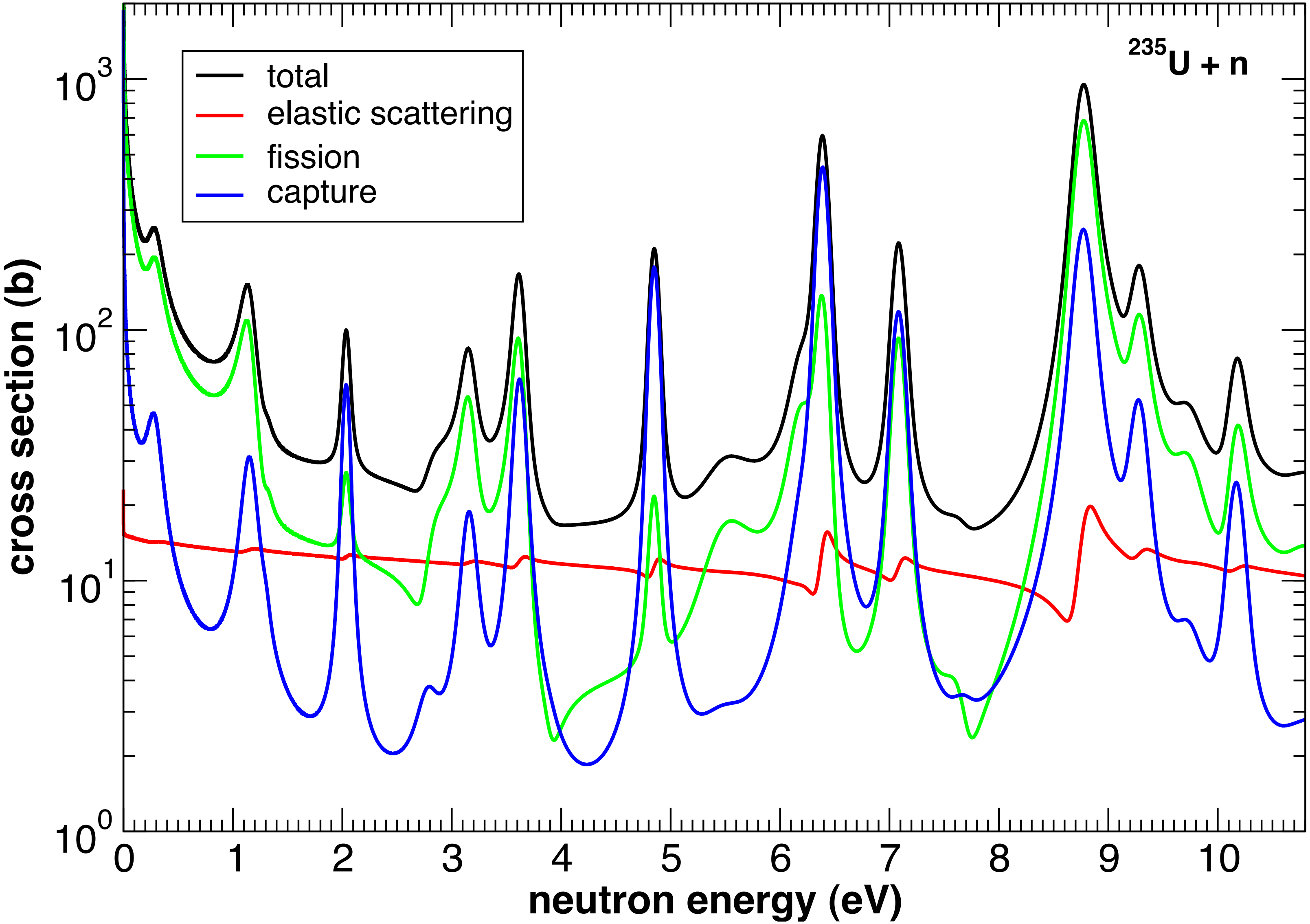
function of the kinetic energy of the particle a and function of the kinetic energy **or** the angle of the particle b

$$\frac{d\sigma(E_a, E_b)}{dE_b} \qquad \frac{d\sigma(E_a, \Omega)}{d\Omega}$$

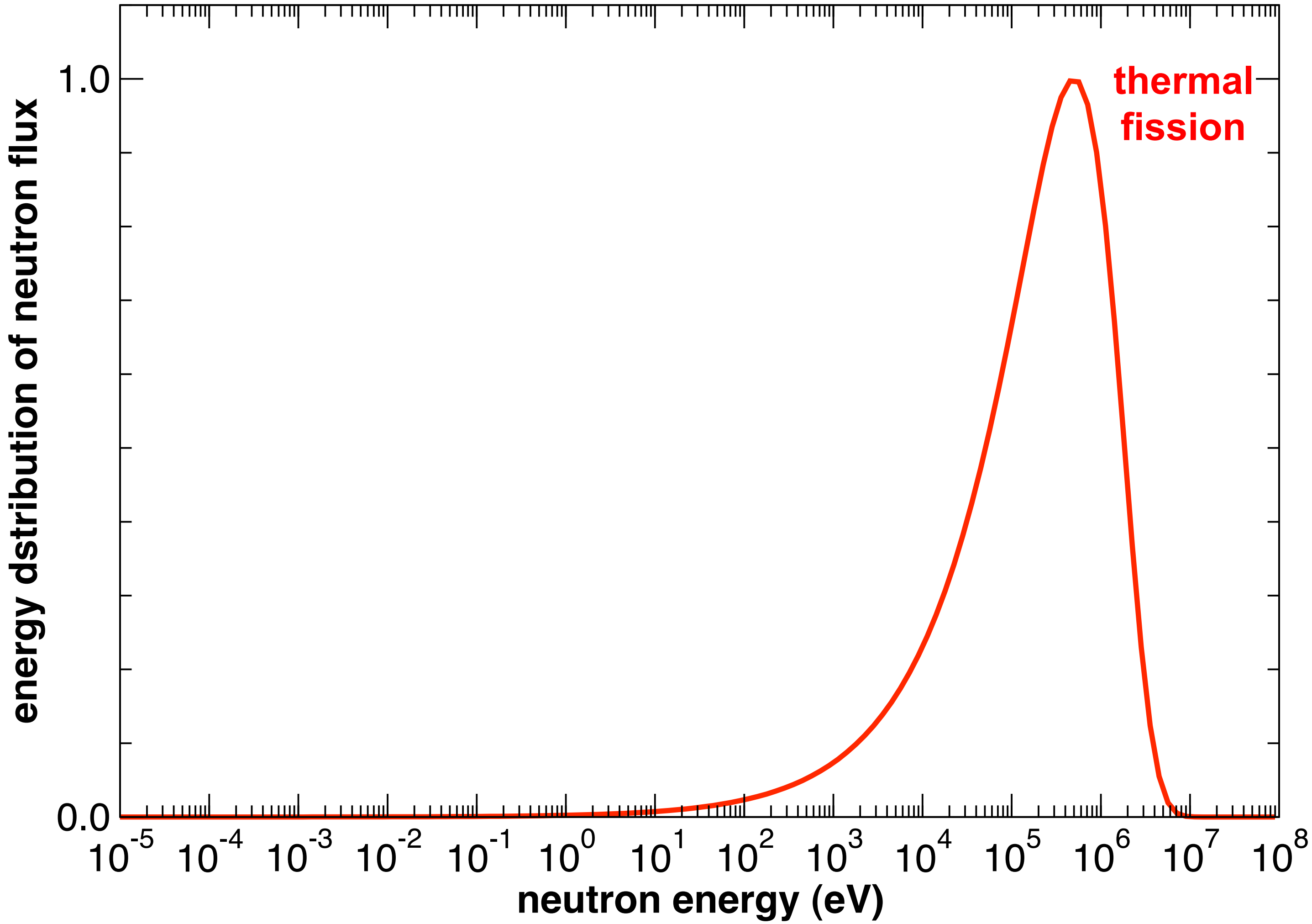
Double differential cross section:
function of the kinetic energy of the particle a and function of the kinetic energy **and** the angle of the particle b

$$\frac{d^2\sigma(E_a, E_b, \Omega)}{dE_b d\Omega}$$

Cross sections σ_T, σ_γ, σ_n et σ_f



Neutron fluxes and cross sections



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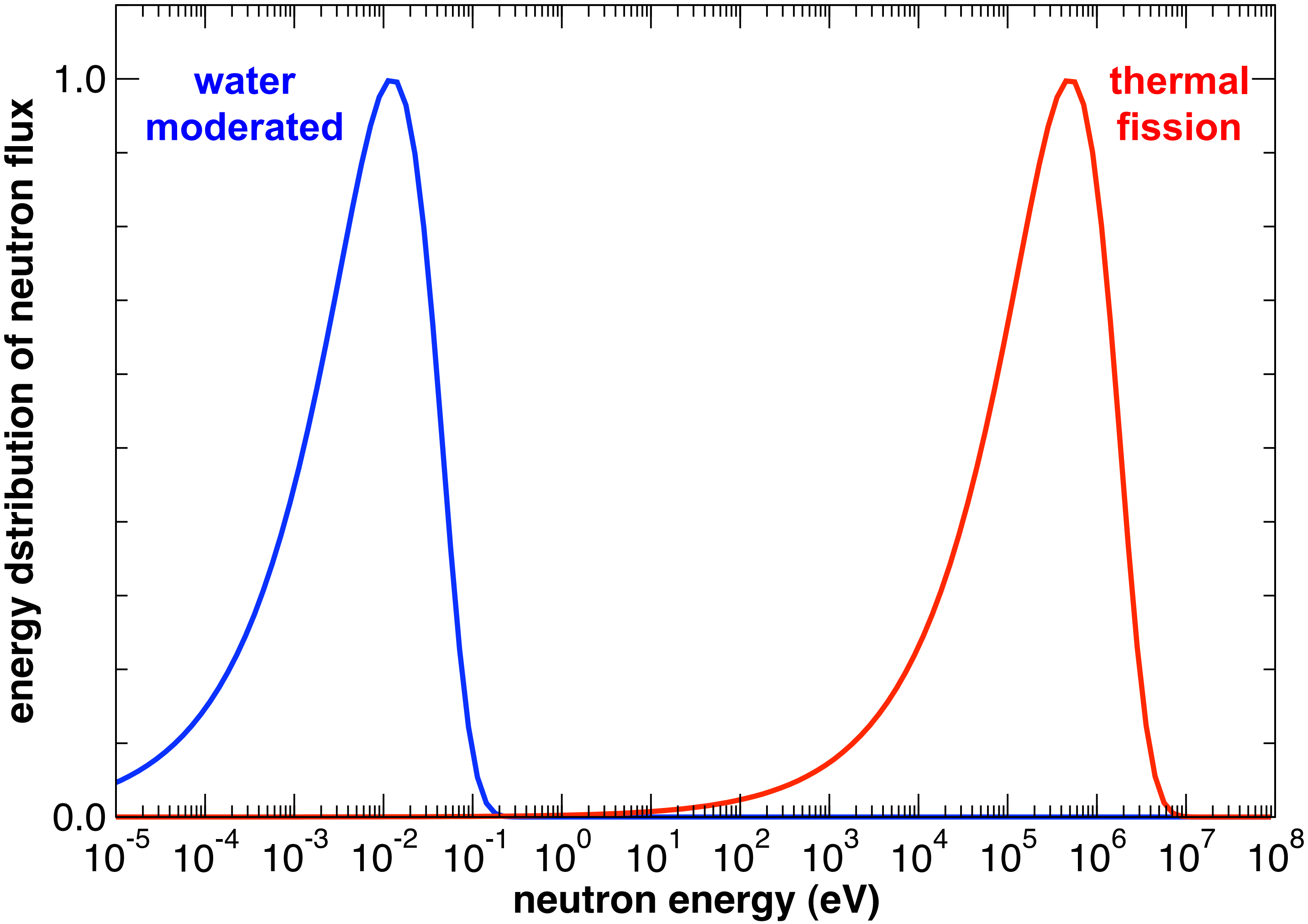

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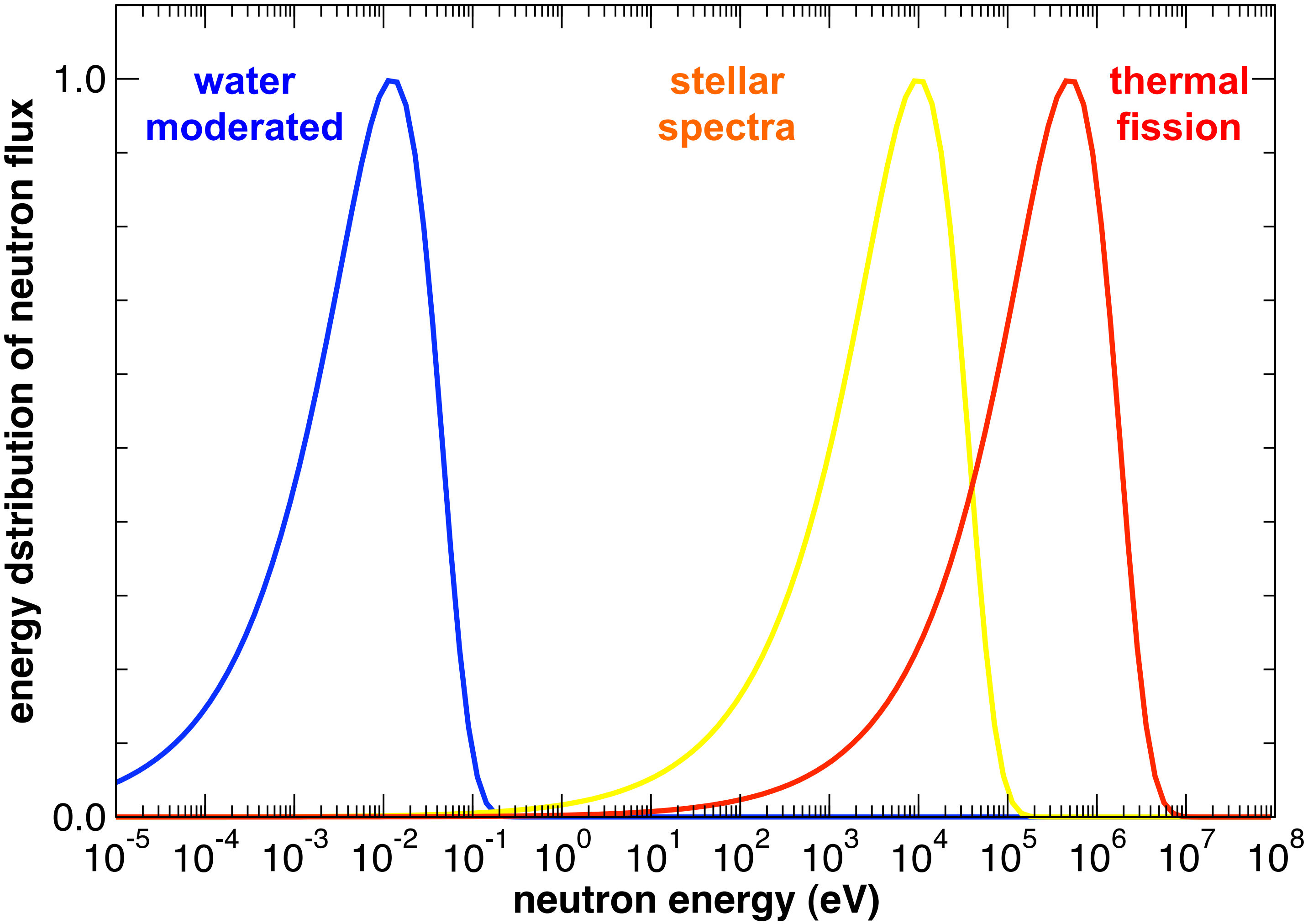

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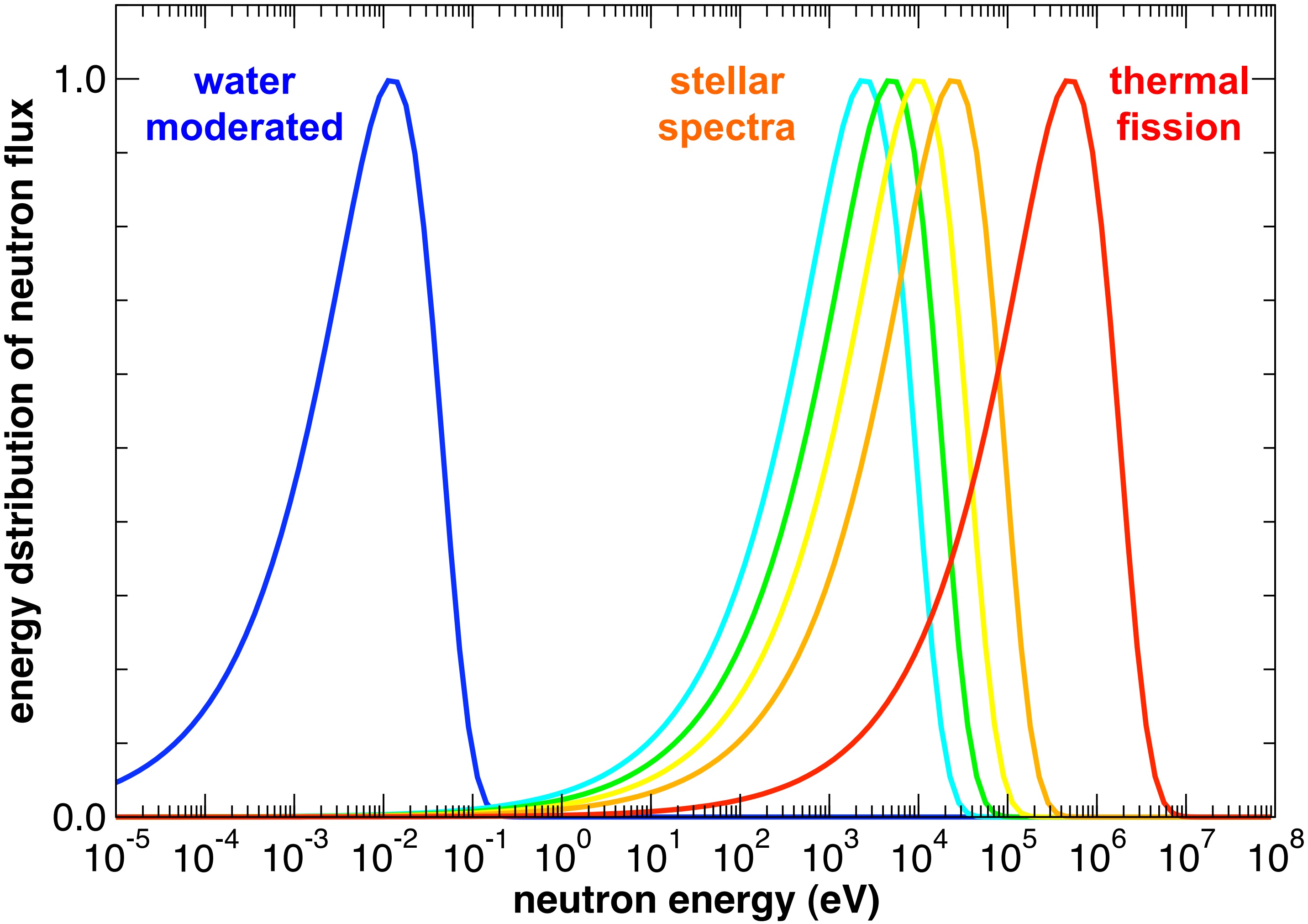
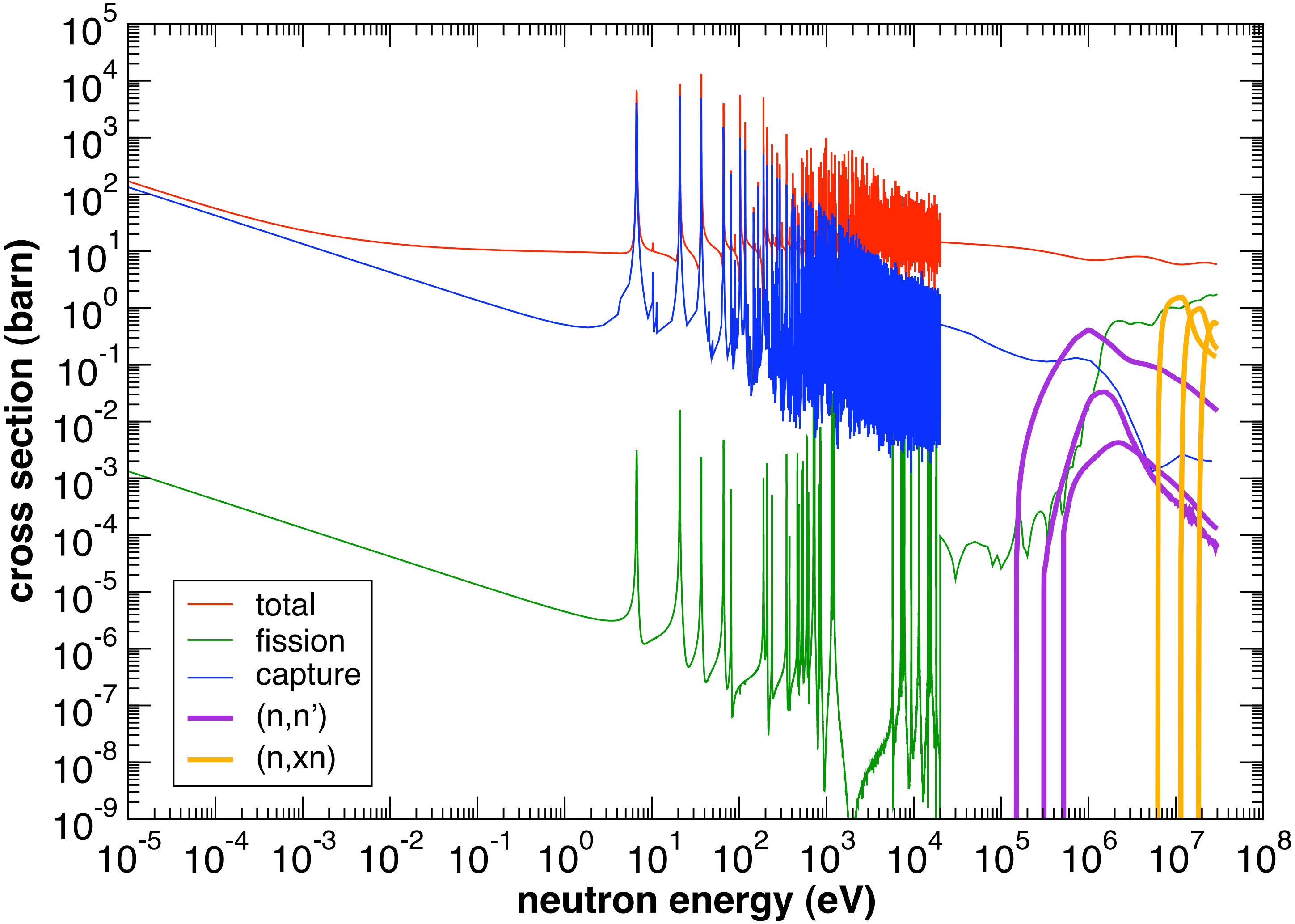


Neutron induced reaction cross sections



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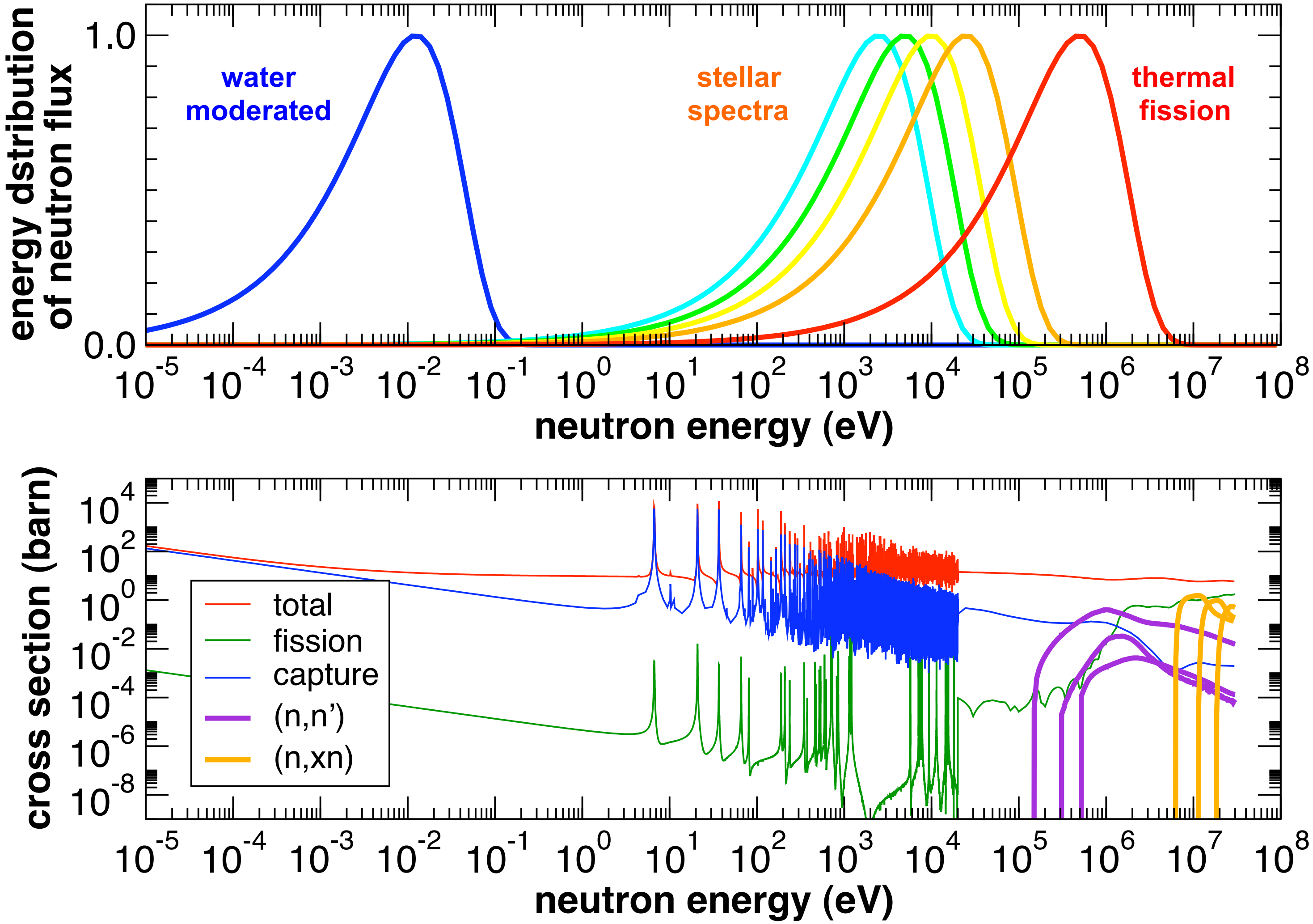

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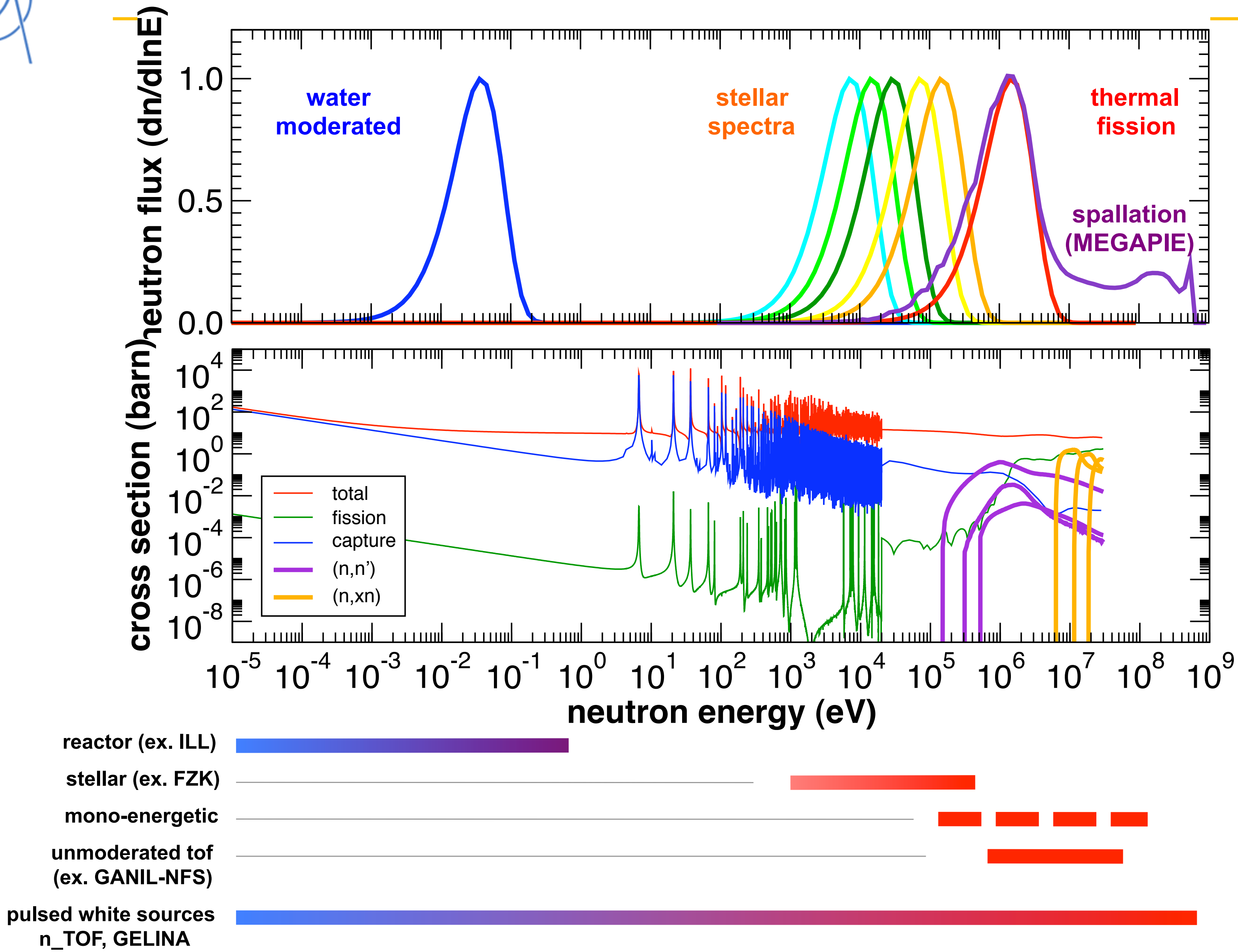
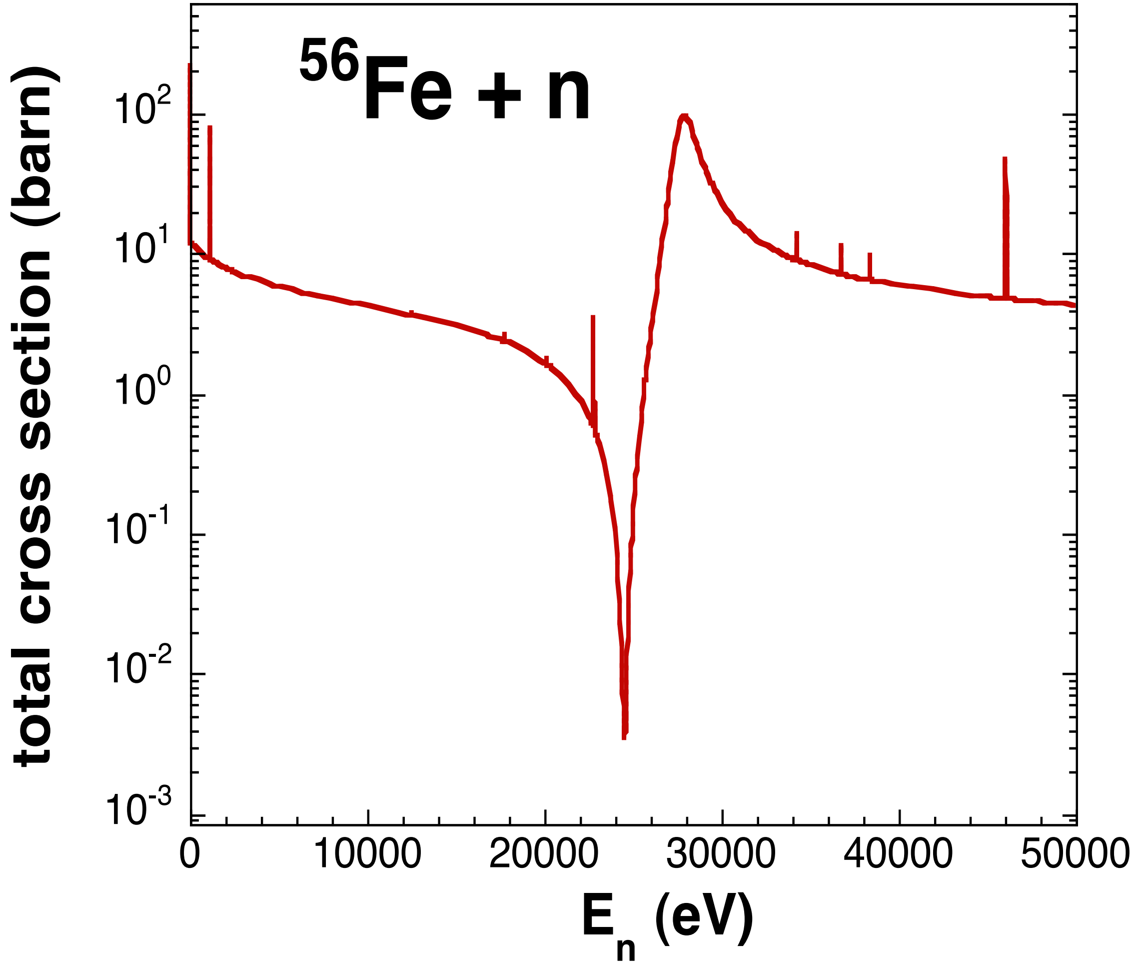
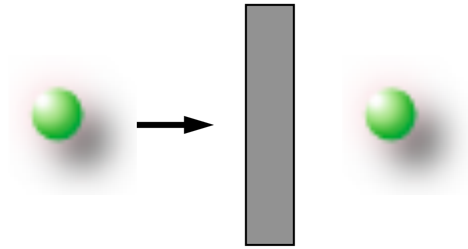


Interference of $\sigma_{potential}$ and σ_n





transmission
$T = \exp(-n \cdot \sigma_T)$
$0<T<1$

Classical – Quantum Physics

Classical physics
- particles, Newton's law of motion
- electromagnetic waves, Maxwell's laws of electromagnetism

Quantum physics
- particles (momentum) and waves (wavelength) are different descriptions of the same thing. Related by Planck's constant *h*.

De Broglie wavelength: $\lambda = \frac{h}{p}$

From 1900, observations of electrons, photons behaving as particles or waves in different experiments (black body radiation, photo-electric effect, crystal diffraction).

Probability of a particle being at time *t*, having position *x* is related to a "wave function".

Probability (Born interpretation): $\psi^* \psi$

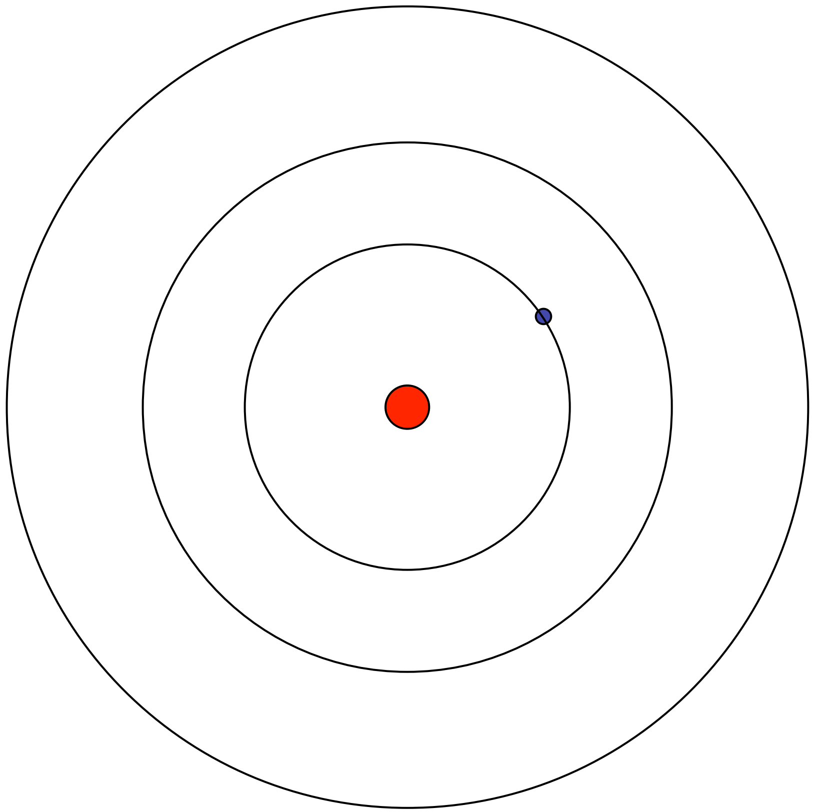
The wave function is a solution of the Schrödinger equation (postulate).

The hydrogen atom

Hydrogen atom

- quantum system of one proton, one electron
- the system can be in well-defined energy states (electron orbits).
- transitions between these states can be observed as electromagnetic radiation
- Observed: energy states: $E_n = -13.6/n^2$ eV, with n=1 the ground state
- wavelengths observed corresponding to transitions between these states ($\Delta E = hc/\lambda$)

The hydrogen atom – Bohr model



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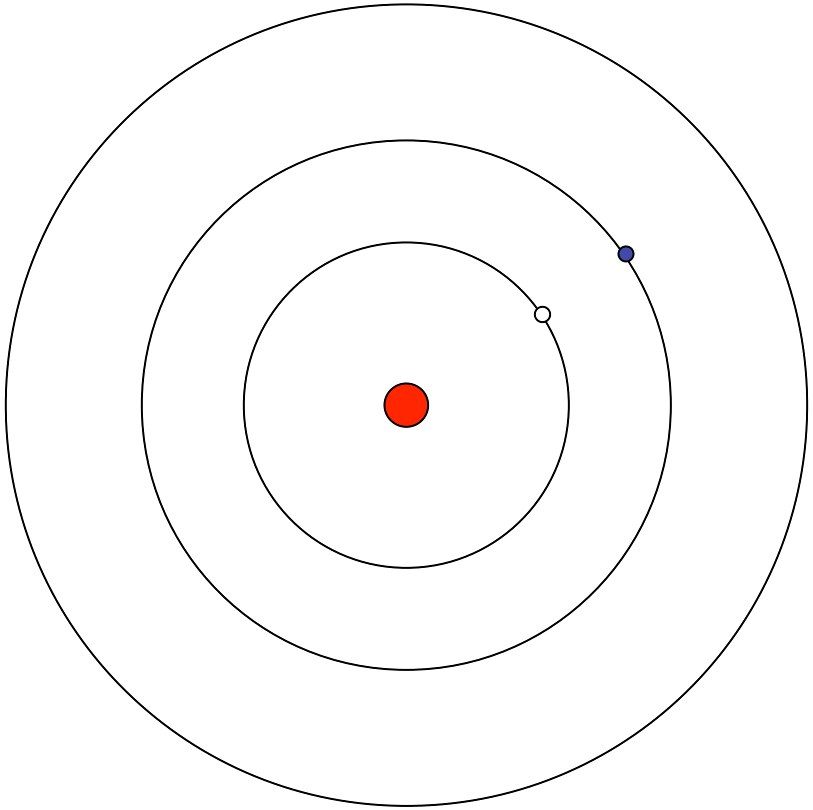

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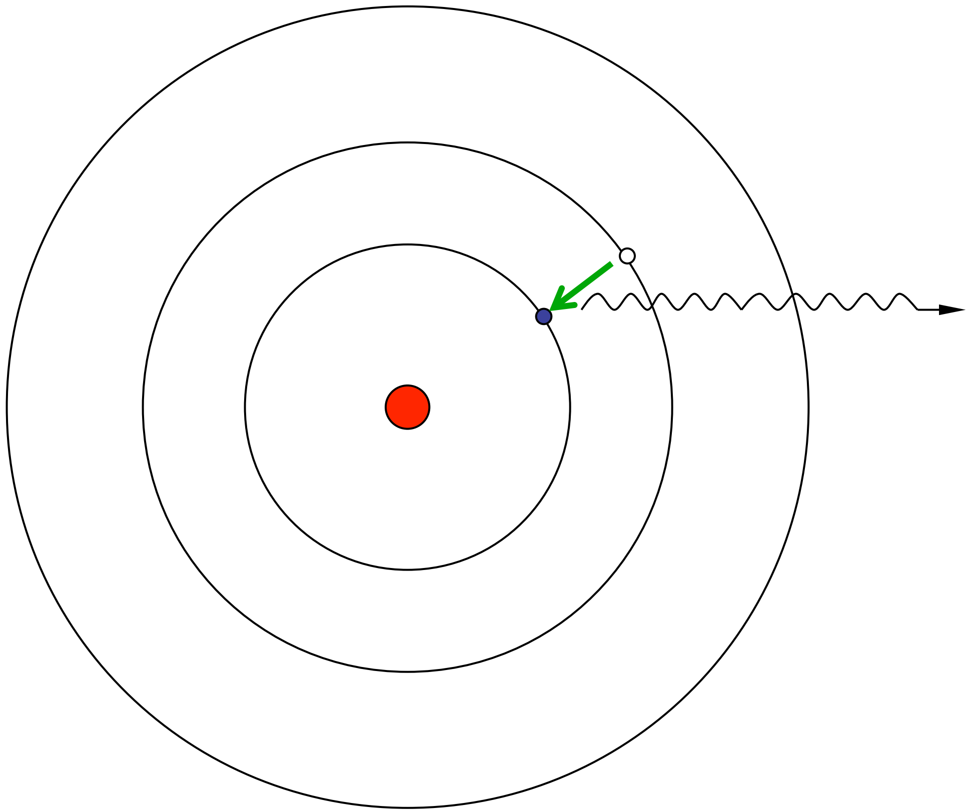
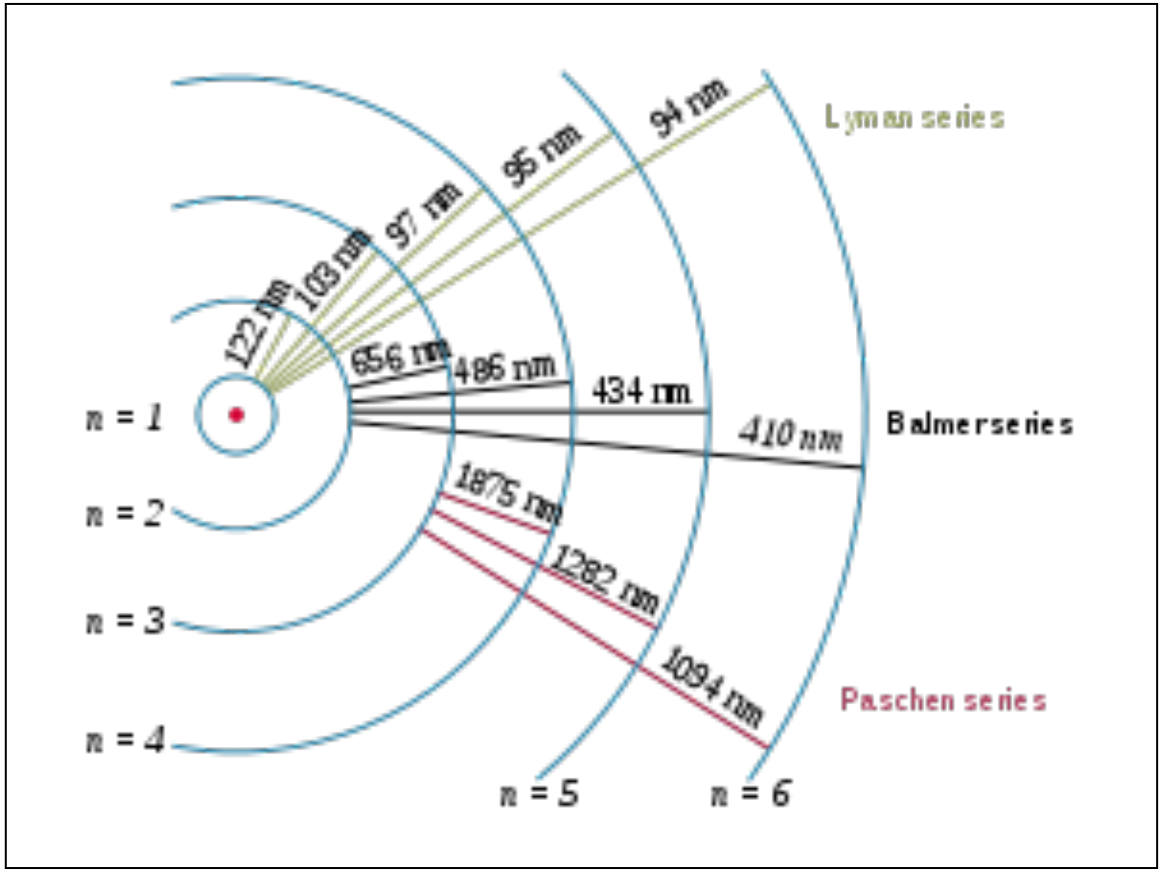
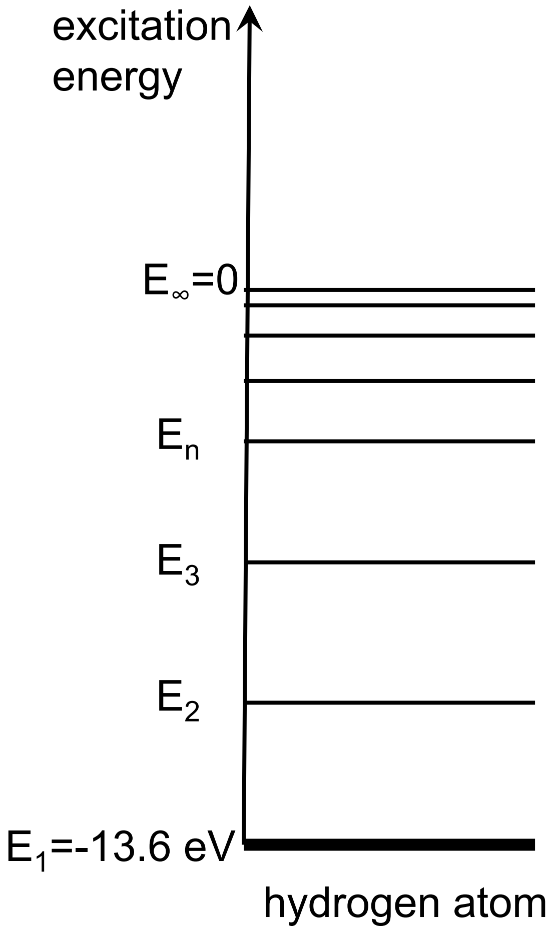


Discrete states of the hydrogen atom





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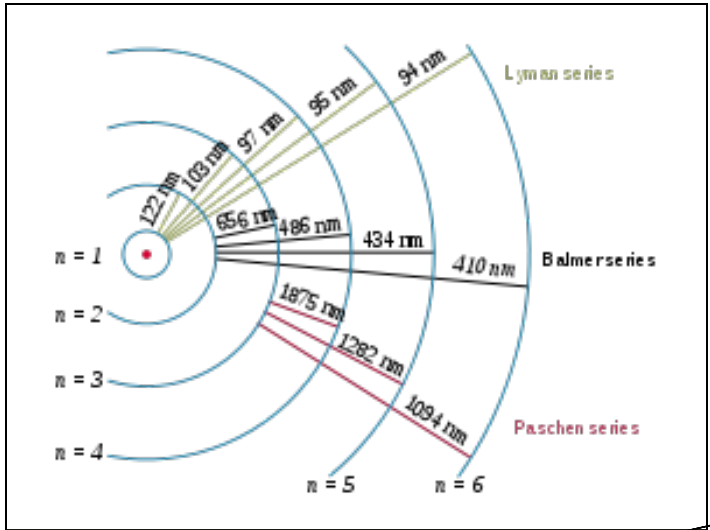

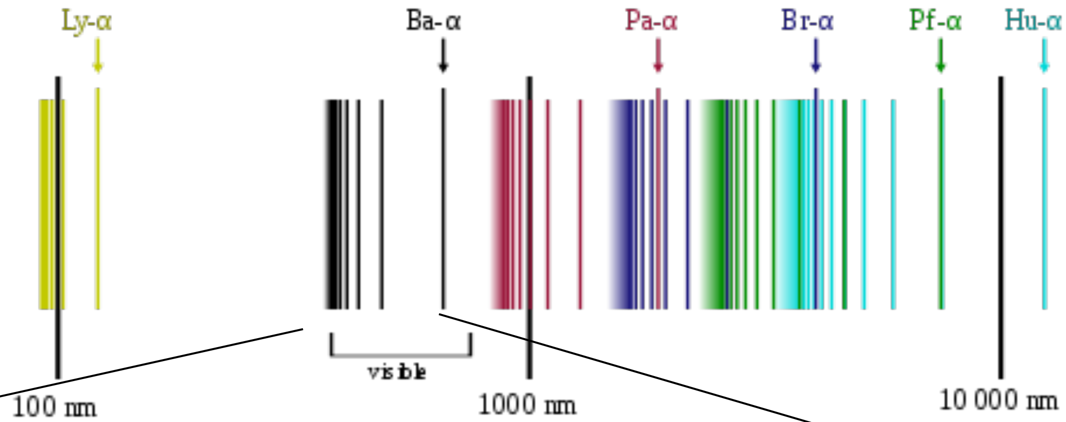

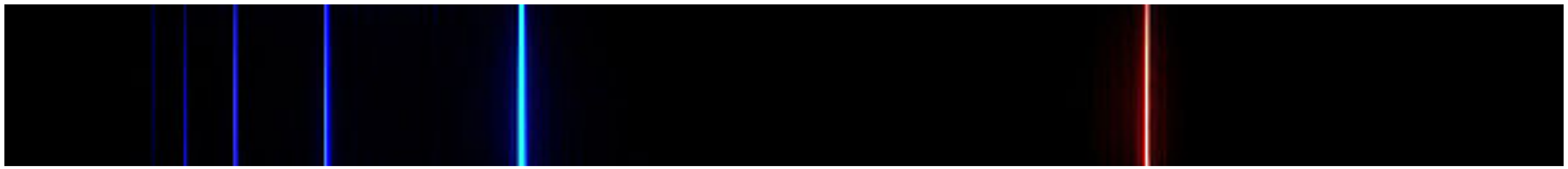

The Schrödinger equation

Time-independent, for a spinless, onedimensional particle:

wave function

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

potential

energy

Solutions: ψ, E

$\psi^*\psi$ Interpreted as probabitliy

The Schrödinger equation

Time-independent. Equation for a spinless, onedimensional particle:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right]\psi(x) = E\psi(x)$$

$$\hat{H}\psi(x) = E\psi(x)$$

↓

Hamiltonian

Quantum system: the infinite well

Solve Schrödinger equation,
for a spinless, onedimensional particle

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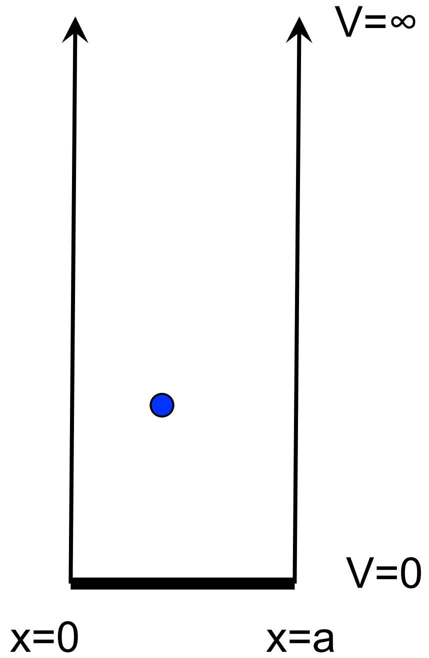
Probability $\psi^*\psi$

Example: The infinite well
for a spinless, onedimensional particle:

$$V(x) = 0 \text{ for } 0 < x < a$$
$$V(x) = \infty \text{ elsewhere}$$



Solution:

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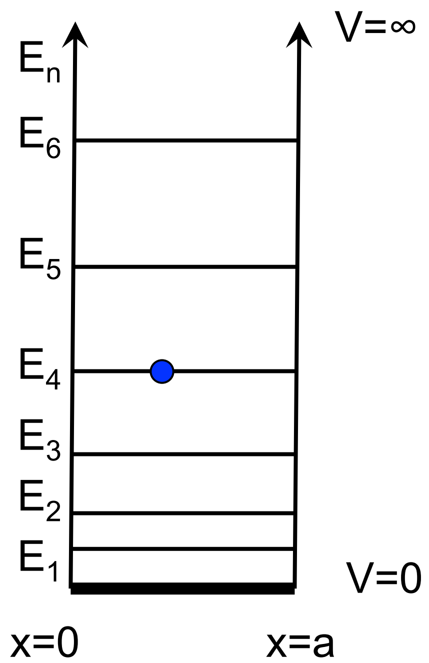
Solution:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$$
$$E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2$$



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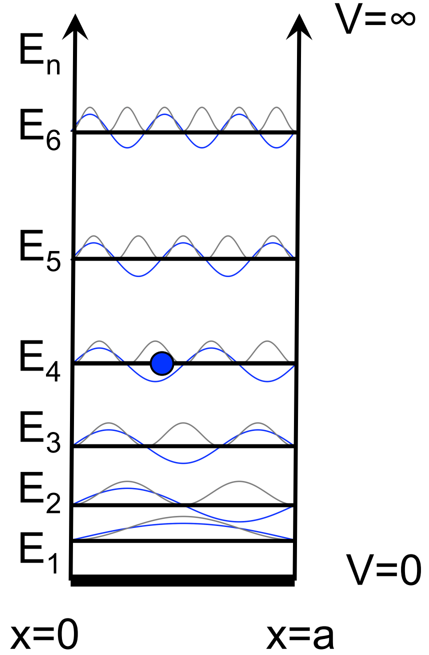

Quantum system: the finite well

Solve Schrödinger equation in two regions:

- inside and outside the well
- normalize solutions to match value and derivative and borders x=0 and x=a

Now the wave function exists also outside the well at x<0 and x>a

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$0 < x < a$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V_0\psi(x) = E\psi(x)$$

$x < 0, x > a$

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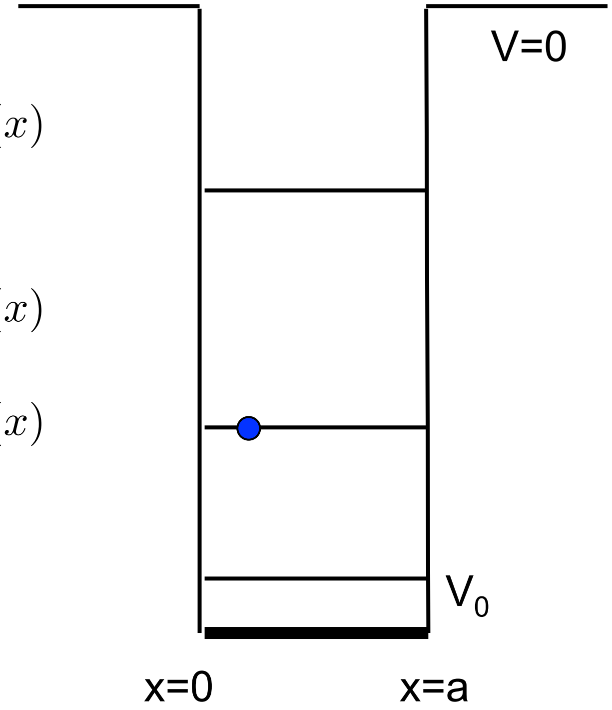

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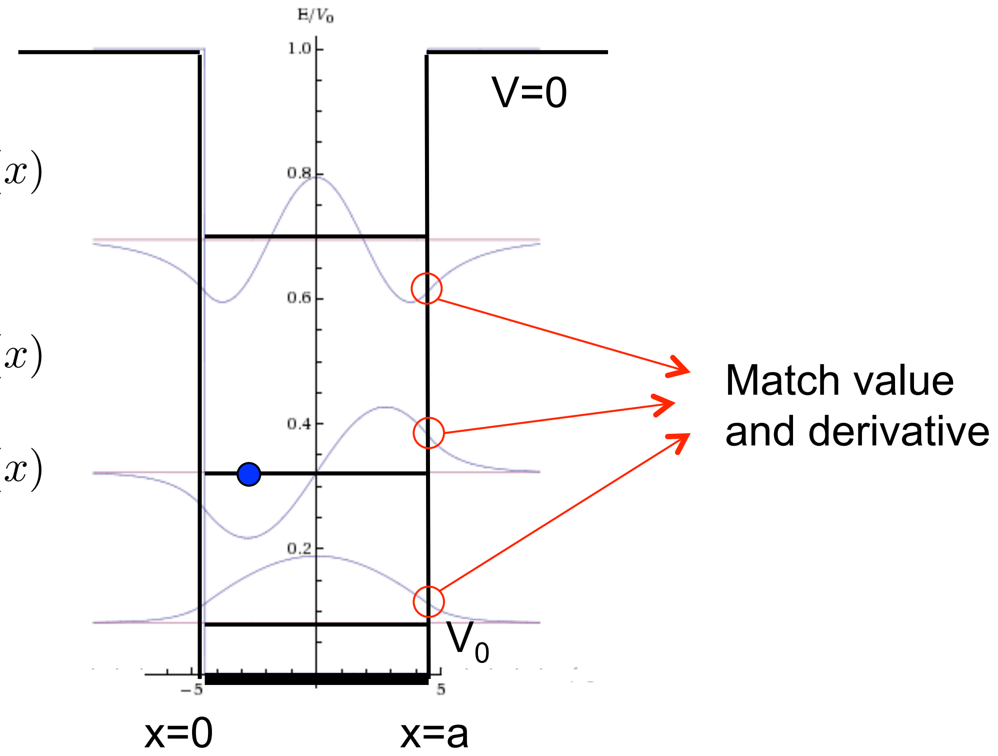

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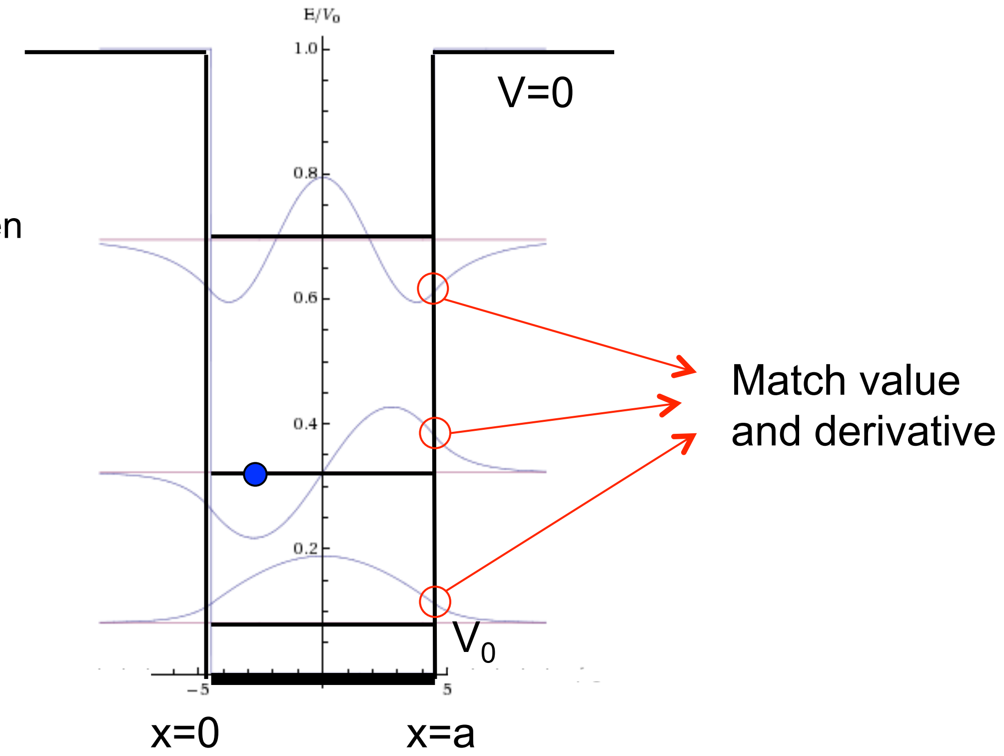
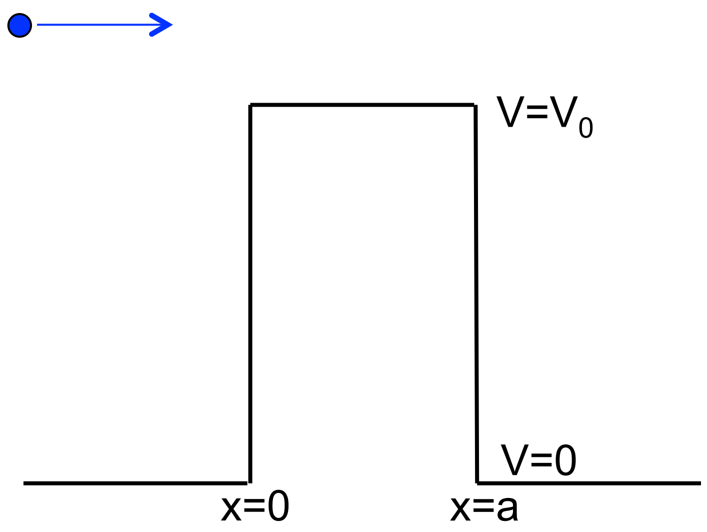
In general, a generic state can be written as a linear expansion of it eigenstates:

$$\psi(x) = \sum_k c_k \psi_k(x)$$



Quantum system: the potential barrier

Solve Schrödinger equation in three regions:
- free travelling particle of energy E
- inside and outside the well
- normalize solutions to match value and derivative and borders x=0 and x=a
- **transmission and reflection**



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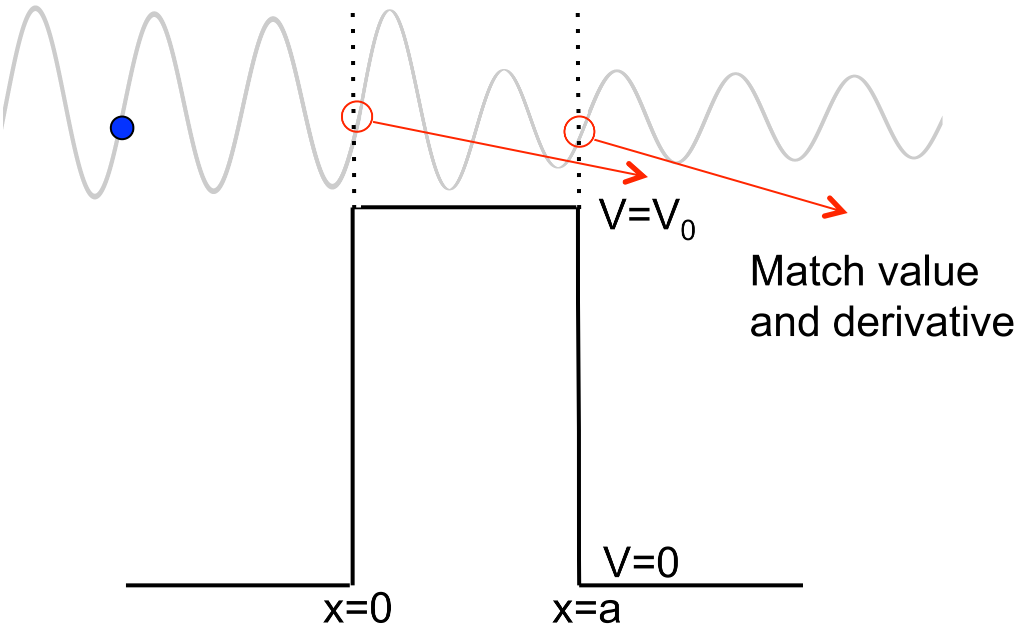

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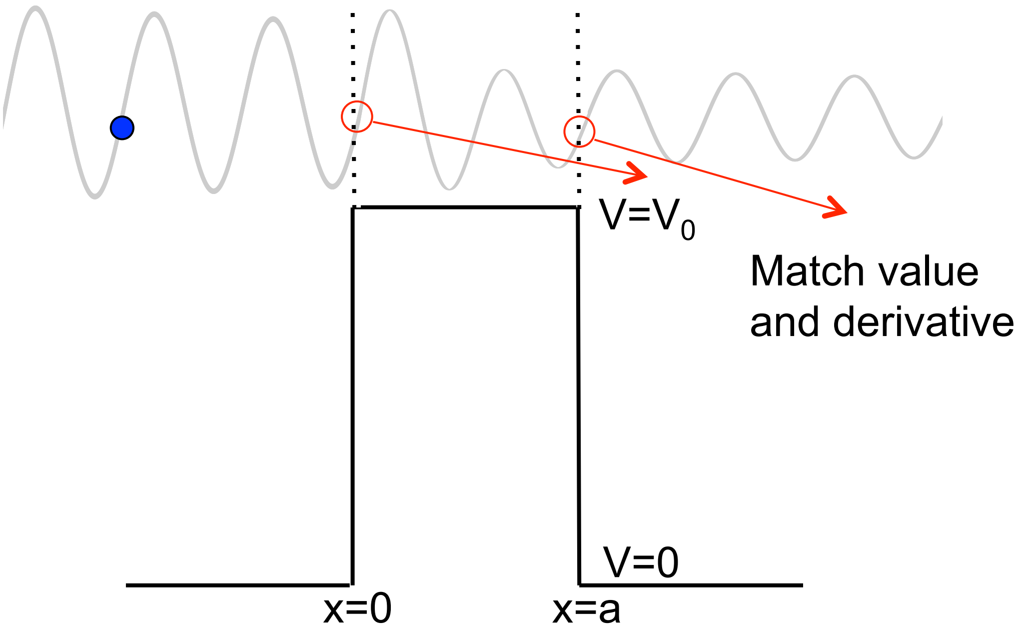
$$\psi_2(x) = Ce^{ik(x)x} + De^{-ik(x)x}$$

$$\psi_3(x) = Ee^{ik(x)x} + Fe^{-ik(x)x}$$

$$k(x) = \sqrt{2m(E - V_0)/\hbar^2}$$



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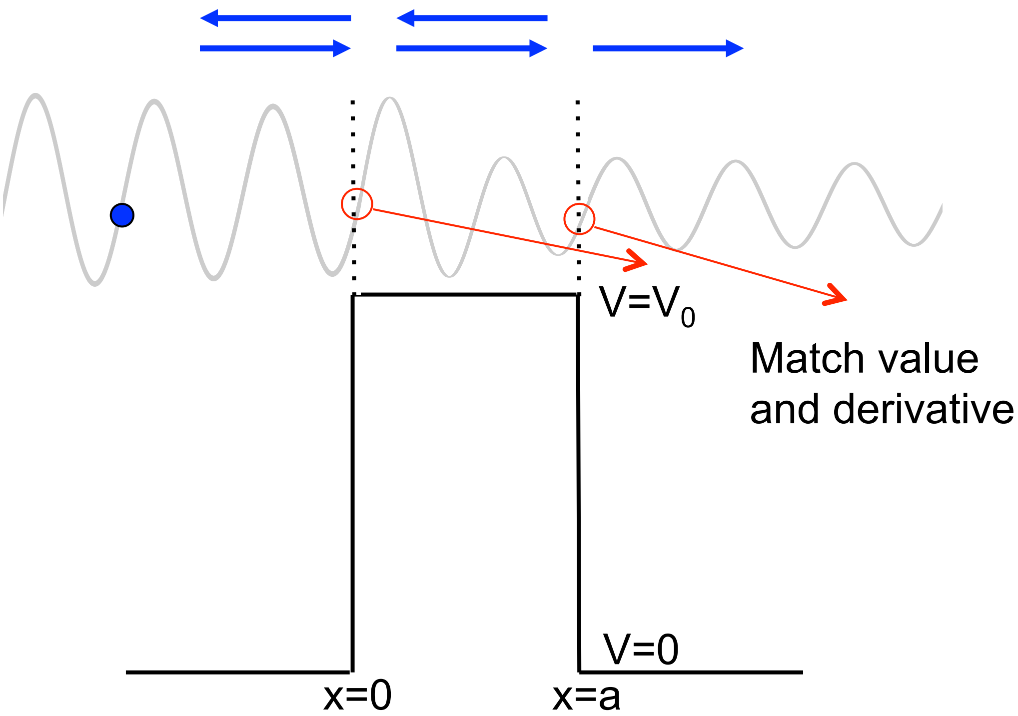

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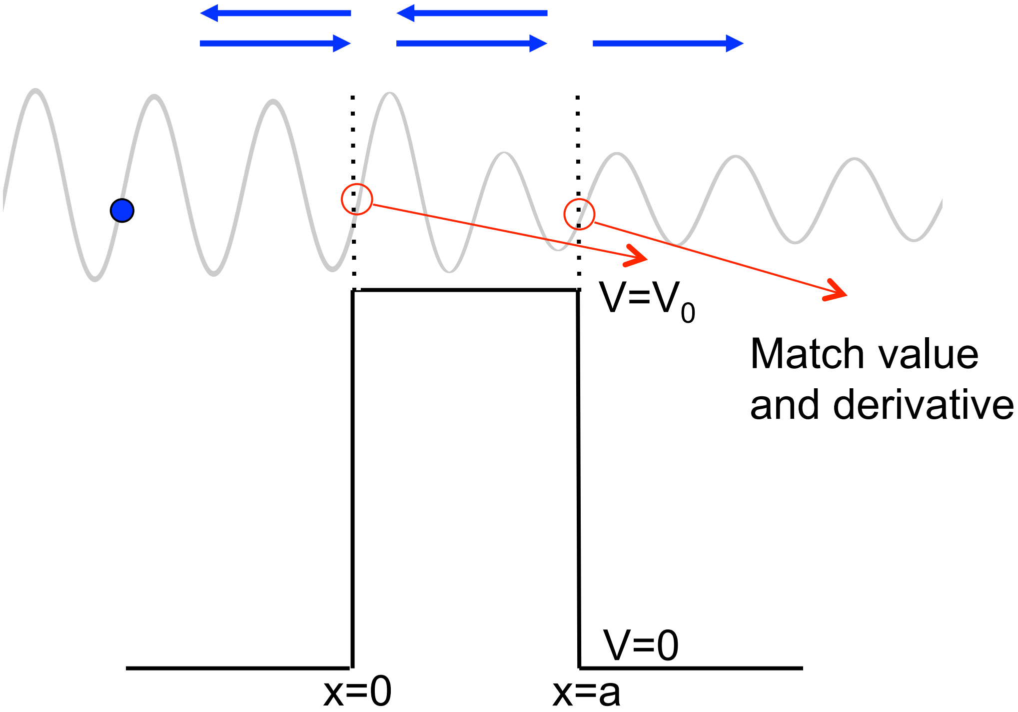

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$$j = \frac{\hbar}{2mi}\left(\psi^* \nabla\psi - \nabla\psi^* \psi\right)$$

transmission $T = |F|^2/|A|^2 = j_{\text{trans}}/j_{\text{inc}}$

Other useful excercises in 1D:
• barrier potential
• finite potential well
• harmonic oscillator

More complicated in 3D, V=V(r), more particles, degeneracy:
• cartesian well
• spherical well
• harmonic oscillator
• realistic potentials (Wood-Saxon),

→ No analytical solution possible,
numerical solutions

Apply to real quantum systems:
atoms (hydrogen) but also to nuclei.

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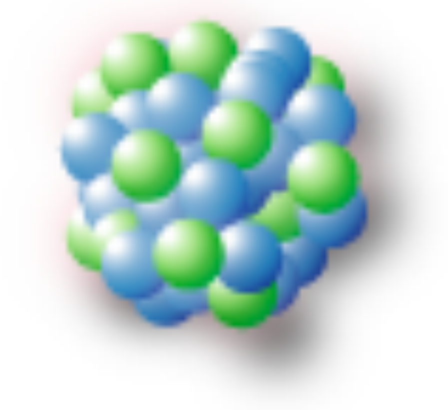
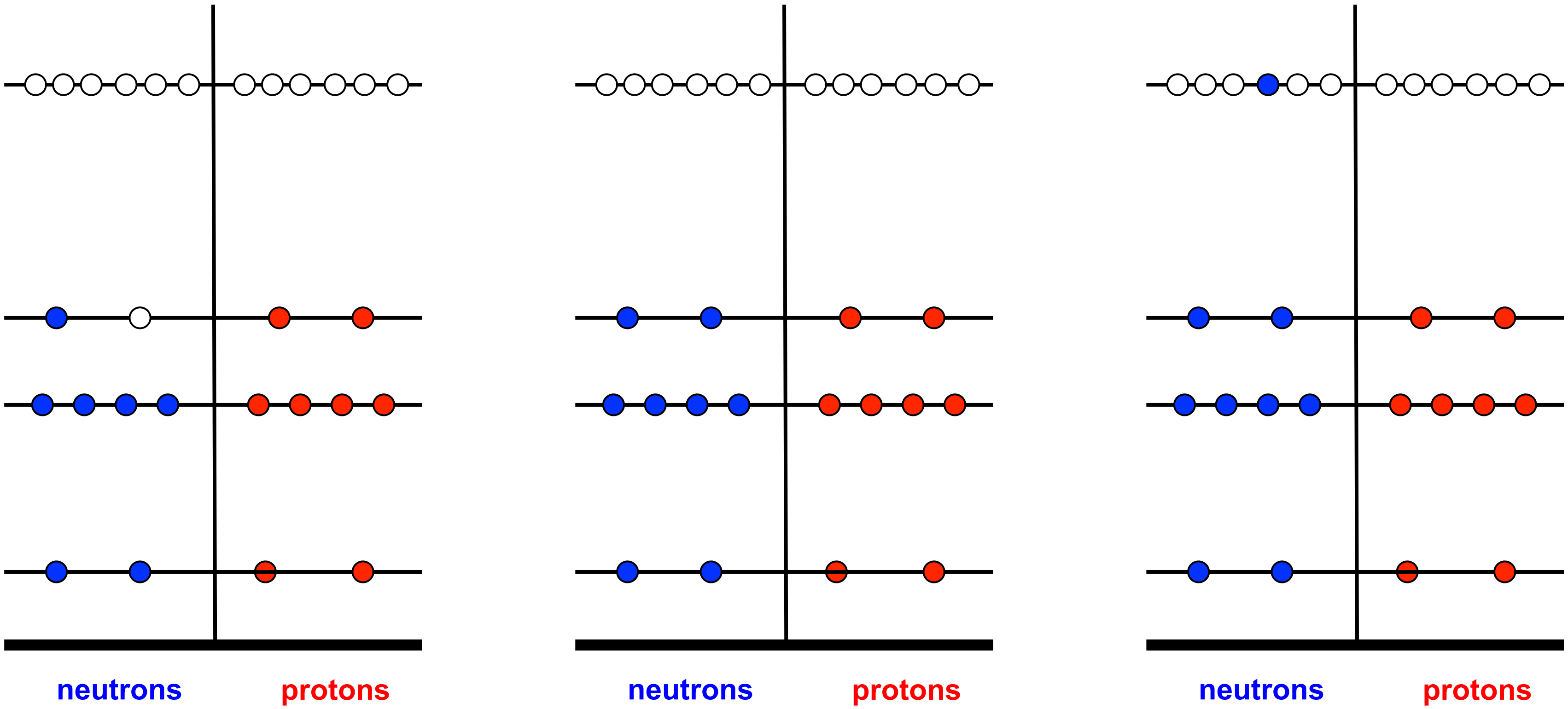


Apply to real quantum systems:
atoms (hydrogen) but also to nuclei.

A nucleus is a quantum system of nucleons (protons and neutrons), bound together by the strong force.

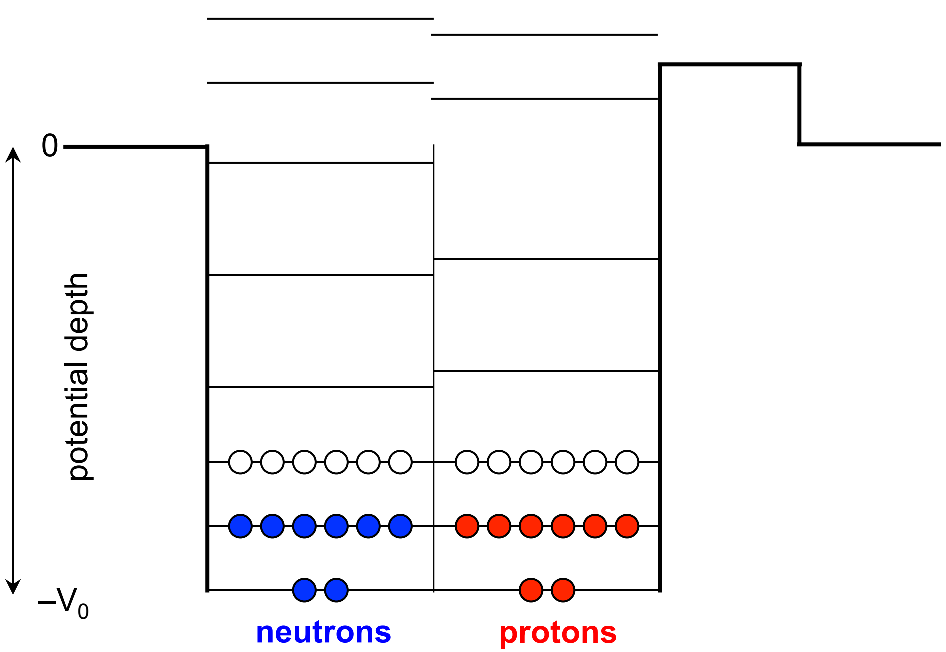
The nucleus as a quantum system, shell model



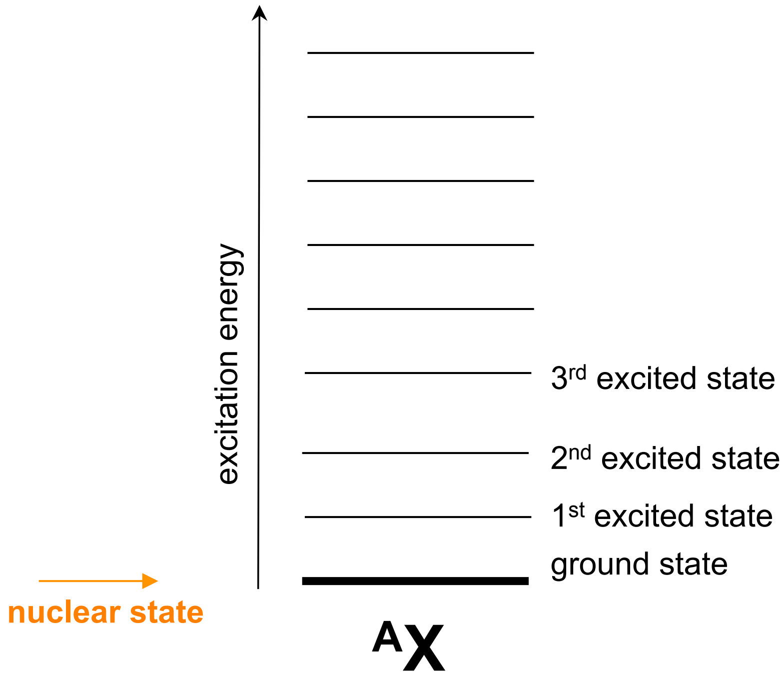
$^{15}_{8}\mathrm{O}$ $^{16}_{8}\mathrm{O}$ $^{17}_{8}\mathrm{O}$

The nucleus as a quantum system

shell model representation:
configuration of nucleons in their potential



level scheme representation:
excited states of a nucleus
(shell model and other states)



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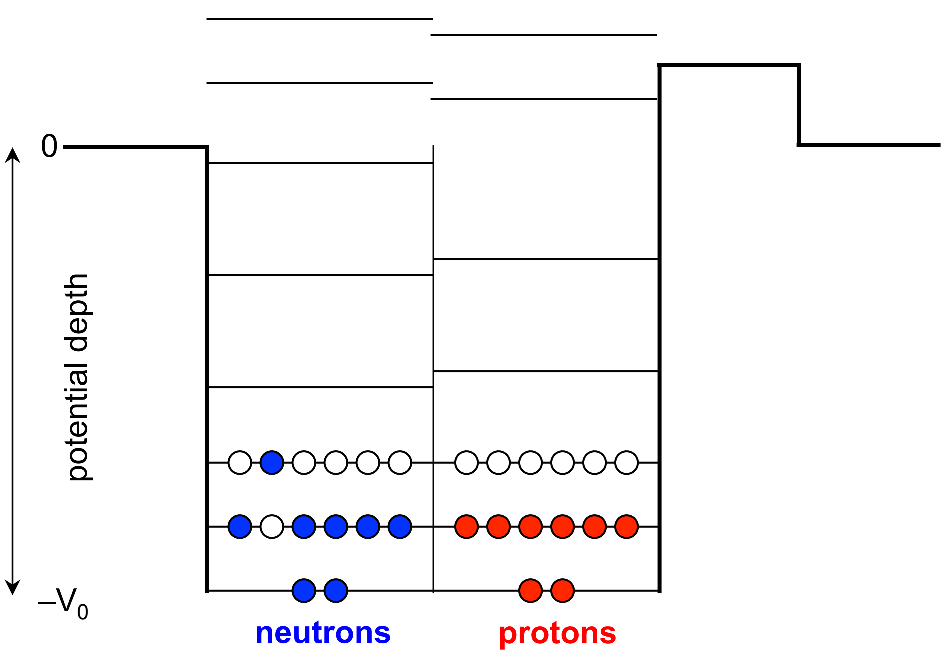

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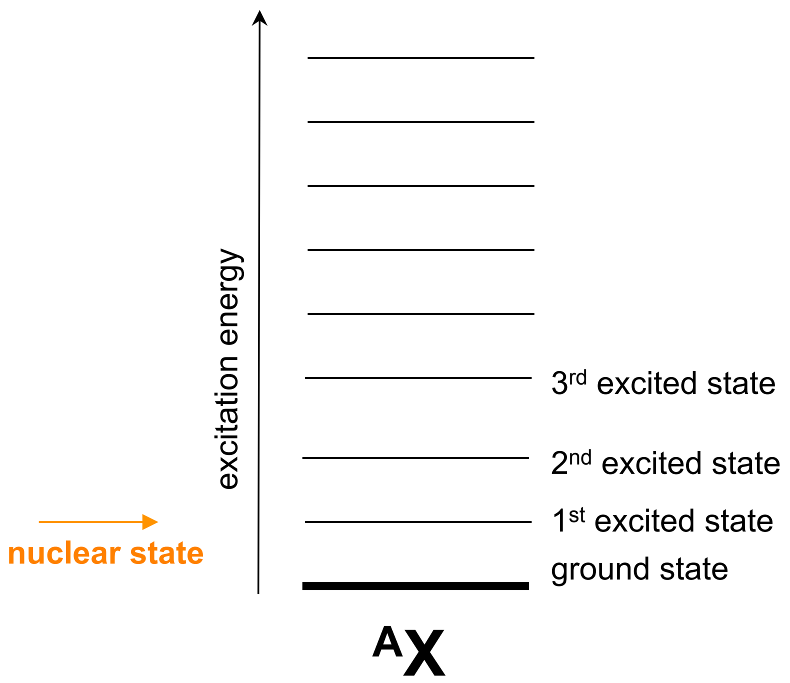

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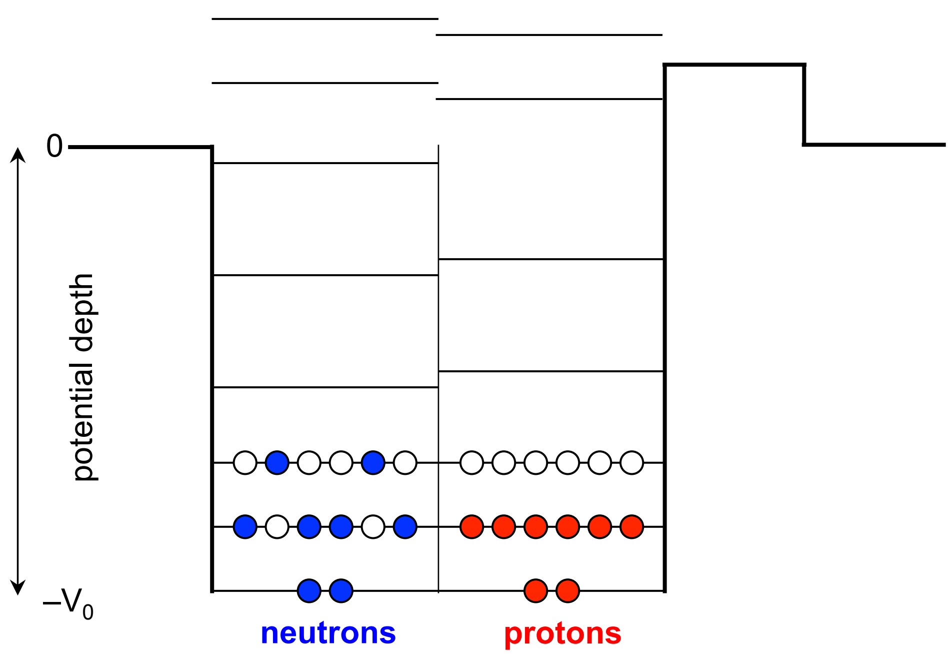

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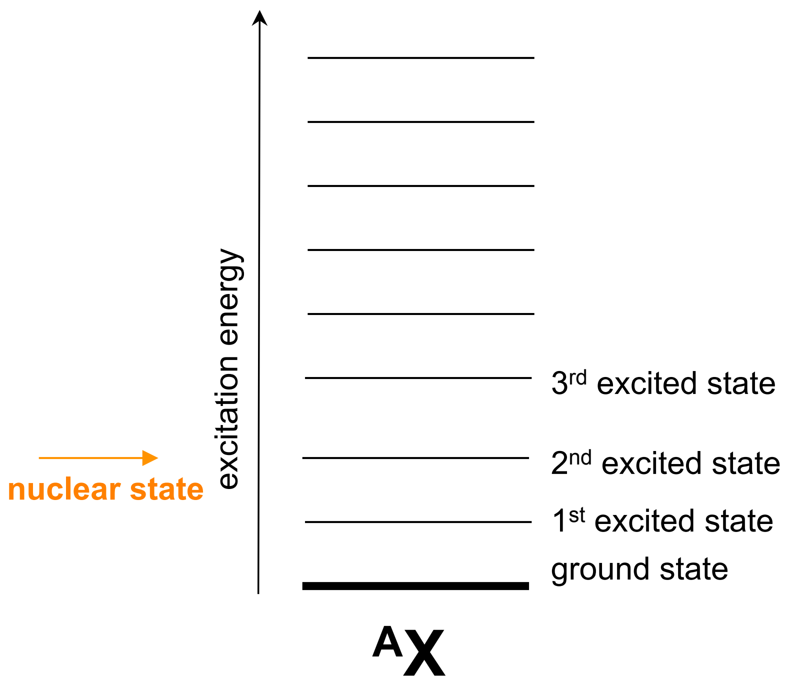

The nucleus as a quantum system

shell model representation:
configuration of nucleons in their potential

level scheme representation:
excited states of a nucleus
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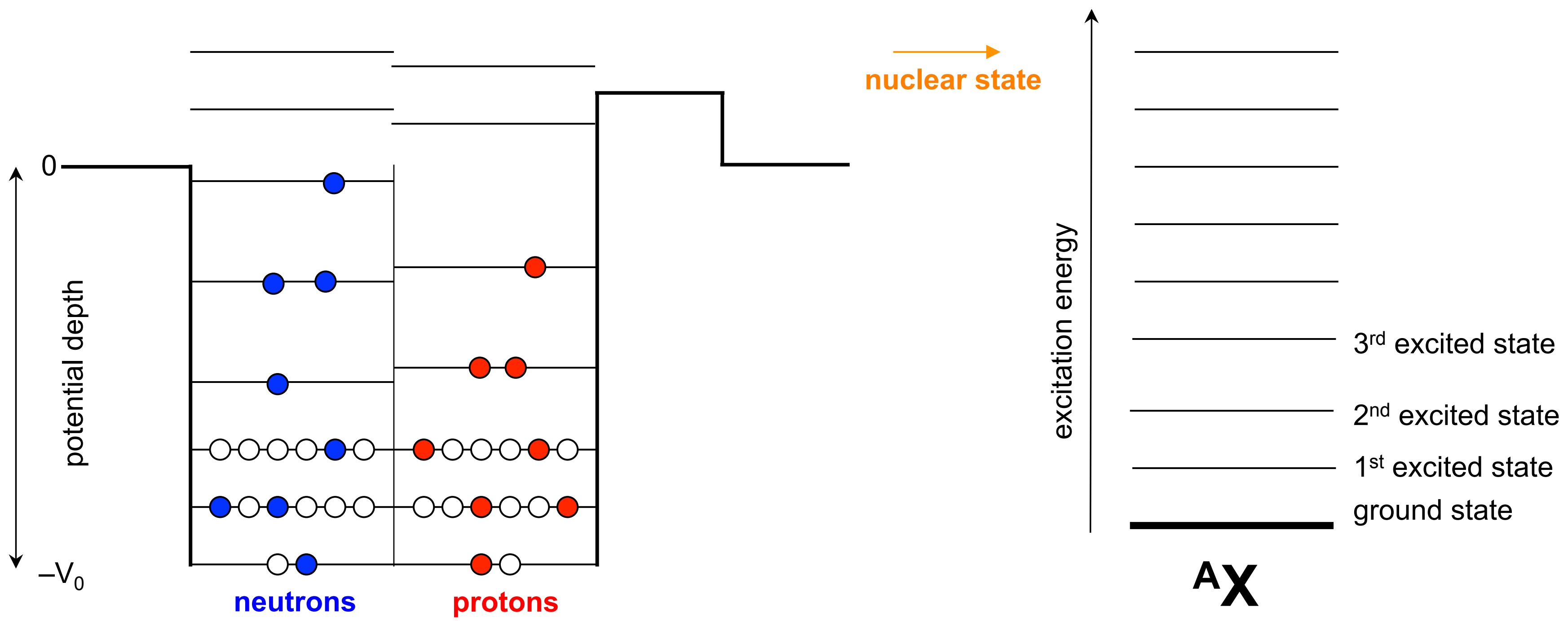
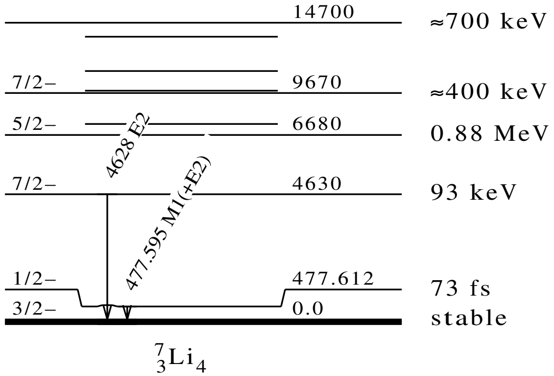
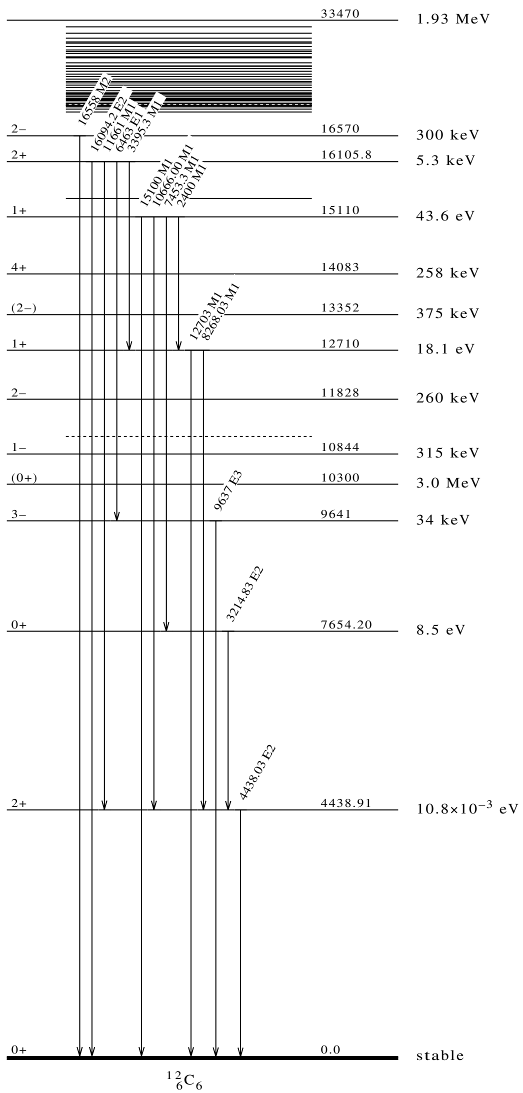


The nucleus as a quantum system

Level schemes from ENSDF
www.nndc.bnl.gov/ensdf

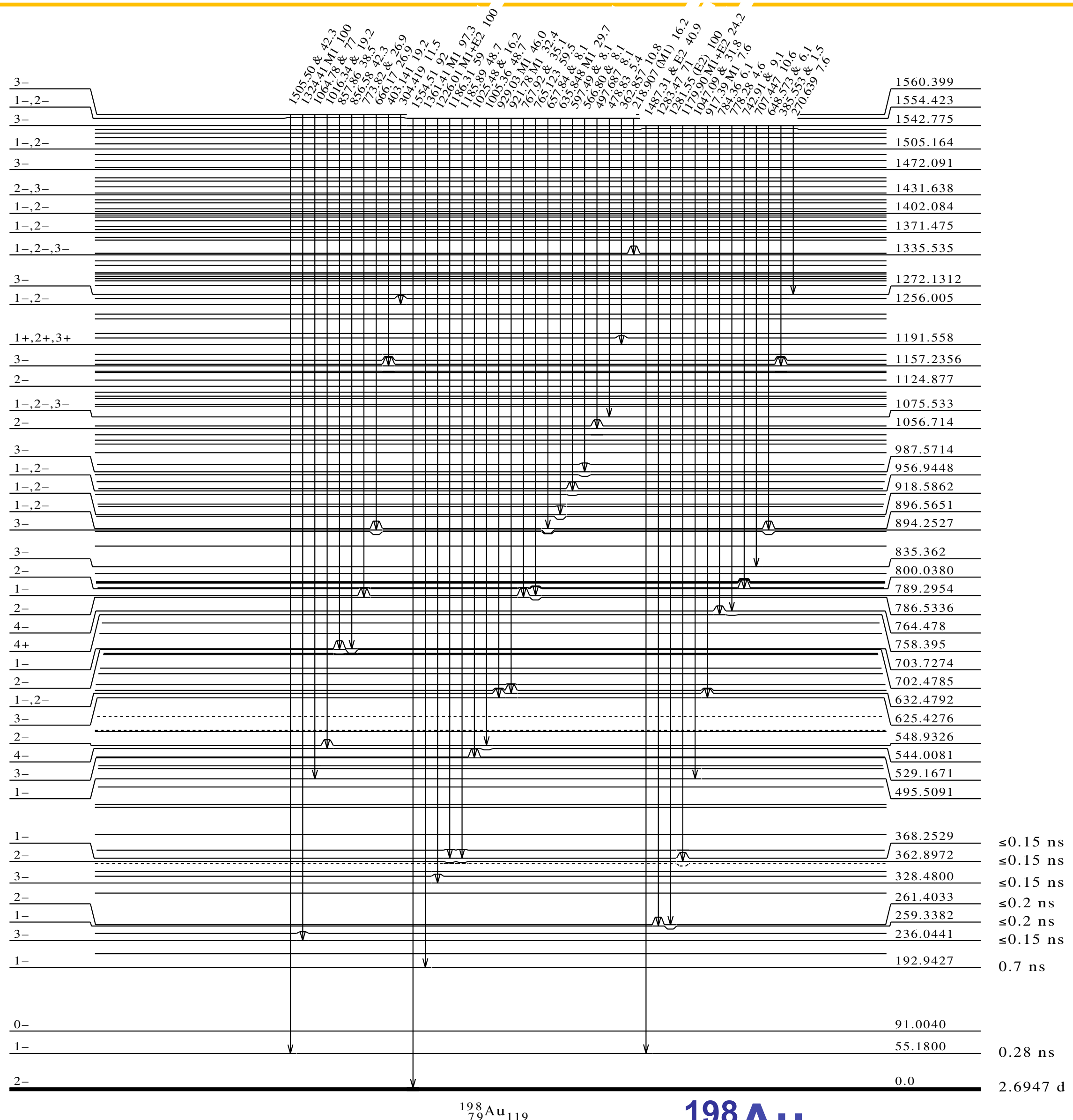


^{7}Li



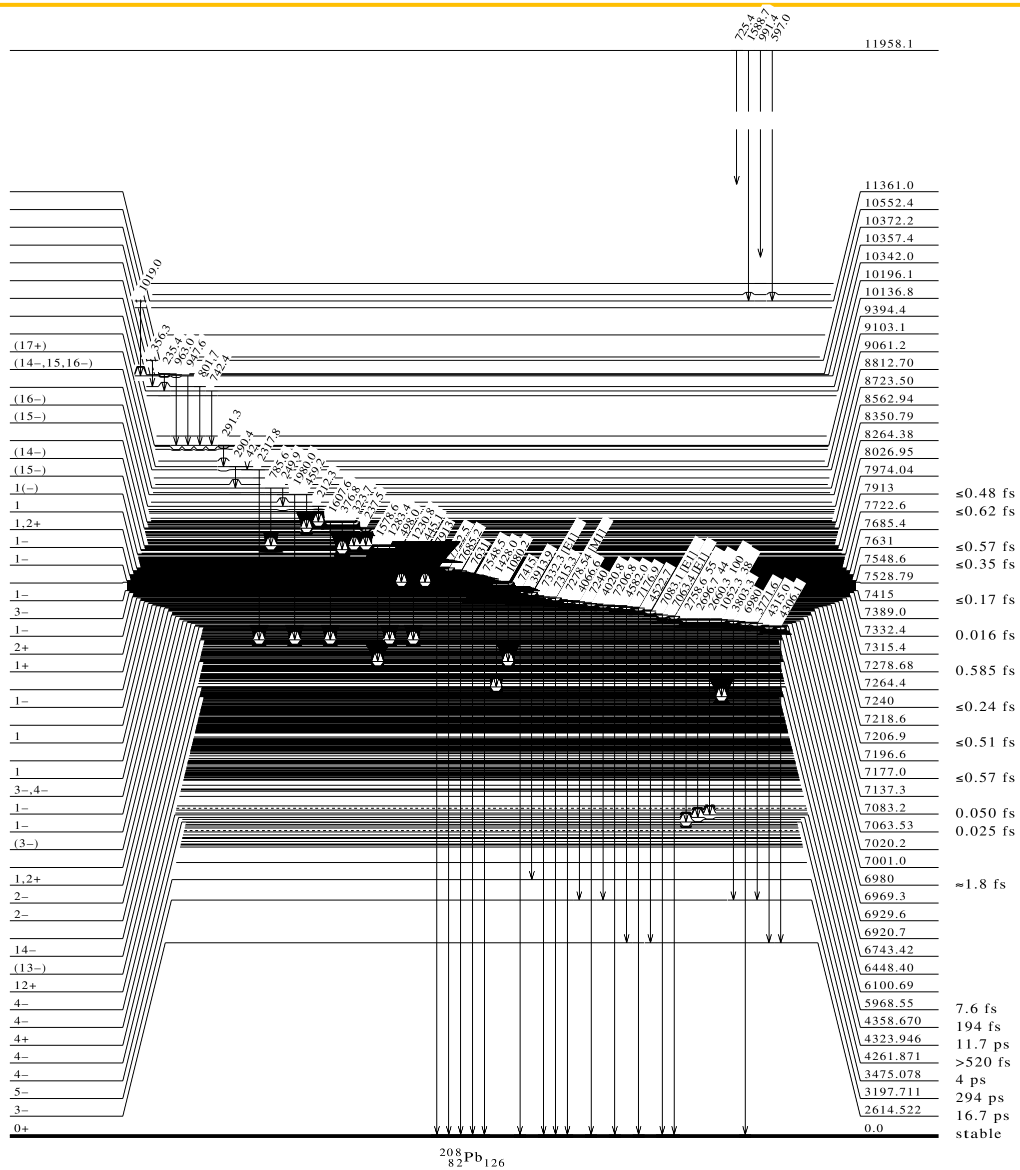
^{12}C

The nucleus as a quantum system



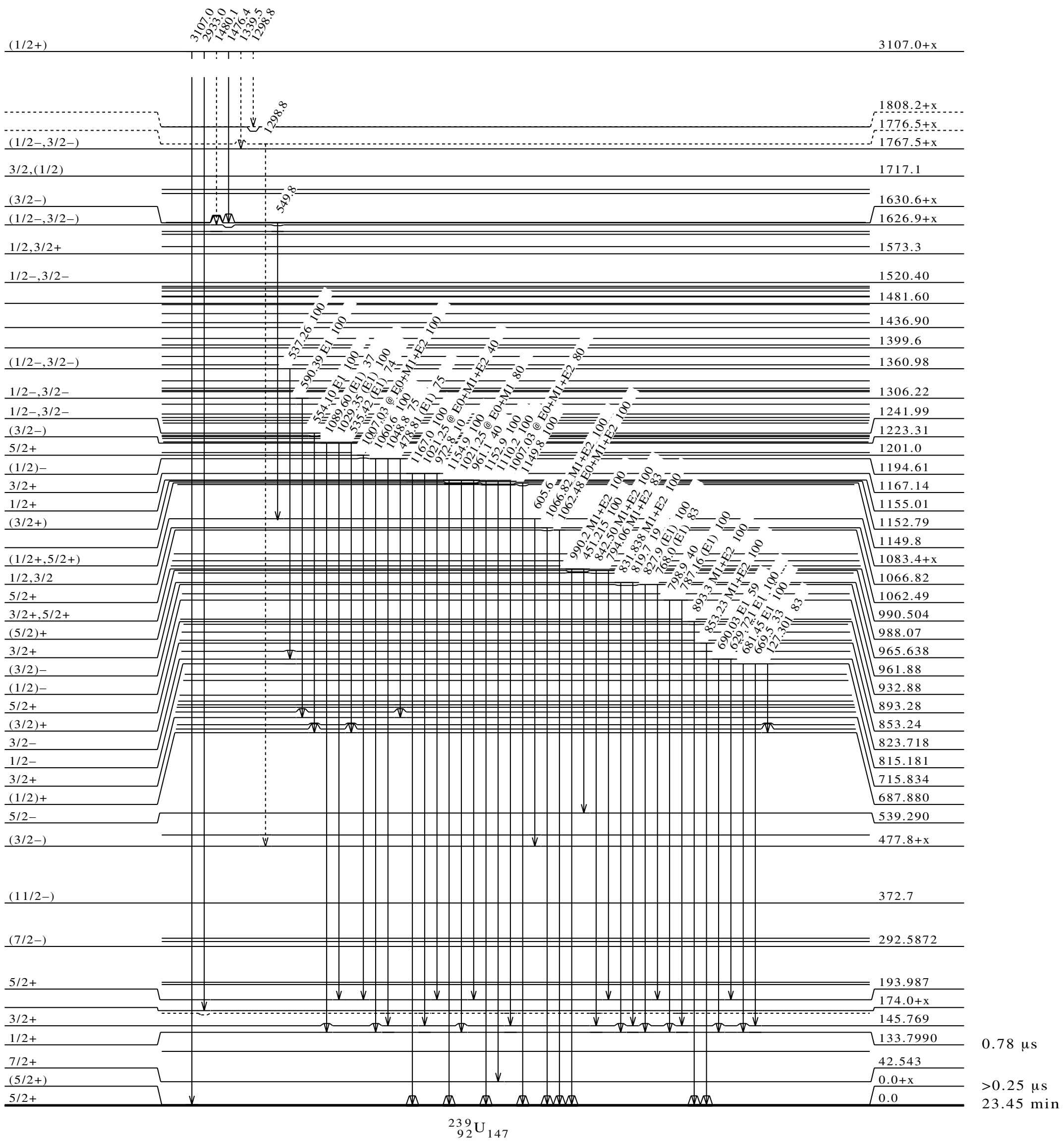
^{198}Au

The nucleus as a quantum system



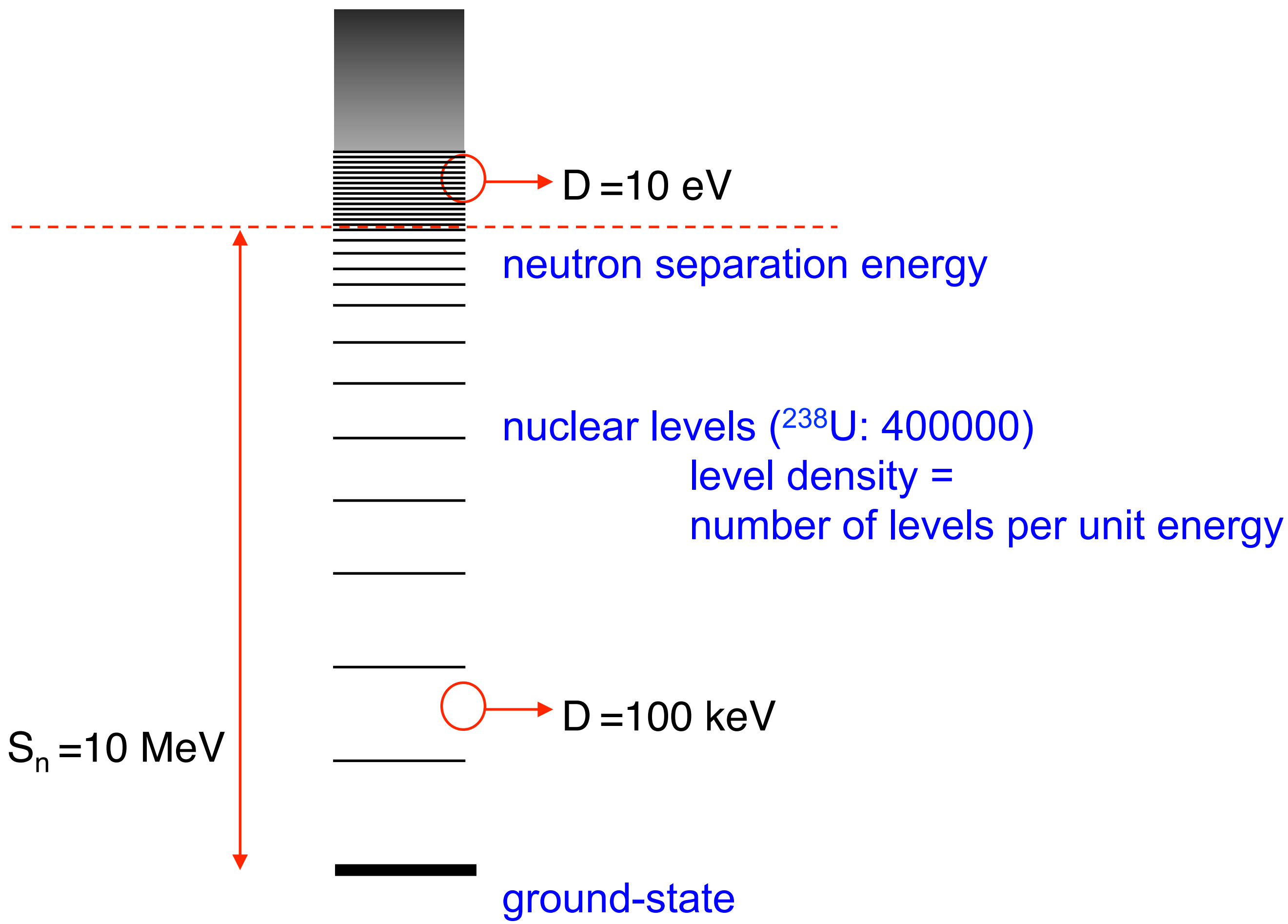
^{208}Pb

The nucleus as a quantum system

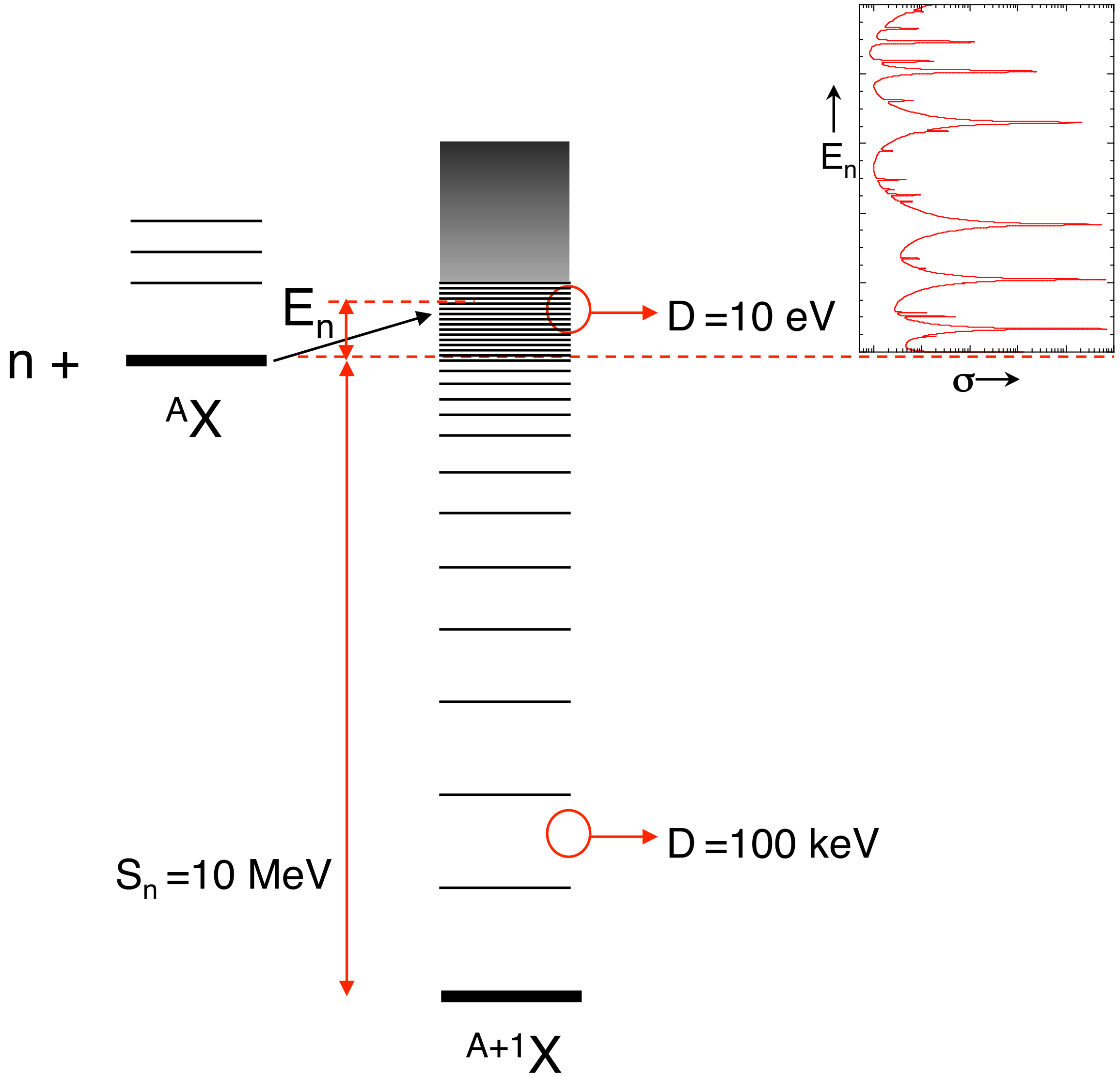


^{239}U

Nuclear levels

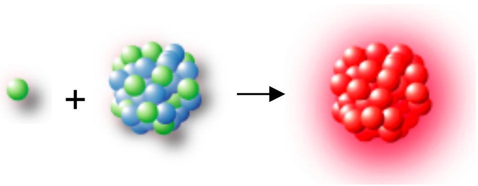


Compound neutron-nucleus reactions



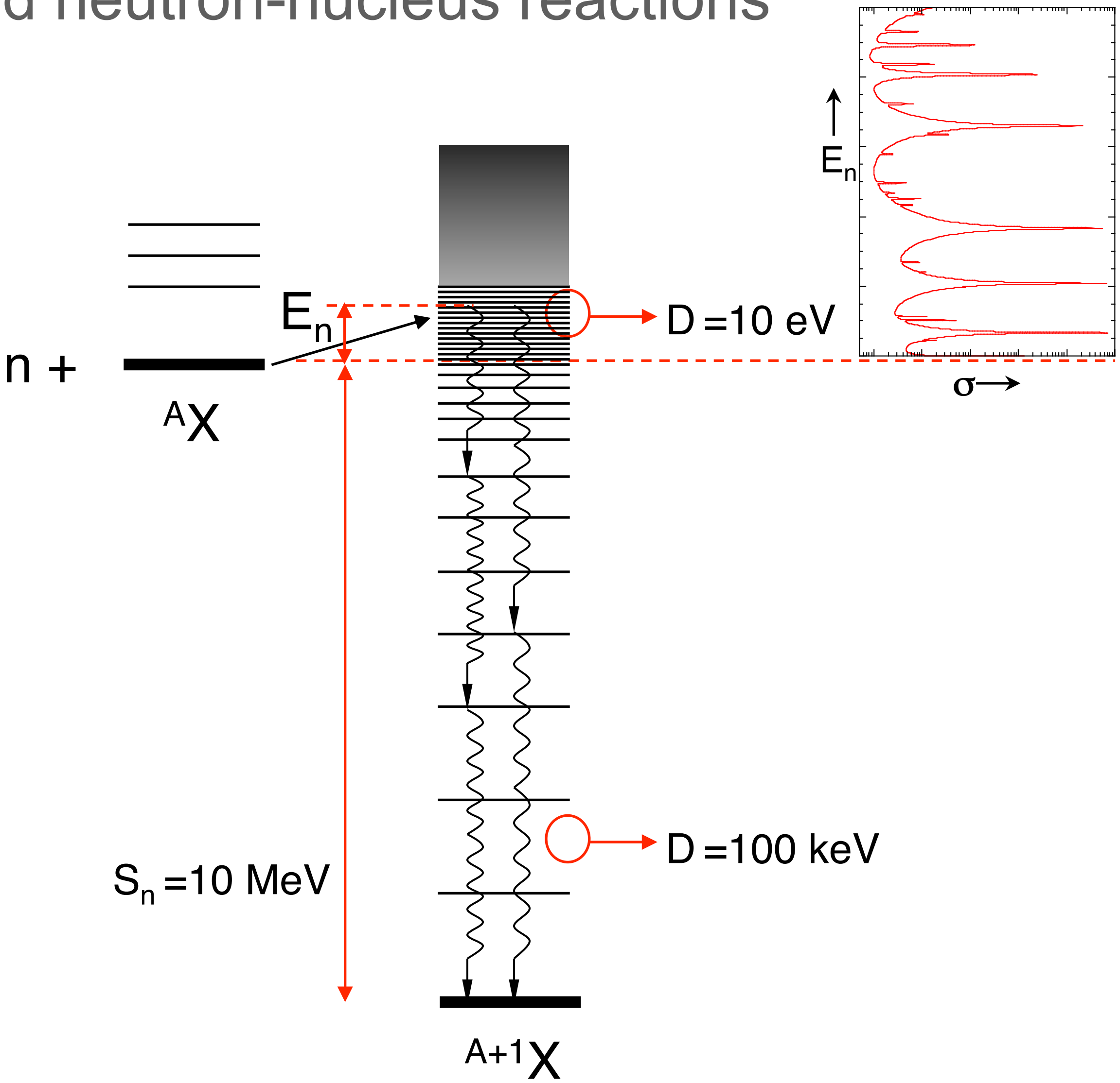
Compound neutron-nucleus reactions

compound
nucleus reaction

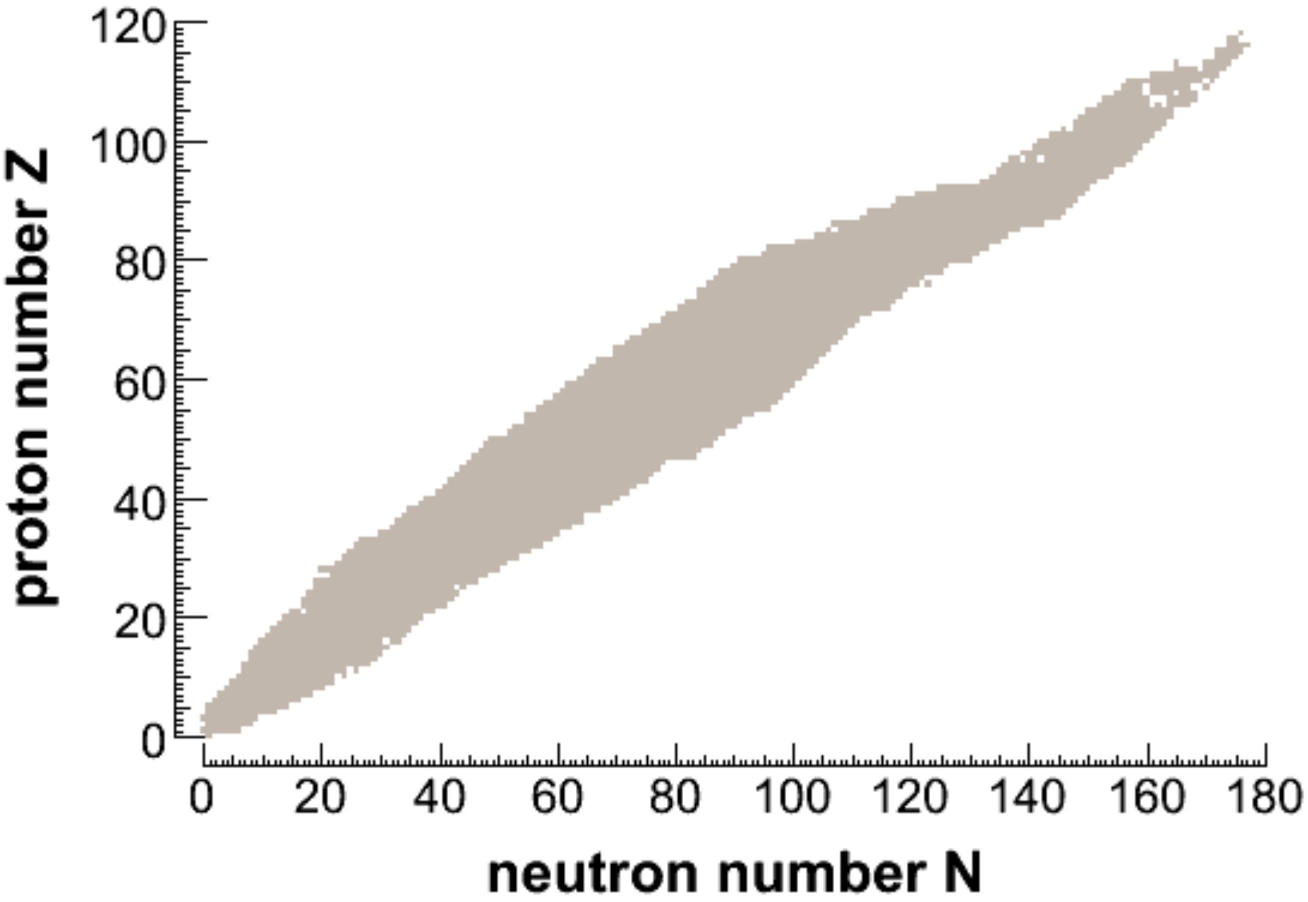


$\tau \sim 10^{-16}$

$E_n < 10$ MeV



Neutron induced reactions: Chart of nuclides



Neutron induced reactions: Chart of nuclides

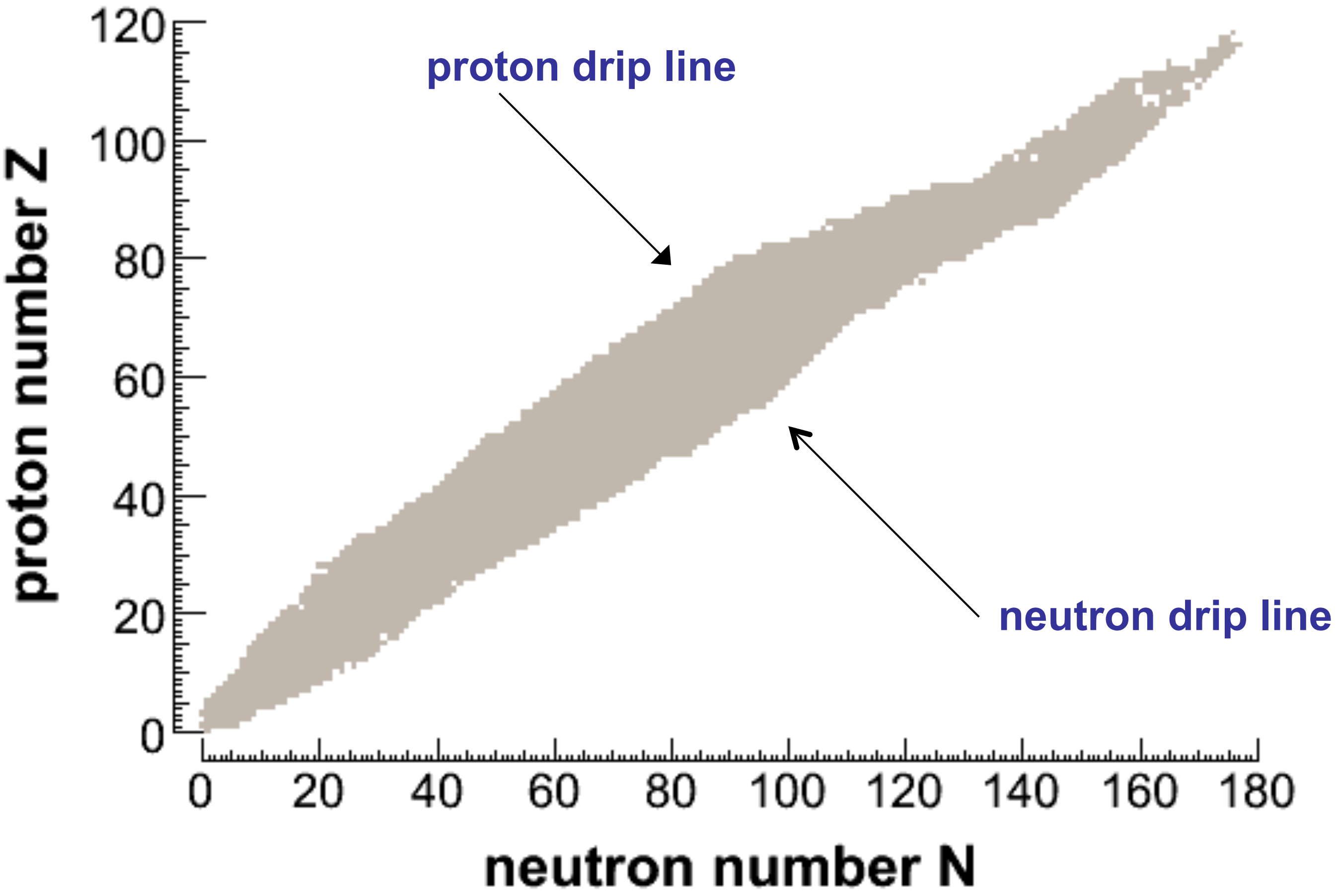


Chart of nuclides

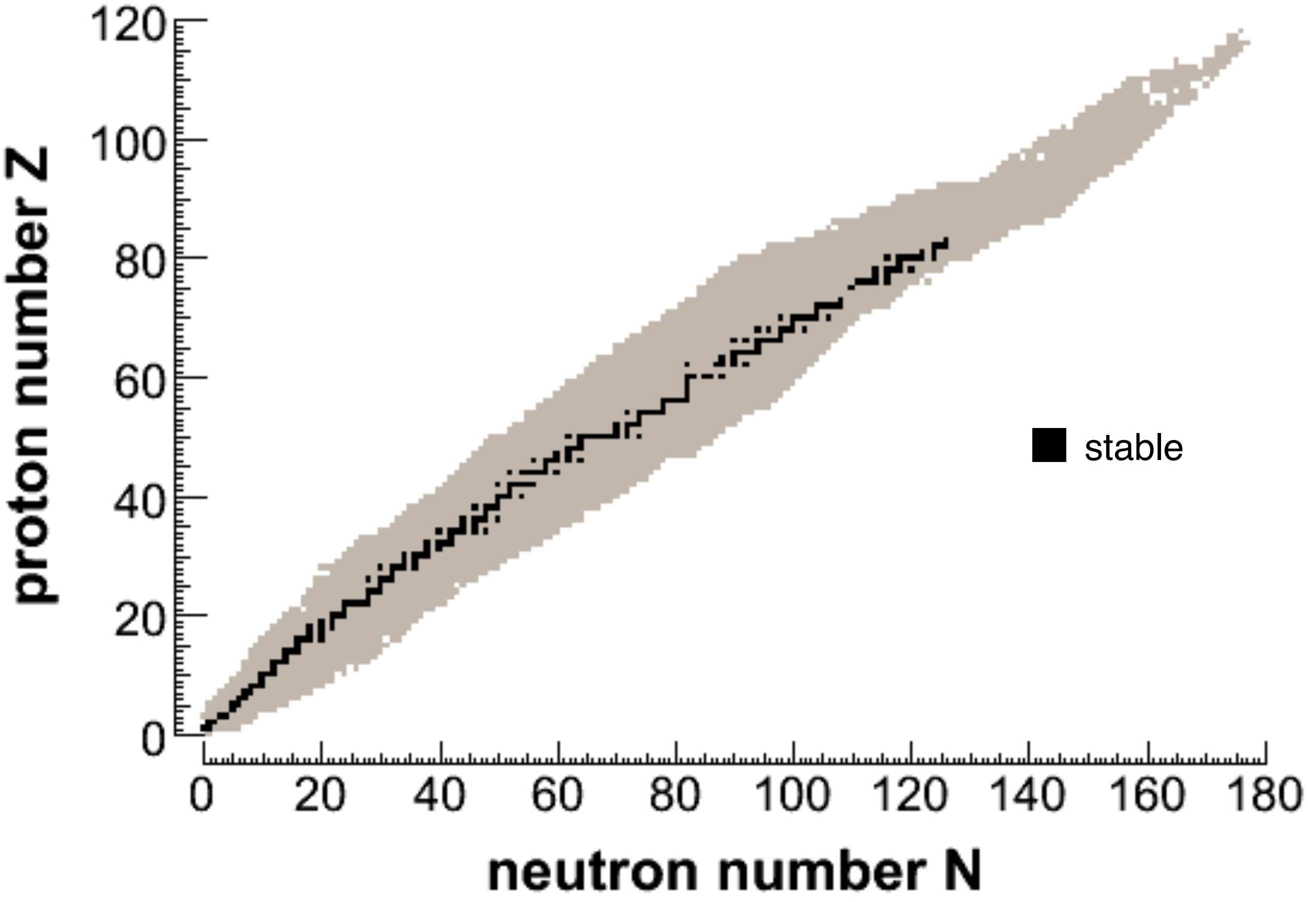
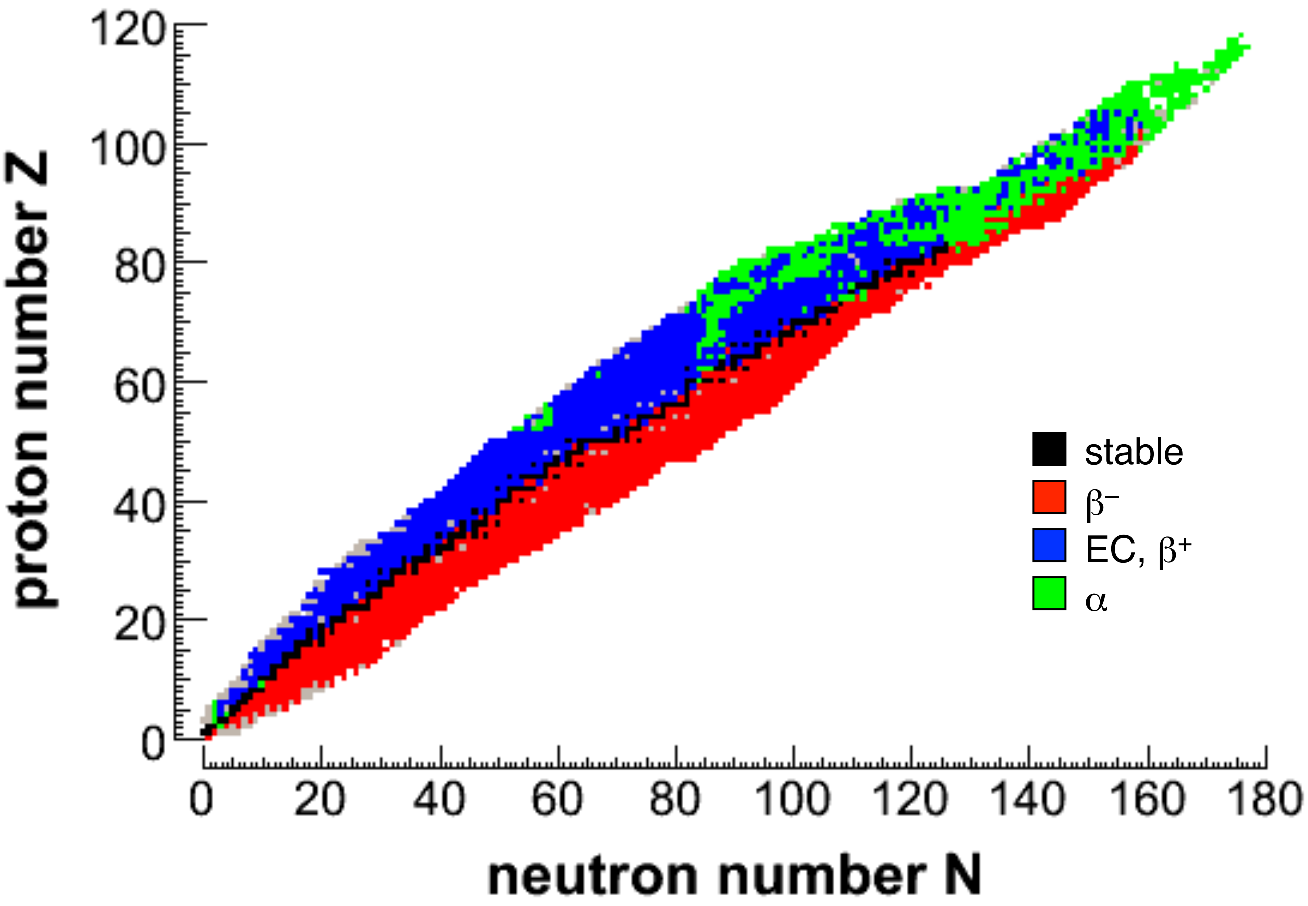
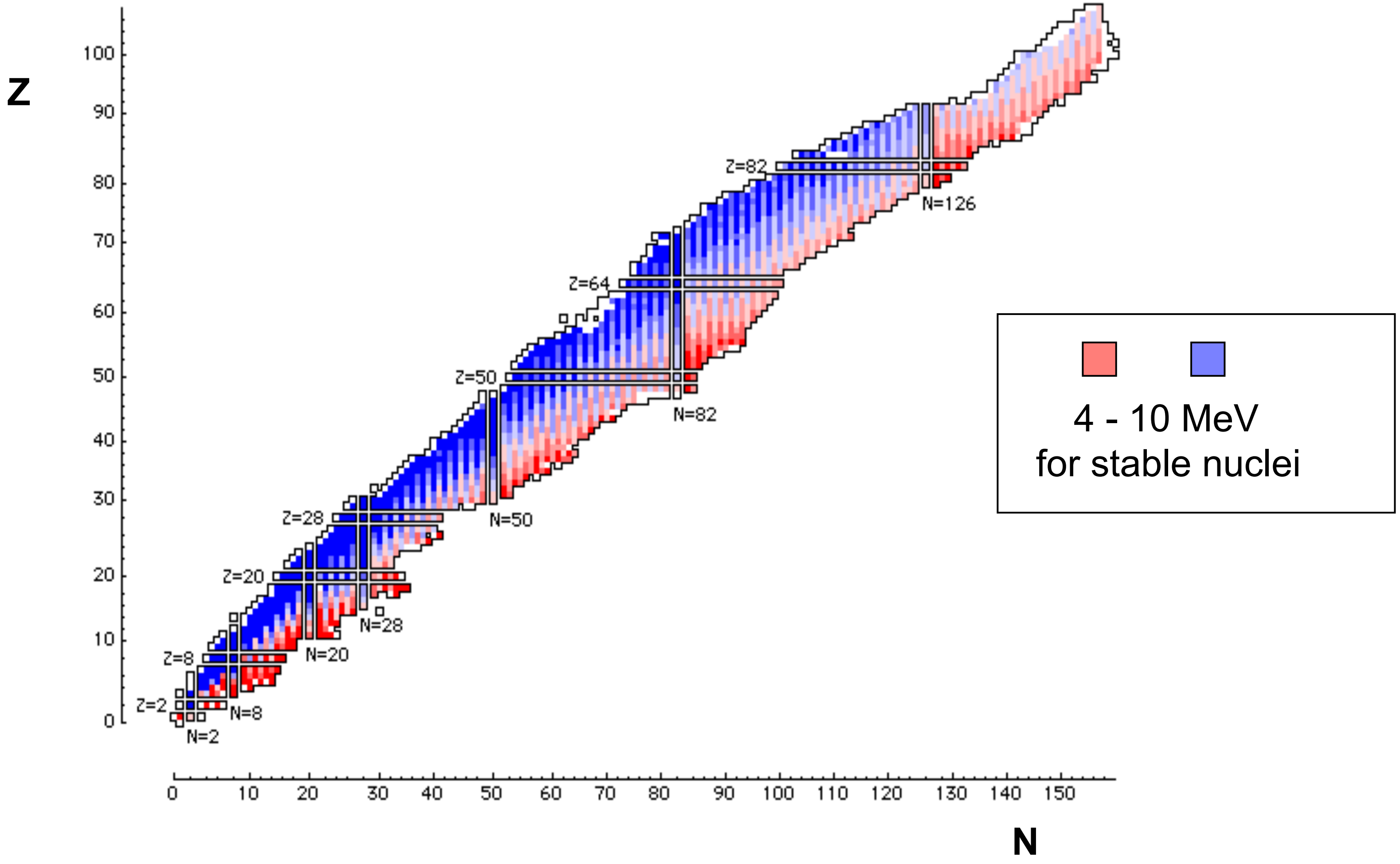


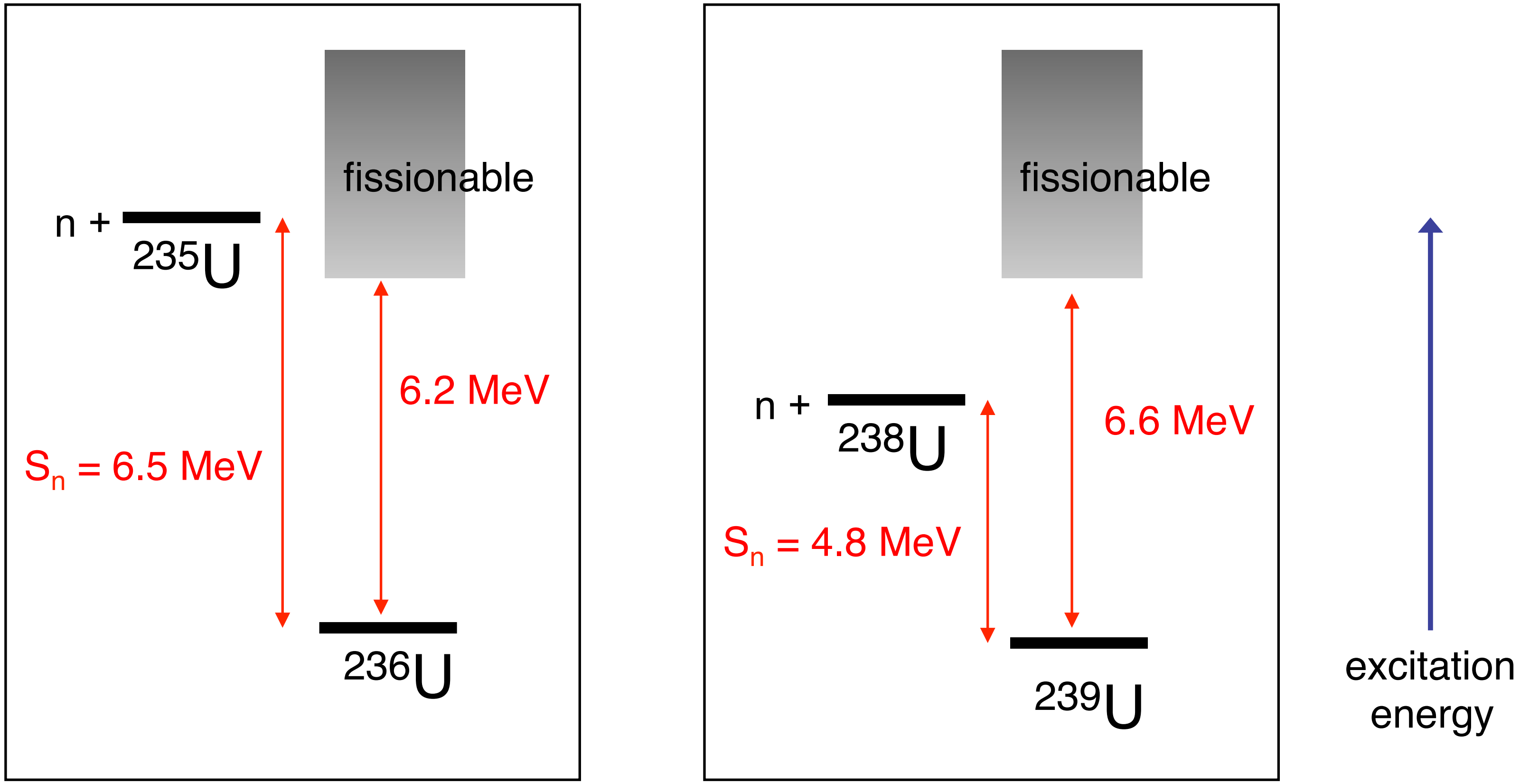
Chart of nuclides



Neutron separation energy



Fission of ^{235}U+n et ^{238}U+n



Decay of a nuclear state

state with a life time τ:

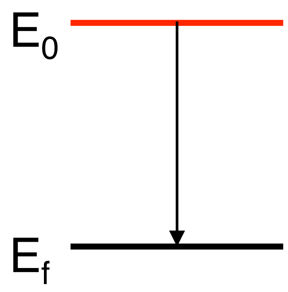
$$\Psi(t) = \Psi_0 e^{-iE_0 t/\hbar} e^{-t/2\tau}$$

definition (Heisenberg):

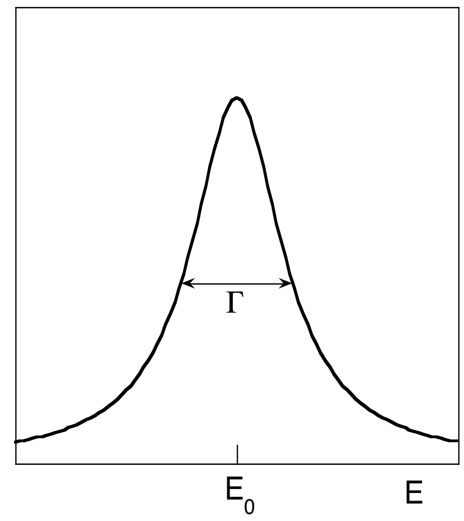
$$\Gamma = \frac{\hbar}{\tau}$$

Fourier transform gives energy profile:

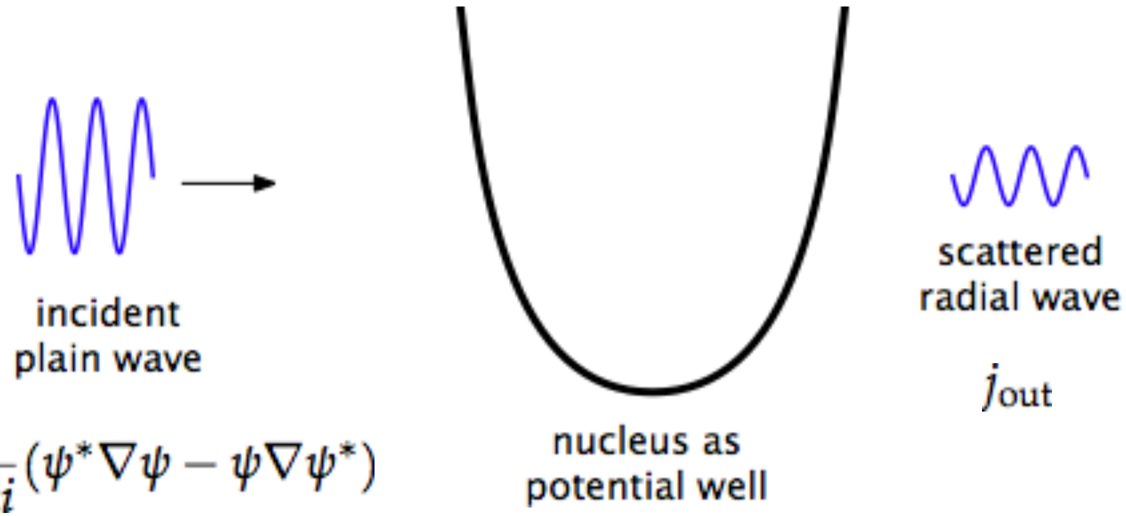
$$I(E) = \frac{\Gamma / 2\pi}{(E - E_0)^2 + \Gamma^2 / 4}$$



eigen state,
transitition $E_0 \rightarrow E_f$
life time τ



Neutron-nucleus reactions



$$j_{\text{inc}} = \frac{\hbar}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*)$$

Conservation of probability density: $\sigma(\Omega) = \dfrac{r^2 j_{\text{out}}(r,\Omega)}{j_{\text{inc}}}$

Solve Schrödinger equation of system to get cross sections.
Shape of wave functions of in- and outgoing particles are known,
potential is unknown. Two approaches:

- **calculate potential (optical model calculations, smooth cross section)**
- **use eigenstates (R-matrix, resonances)**

R-matrix formalism

partial incoming wave functions: $\mathcal{I}_c$

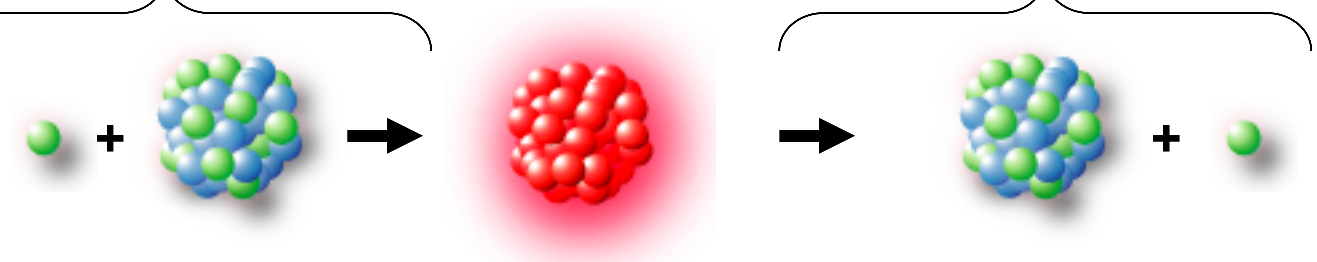
partial outgoing wave functions: $\mathcal{O}_{c'}$

related by collision matrix: $U_{cc'}$

cross section:

$$\sigma_{cc'} = \pi \lambda_c^2 |\delta_{c'c} - U_{c'c}|^2$$

External region (r>a_c, well separated particles):
• no interaction, Schrödinger equation solvable.



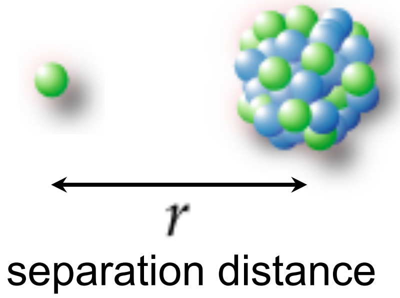
entrance channel
$c = \{\alpha, \ell, j, J, m_J\}$

compound nucleus

exit channel
$c' = \{\alpha', \ell', j', J', m_J'\}$

Internal region (r<a_c compound nucleus):
• wave function is expansion of eigenstates λ.

Find the wave functions



$r > a_c$ external region

$r < a_c$ internal region

$r = a_c$ match value and derivate of Ψ

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - \frac{2m_c}{\hbar^2}(V-E)\right] rR(r) = 0$$

External region: **easy**, solve Schrödinger equation
central force, separate radial and angular parts.
solution: solve Schrödinger equation of relative motion:

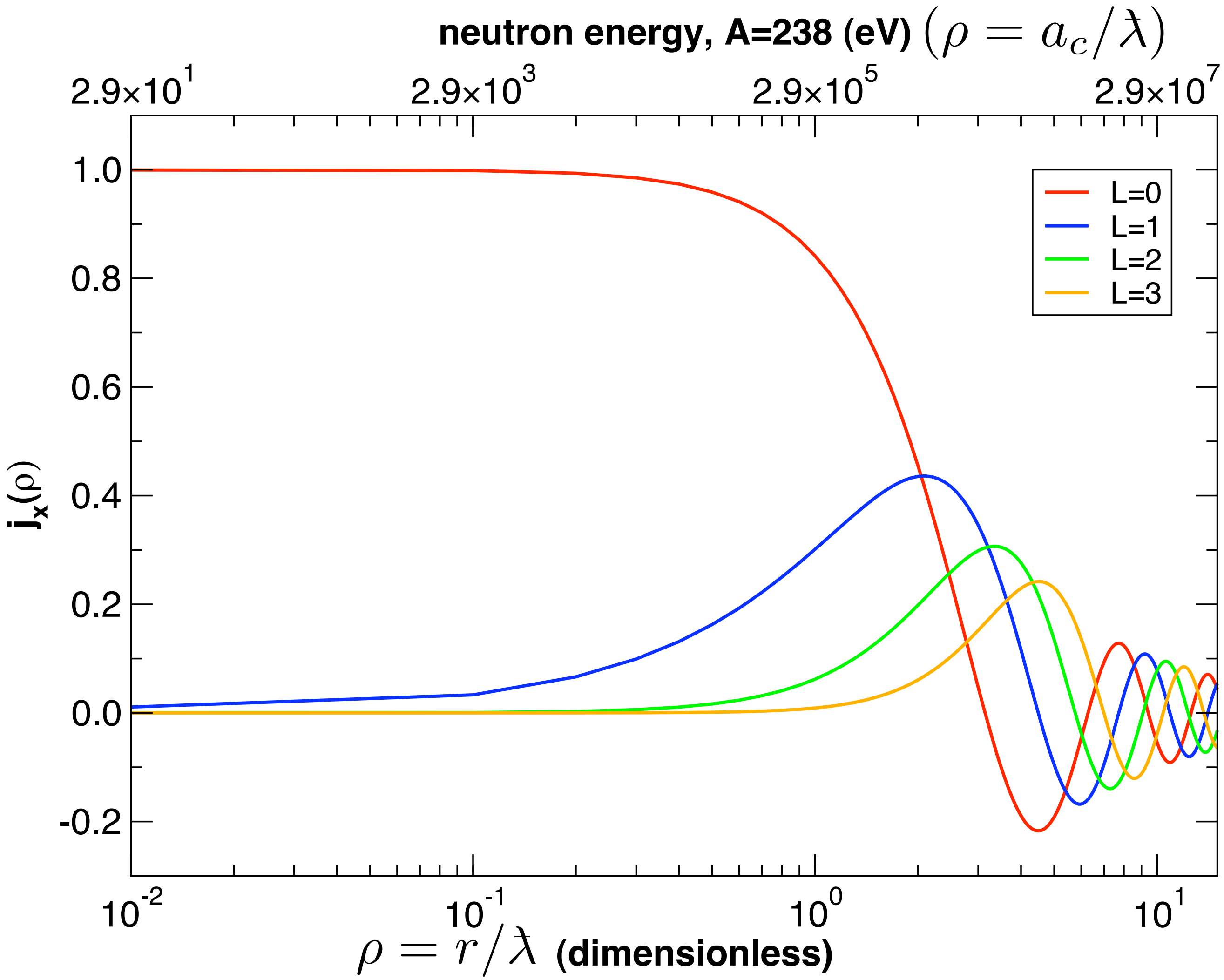
$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

- Coulomb functions
- special case of neutron particles (neutrons): fonctions de Bessel

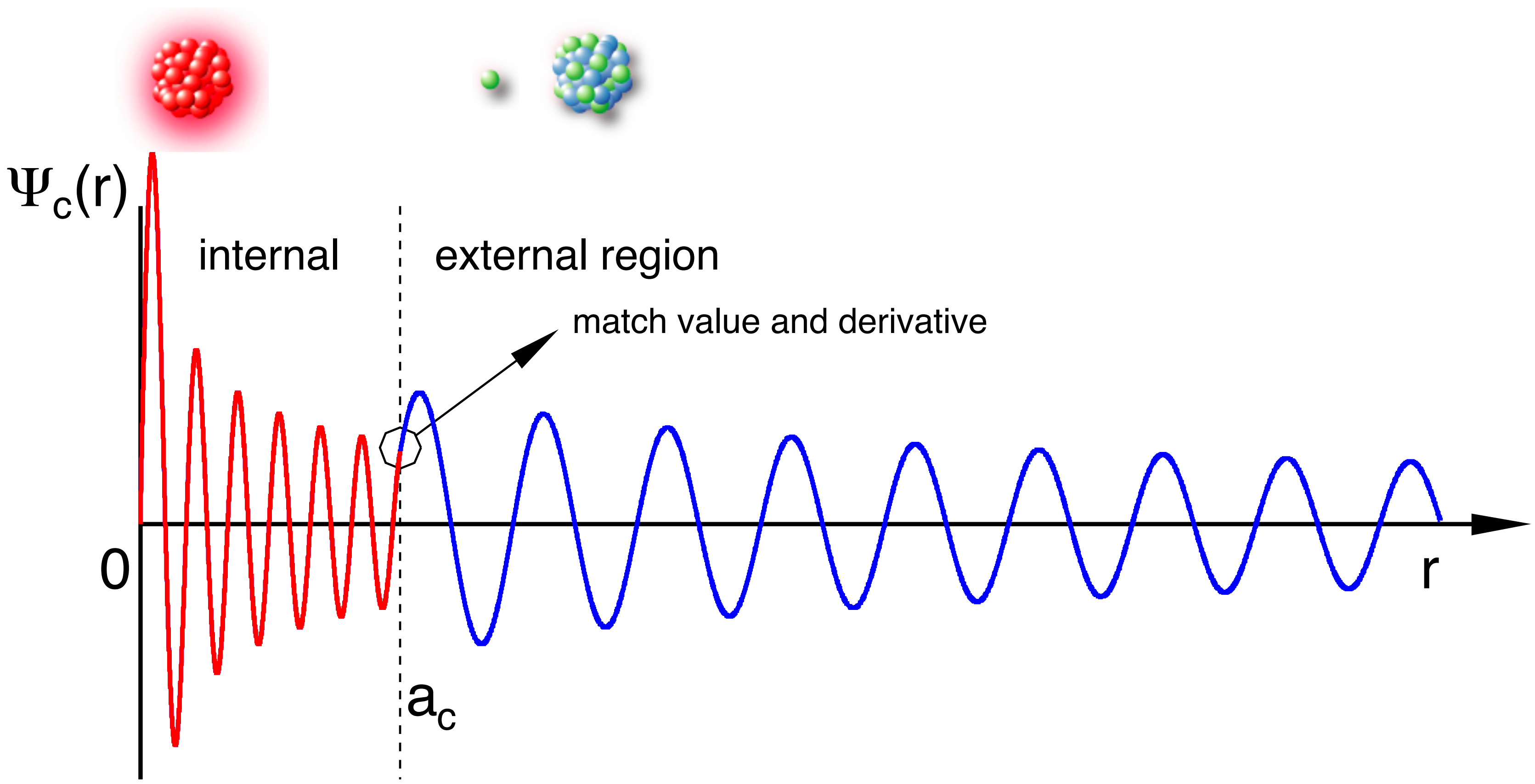
Internal region: **very difficult**, Schrödinger equation cannot be solved directly
solution: expand the wave function as a linear combination of its eigenstates.
using the R-matrix:

$$R_{cc'} = \sum_{\lambda} \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E}$$

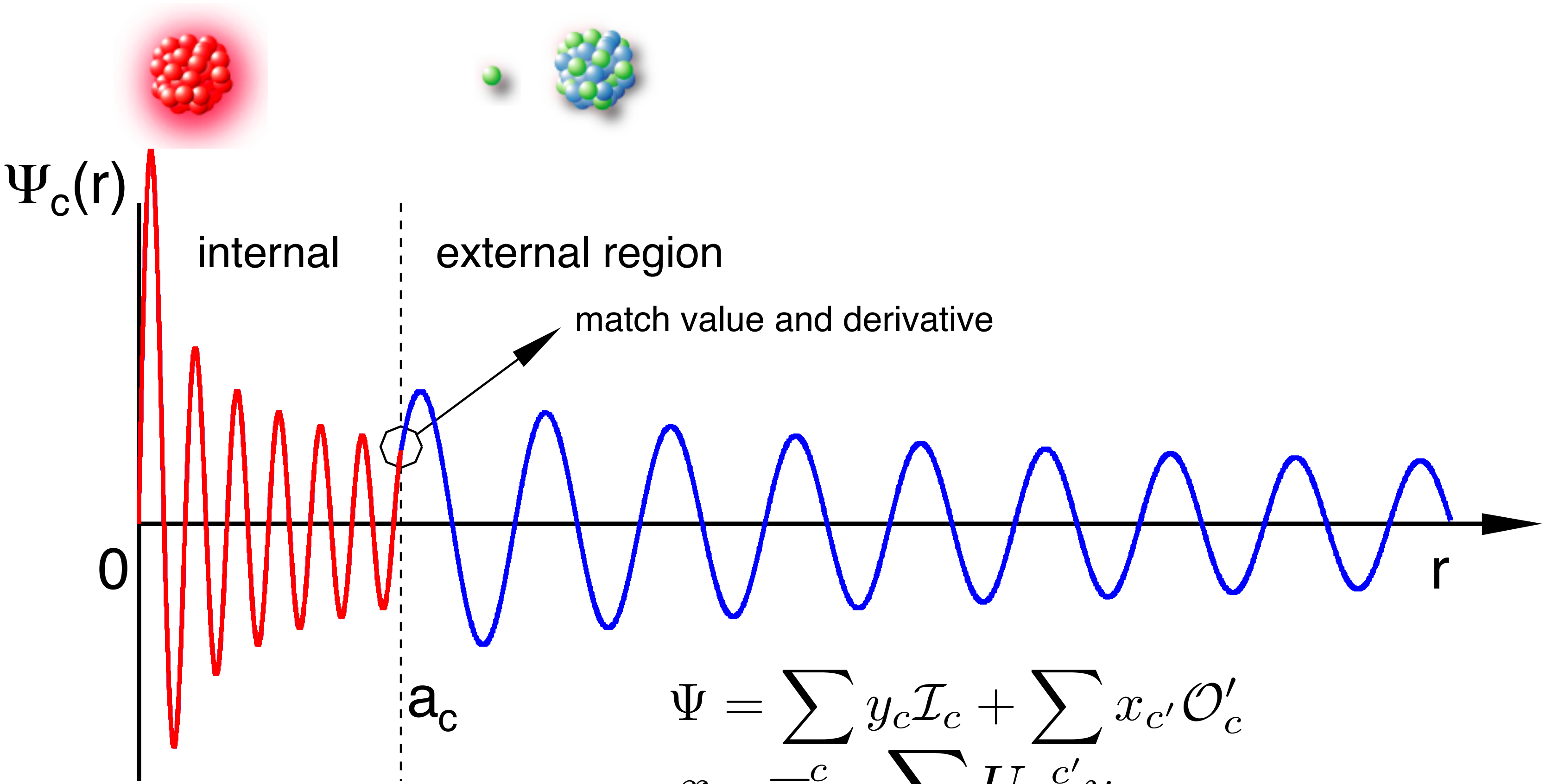
The R-matrix formalism



The R-matrix formalism



The R-matrix formalism



$$\Psi = \Psi(R_{cc'})$$

$$R_{cc'} = \sum_{\lambda} \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E}$$

$$\Psi = \sum y_c \mathcal{I}_c + \sum x_{c'} \mathcal{O}'_c$$

$$x_{c'} \equiv^{c} - \sum U_{c'c}^{c'} y_c$$

$$\mathcal{I}_c = I_c r^{-\epsilon 1} \varphi_c i^\ell Y^\ell_{m_\ell}(\theta,\phi)/\sqrt{v_c}$$

$$\mathcal{O}_c = O_c r^{-1} \varphi_c i^\ell Y^\ell_{m_\ell}(\theta,\phi)/\sqrt{v_c}$$

The R-matrix formalism

The wave function of the system is a superposition of incoming and outgoing waves:

$$\Psi = \sum_c y_c \mathcal{I}_c + \sum_{c'} x_{c'} \mathcal{O}'_c$$

Incoming and outgoing wavefunctions have form:

$$\mathcal{I}_c = I_c r^{-1} \varphi_c i^\ell Y^\ell_{m_\ell}(\theta,\phi)/\sqrt{v_c}$$

$$\mathcal{O}_c = O_c r^{-1} \varphi_c i^\ell Y^\ell_{m_\ell}(\theta,\phi)/\sqrt{v_c}$$

The physical interaction is included in the collision matrix **U**:

$$x_{c'} \equiv -\sum_c U_{c'c} y_c$$

The wave function depends on the R-matrix, which depends on the widths and levels of the eigenstates.

$$\Psi = \Psi(R_{cc'})$$

$$R_{cc'} = \sum_\lambda \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E}$$

The R-matrix formalism

The relation between the R-matrix and the collision matrix:

$$\mathbf{U} = \mathbf{\Omega} \mathbf{P}^{1/2} [\mathbf{1} - \mathbf{R}(\mathbf{L} - \mathbf{B})]^{-1} [\mathbf{1} - \mathbf{R}(\mathbf{L}^* - \mathbf{B})] \mathbf{P}^{-1/2} \mathbf{\Omega}$$

with: $L_c = S_c + iP_c = \left(\frac{\rho}{O_c} \frac{dO_c}{d\rho} \right)_{r=a_c}$

The relation between the collision matrix and cross sections:

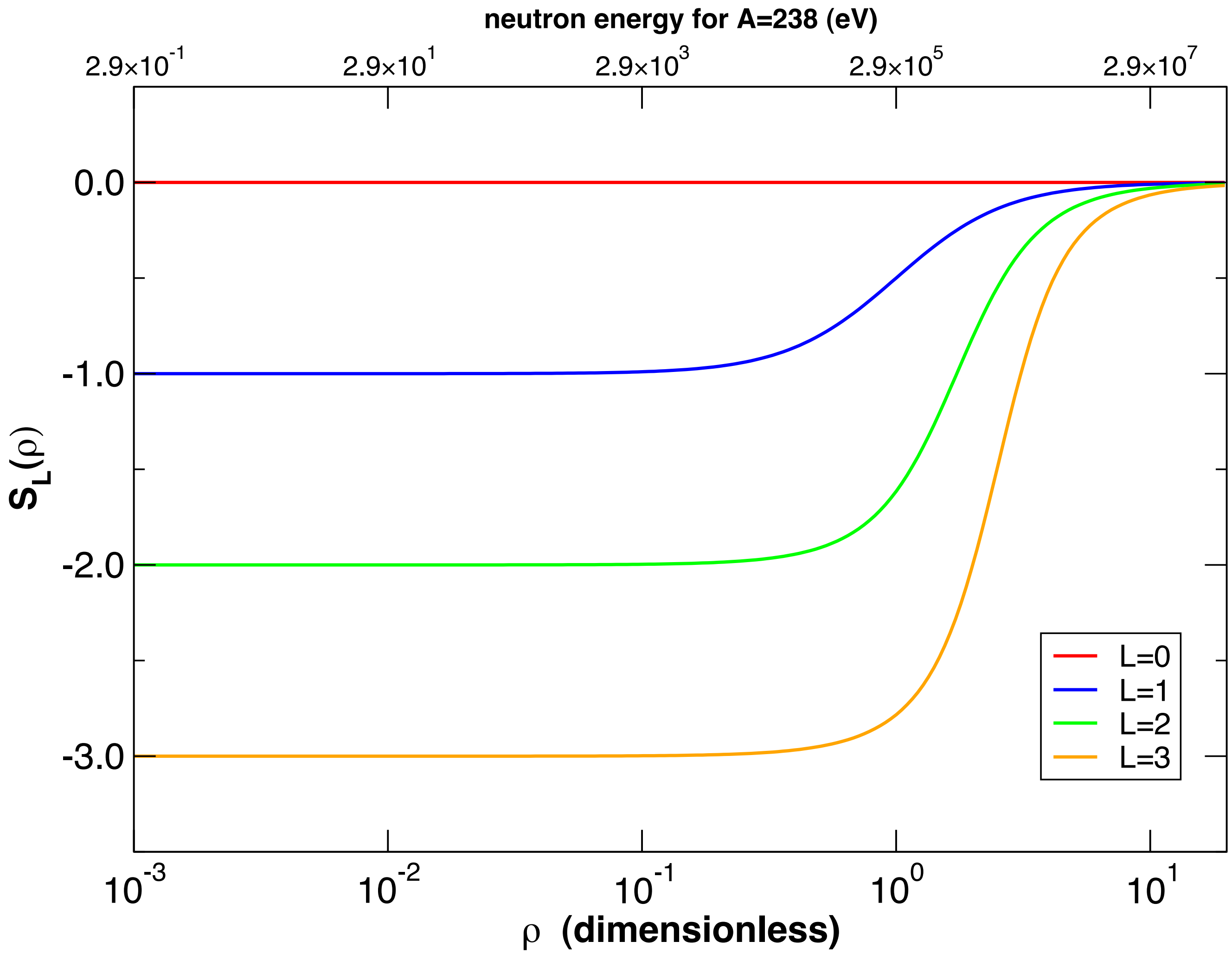
channel to one other channel: $\sigma_{cc'} = \pi \lambda_c^2 |\delta_{c'c} - U_{c'c}|^2$

channel to any other channel: $\sigma_{cr} = \pi \lambda_c^2 (1 - |U_{cc}|^2)$

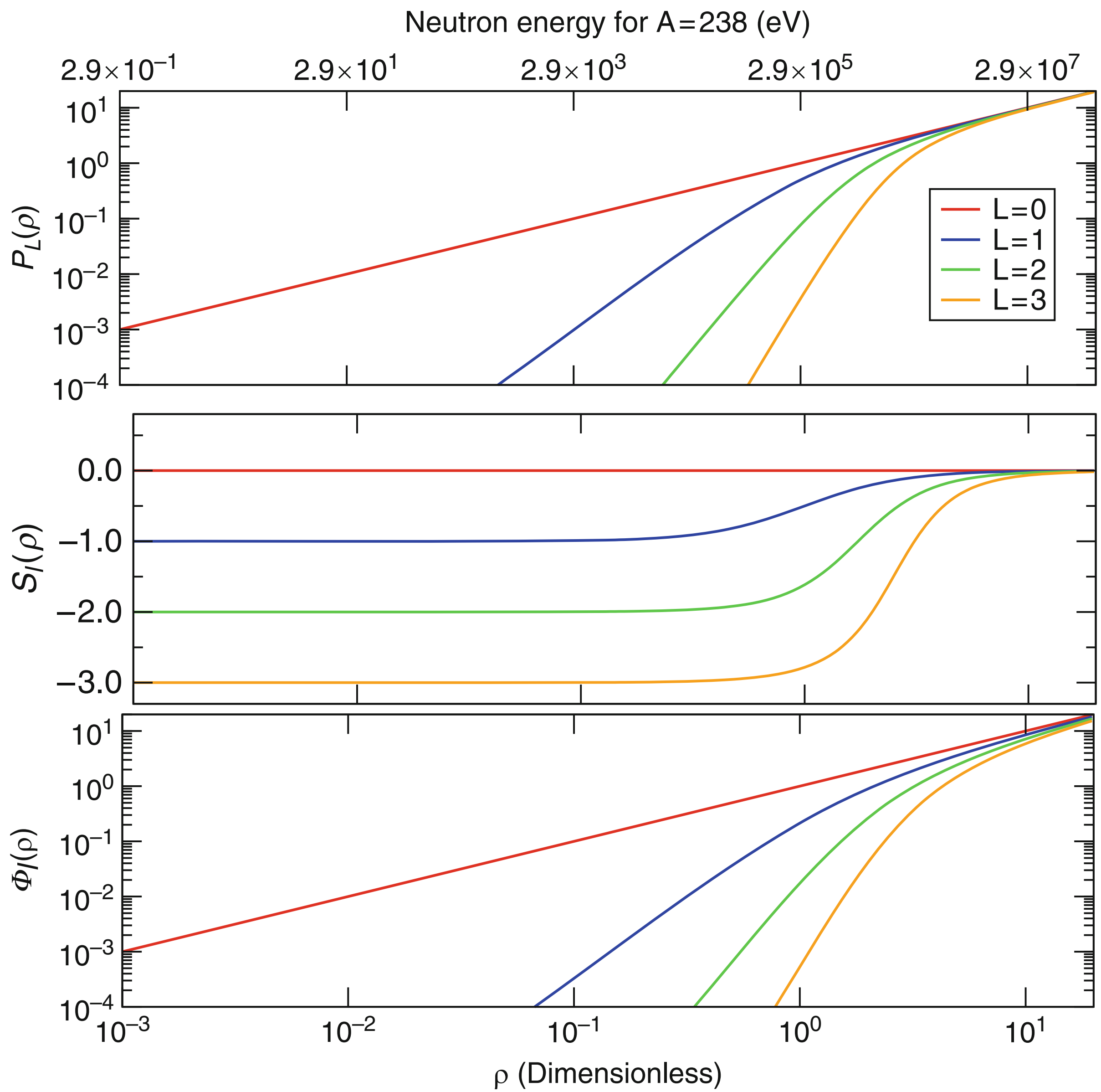
channel to same channel: $\sigma_{cc} = \pi \lambda_c^2 |1 - U_{cc}|^2$

channel to any channel (total): $\sigma_{c,T} = \sigma_c = 2\pi \lambda_c^2 (1 - \mathrm{Re}\, U_{cc})$

The R-matrix formalism



The R-matrix formalism



The R-matrix formalism

The Breit-Wigner Single Level approximation:

total cross section:

$$\sigma_c = \pi \bar{\lambda}_c^2 g_c \left(4\sin^2\phi_c + \frac{\Gamma_\lambda \Gamma_{\lambda c} \cos 2\phi_c + 2(E - E_\lambda - \Delta_\lambda)\Gamma_{\lambda c} \sin 2\phi_c}{(E - E_\lambda - \Delta_\lambda)^2 + \Gamma_\lambda^2/4} \right)$$

neutron channel: $c = n$

only capture, scattering, fission: $\Gamma_\lambda = \Gamma = \Gamma_n + \Gamma_\gamma + \Gamma_f$

other approximations: $\ell = 0 \quad \cos\phi_c = 1 \quad \sin\phi_c = \rho = ka_c \quad \Delta_\lambda = 0$

total cross section:

$$\sigma_T(E) = \overbrace{4\pi R'^2}^{\text{potential}} + \pi \bar{\lambda}^2 g \left(\frac{\overbrace{4\Gamma_n (E - E_0) R'/\bar{\lambda}}^{\text{interference}} + \overbrace{\Gamma_n^2}^{\text{elastic}} + \overbrace{\Gamma_n \Gamma_\gamma}^{\text{capture}} + \overbrace{\Gamma_n \Gamma_f}^{\text{fission}}}{(E - E_0)^2 + (\underbrace{\Gamma_n + \Gamma_\gamma + \Gamma_f +}_{\text{total width}})^2/4} \right)$$

The R-matrix formalism

The Reich-Moore approximation:

Use the fact that there are many photon channels, with Gaussian distributed amplitudes with zero mean:

$$< \gamma_{\lambda c}\gamma_{\mu c} >= \gamma^2_{\lambda c}\delta_{\lambda\mu}$$

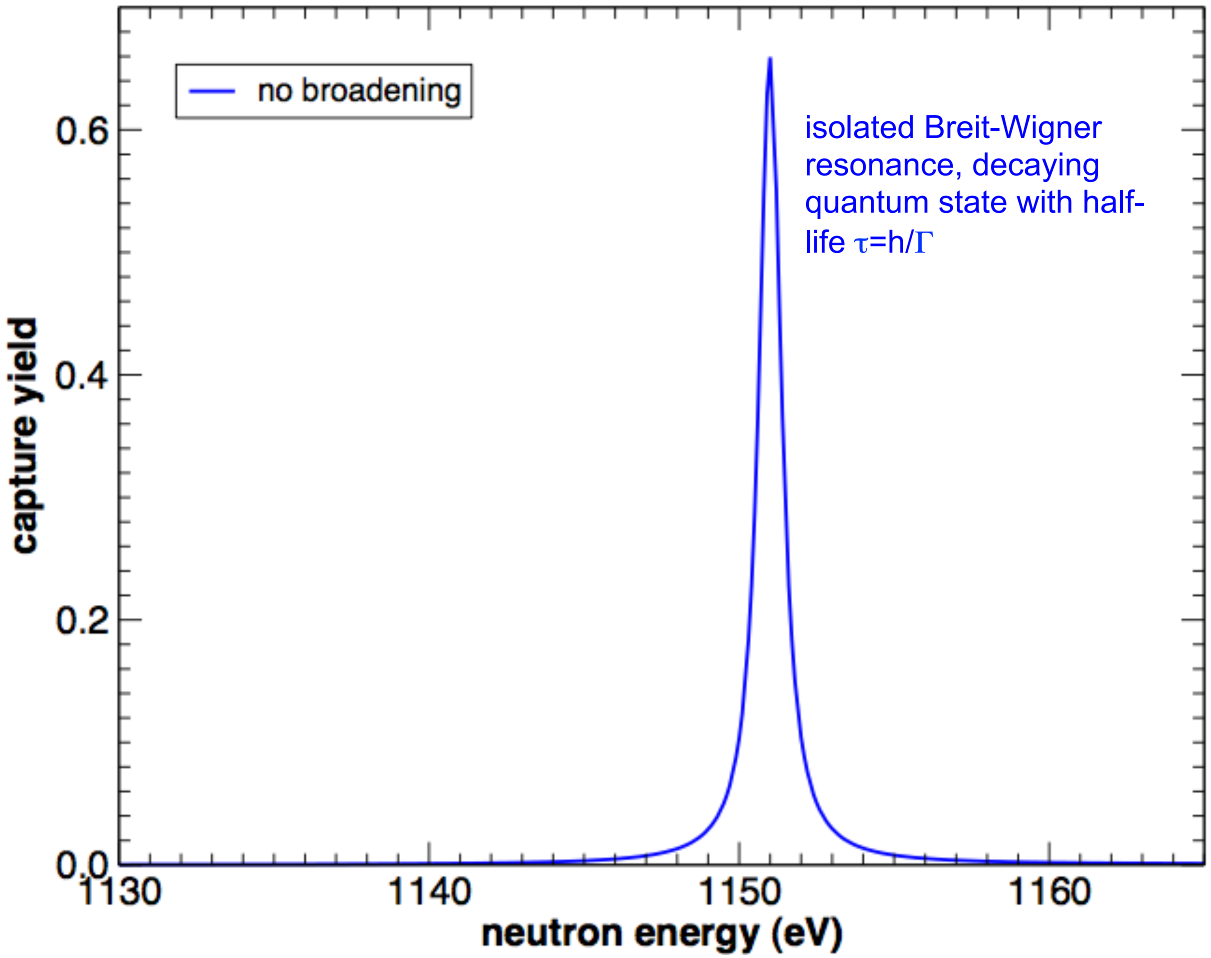
The sum over the amplitudes of the photon channels becomes then:

$$\sum_{c\in\text{photon}} \gamma_{\lambda c}\gamma_{\mu c} = \sum_{c\in\text{photon}} \gamma^2_{\lambda c}\delta_{\lambda\mu} = \Gamma_{\lambda\gamma}\delta_{\lambda\mu}$$

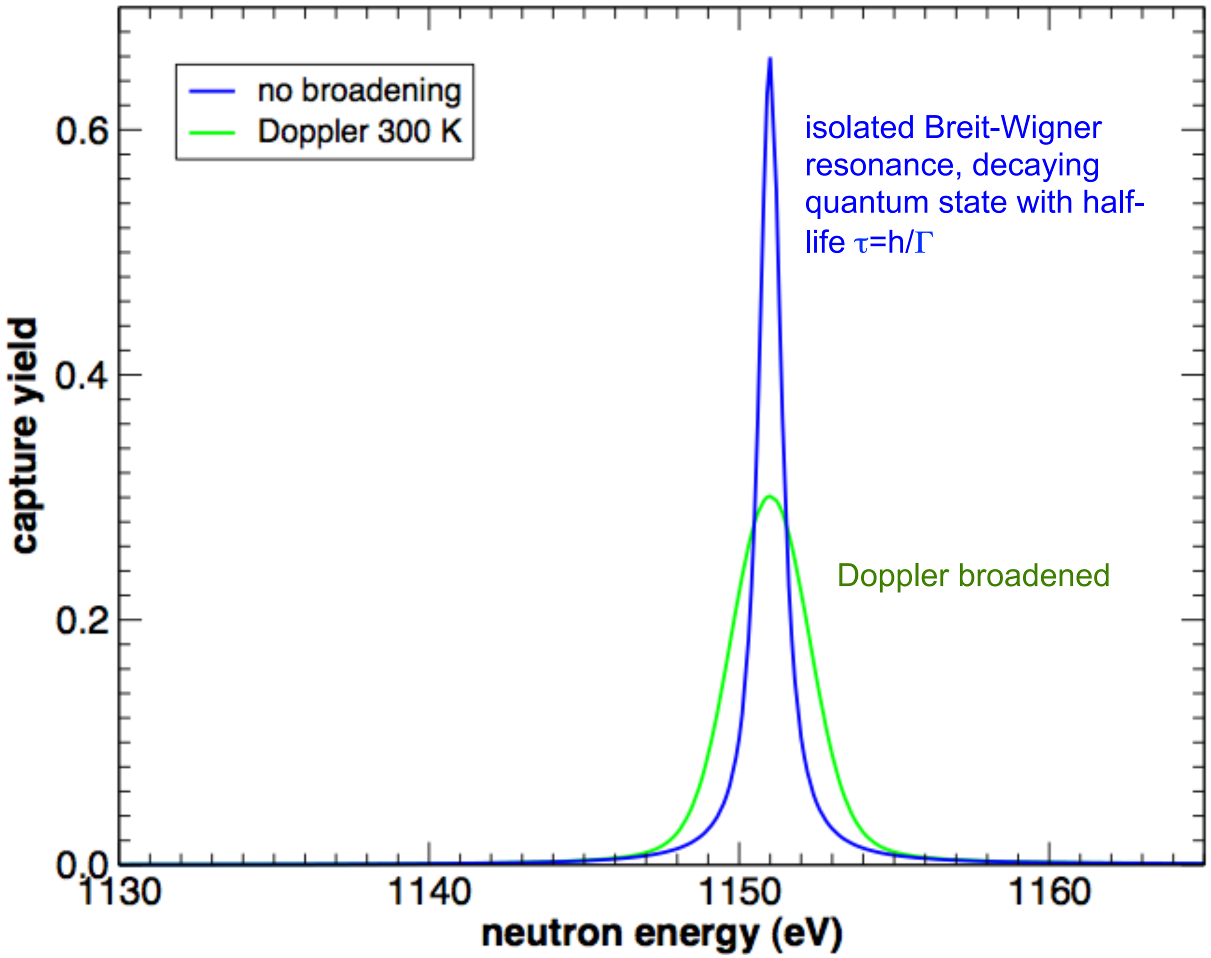
Then photon channels can be eliminated in the R-matrix:

$$R_{cc'} = \sum_{\lambda} \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E - i\Gamma_{\lambda\gamma}/2} \qquad c \notin \text{photon}$$

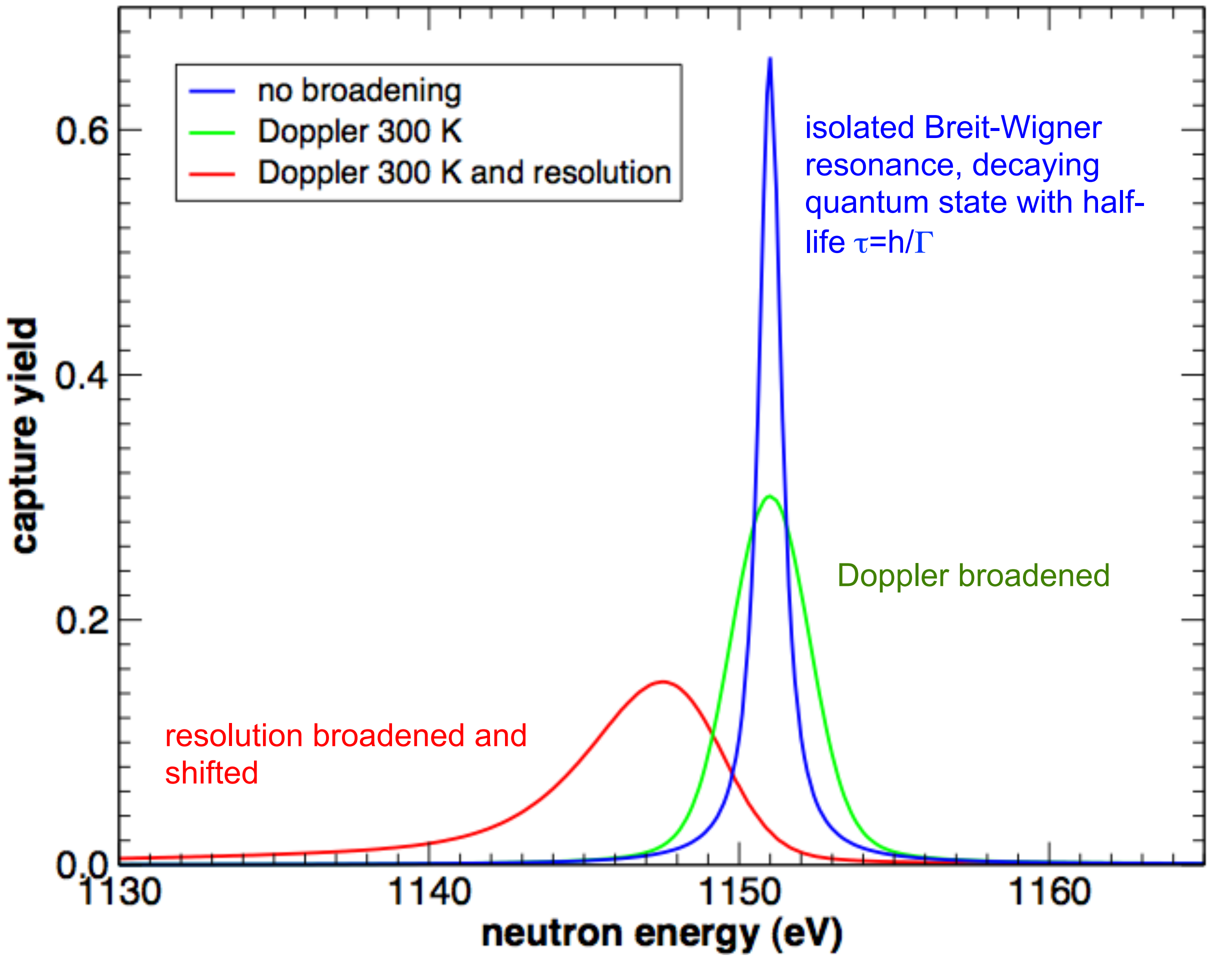
Measured reaction yield



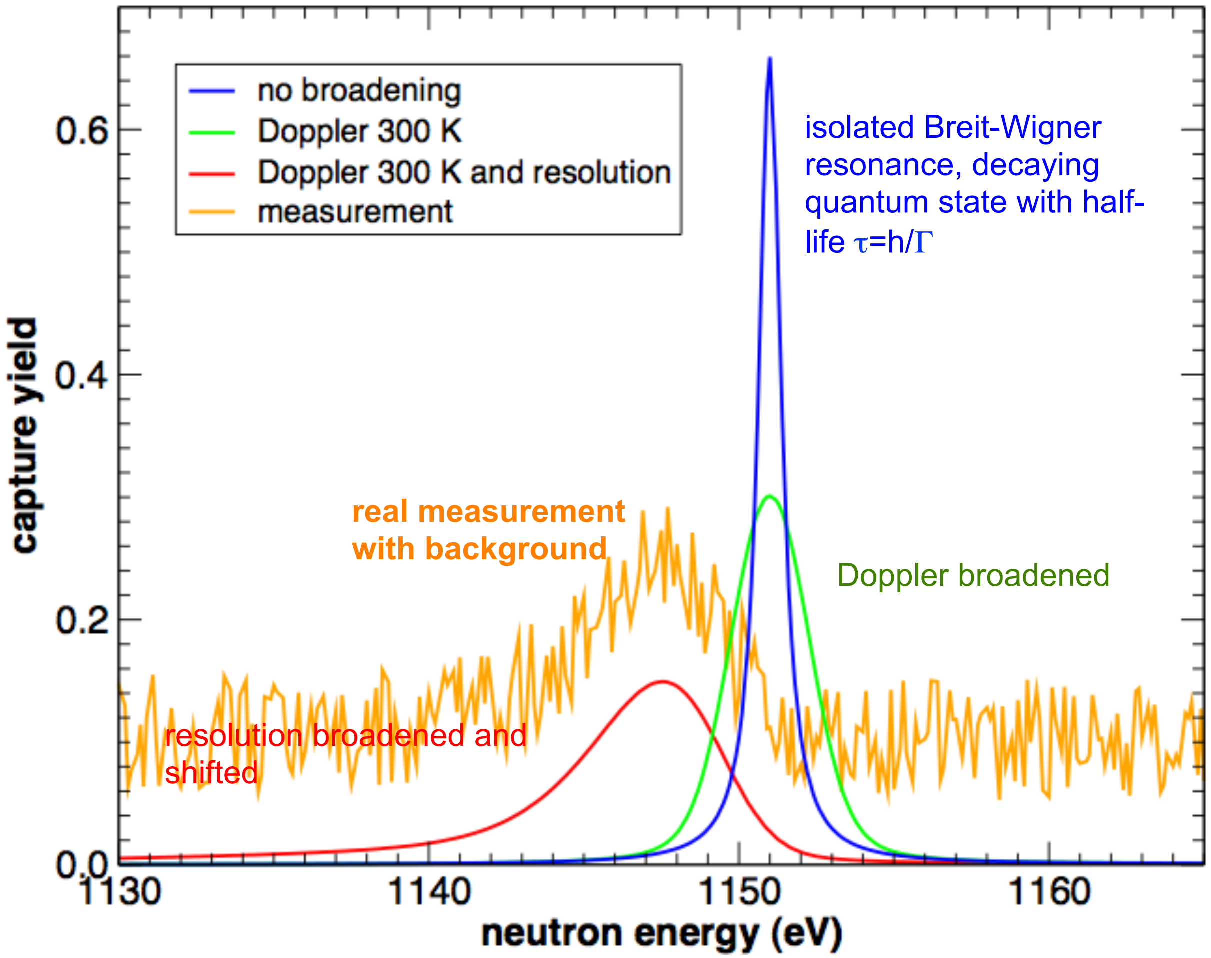
Measured reaction yield



Measured reaction yield



Measured reaction yield



Measured quantities for resolved resonances

- **Experimental quantities are not cross sections but reaction yields and transmission factors**

reaction yield:
$$Y(E_n) = \mu(E_n)\left(1 - e^{-n\sigma_T(E_n)}\right) \cdot \frac{\sigma_\gamma(E_n)}{\sigma_T(E_n)}$$

transmission:
$$T(E_n) = e^{-n\sigma_T(E_n)}$$

- **Cross sections are functions of the resonance parameters**

cross section:
$$\sigma_{cr} = \pi \lambda\!\!\!^{-}{}_c^2 g_c (1 - |U_{cc}|^2)$$

$$\sigma = \sigma(\{E_r, J^\pi, \Gamma, \Gamma_r\}, \ldots)$$

Measured quantities for unresolved resonances

- **Experimental quantities are average yields and average transmission factors**

reaction yield: $$\langle Y \rangle = \left\langle \mu (1 - e^{-n\sigma_T}) \frac{\sigma_\gamma}{\sigma_T} \right\rangle$$

transmission: $$\langle T \rangle = \left\langle e^{-n\sigma} \right\rangle = e^{-n\langle \sigma \rangle} \cdot \left\langle e^{-n(\sigma - \langle \sigma \rangle)} \right\rangle$$

Measured quantities for unresolved resonances

- **Experimental quantities are average yields and average transmission factors**

reaction yield: $$\langle Y \rangle = \left\langle \mu(1 - e^{-n\sigma_T}) \frac{\sigma_\gamma}{\sigma_T} \right\rangle = f_r \times n \times < \sigma_\gamma >$$

transmission: $$\langle T \rangle = \left\langle e^{-n\sigma_T} \right\rangle = e^{-n\langle \sigma_T \rangle} \cdot \left\langle e^{-n(\sigma_T - \langle \sigma_T \rangle)} \right\rangle =$$

$$f_T \times e^{-n\langle \sigma_T \rangle}$$

Measured quantities for unresolved resonances

- **Experimental quantities are average yields and average transmission factors**

reaction yield: $$\langle Y \rangle = \left\langle \mu(1 - e^{-n\sigma_T}) \frac{\sigma_\gamma}{\sigma_T} \right\rangle = f_r \times n \times < \sigma_\gamma >$$

transmission: $$\langle T \rangle = \left\langle e^{-n\sigma_T} \right\rangle = e^{-n\langle \sigma_T \rangle} \cdot \left\langle e^{-n(\sigma_T - \langle \sigma_T \rangle)} \right\rangle = f_T \times e^{-n\langle \sigma_T \rangle}$$

- **change of parameters describing the cross section**

resolved unresolved parameters

$$E, J^\pi \rightarrow \rho_\ell \quad \text{or} \quad D_\ell$$

$$\Gamma_\gamma \rightarrow \langle \Gamma_\gamma \rangle$$

$$g\Gamma_n^\ell \rightarrow \langle g\Gamma_n^\ell \rangle = (2\ell + 1) S_\ell D_\ell$$

Measured quantities for unresolved resonances

- **Experimental quantities are average yields and average transmission factors**

reaction yield: $$\langle Y \rangle = \left\langle \mu(1 - e^{-n\sigma_T})\frac{\sigma_\gamma}{\sigma_T} \right\rangle = f_r \times n \times < \sigma_\gamma >$$

transmission: $$\langle T \rangle = \left\langle e^{-n\sigma_T} \right\rangle = e^{-n\langle \sigma_T \rangle} \cdot \left\langle e^{-n(\sigma_T - \langle \sigma_T \rangle)} \right\rangle = f_T \times e^{-n\langle \sigma_T \rangle}$$

- **change of parameters describing the cross section**

resolved → unresolved parameters

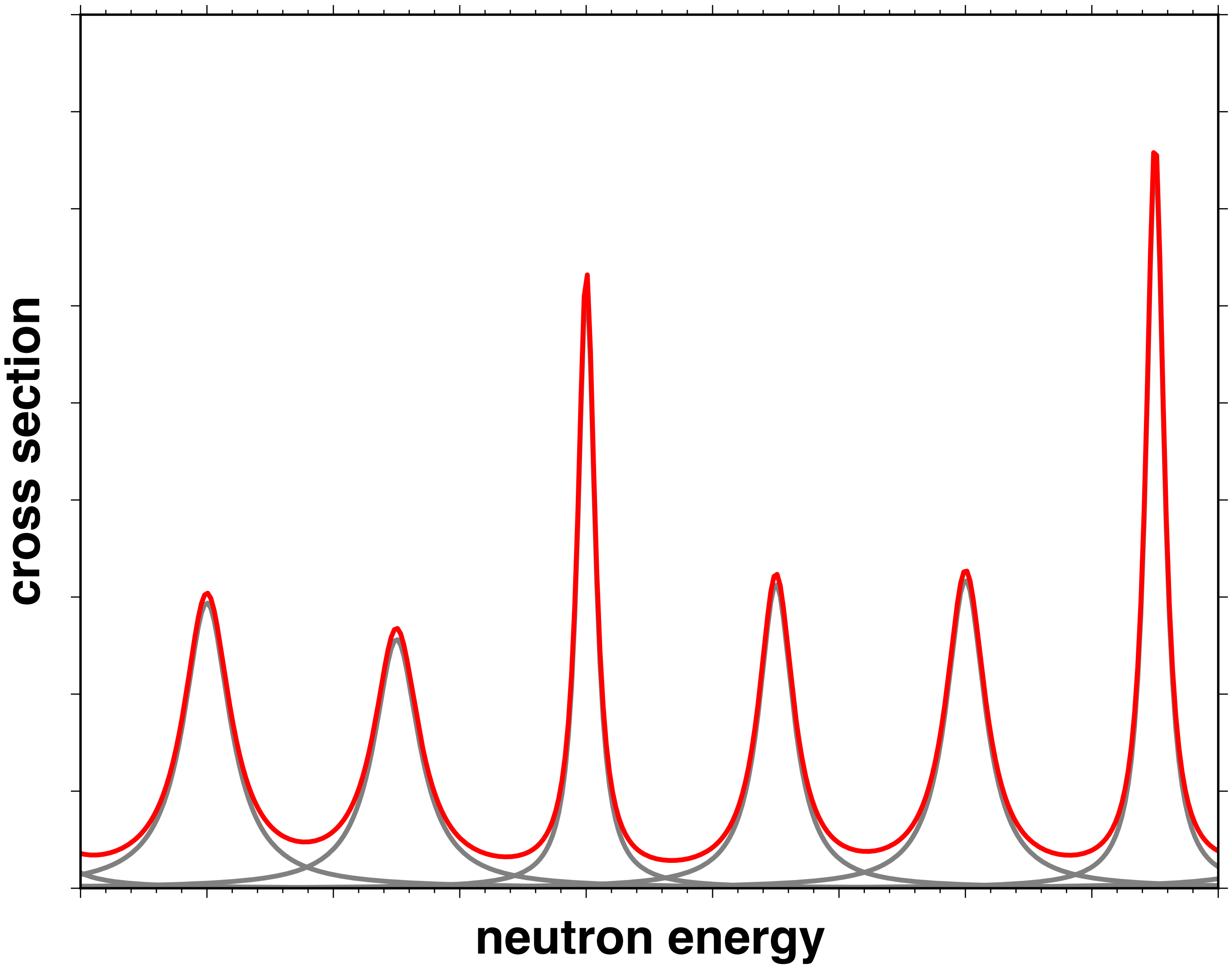
$$E, J^\pi \rightarrow \rho_\ell \quad \text{or} \quad D_\ell$$

$$\Gamma_\gamma \rightarrow \langle \Gamma_\gamma \rangle$$

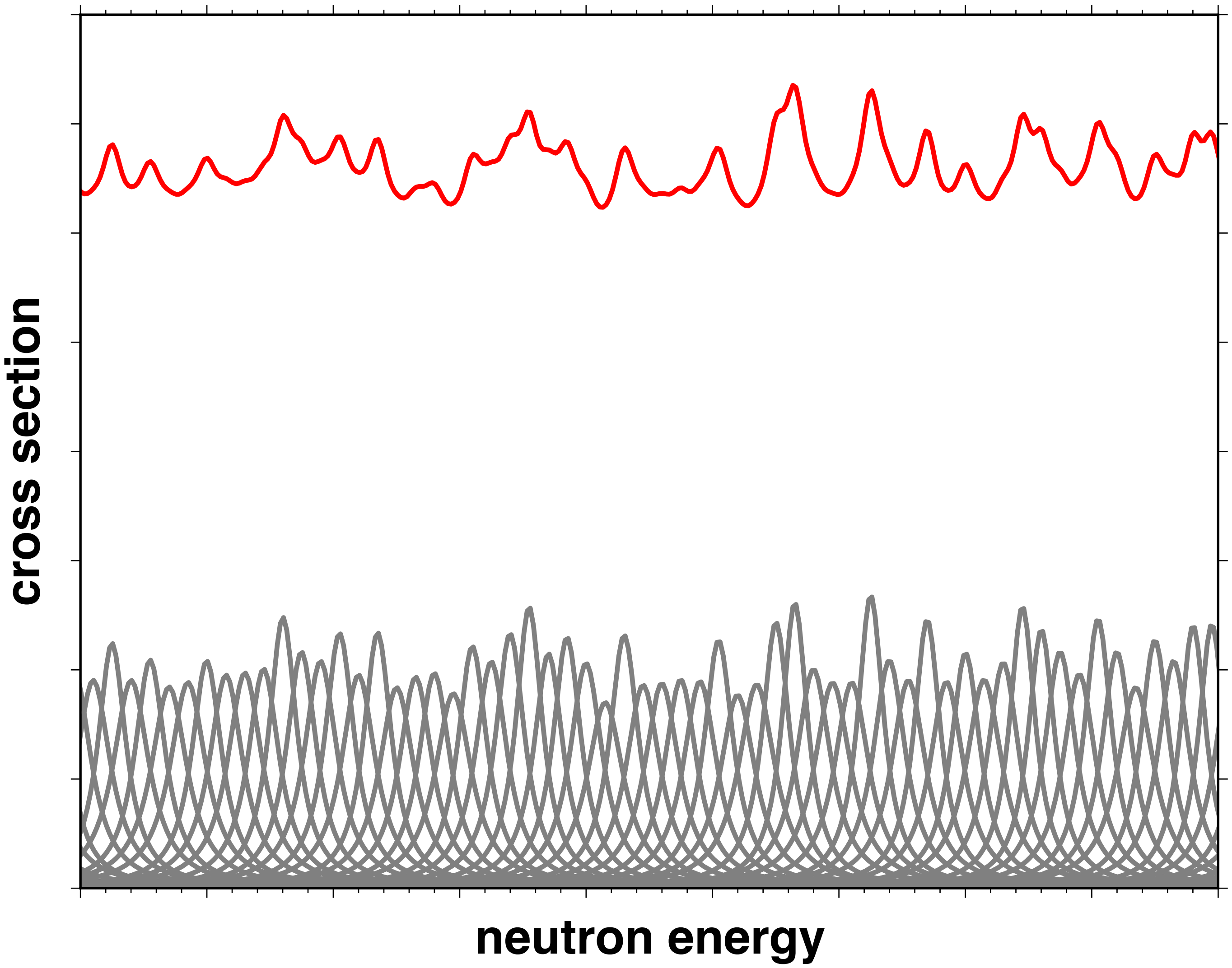
$$g\Gamma_n^\ell \rightarrow \langle g\Gamma_n^\ell \rangle = (2\ell + 1) S_\ell D_\ell$$

Neutron strength function: S_ℓ

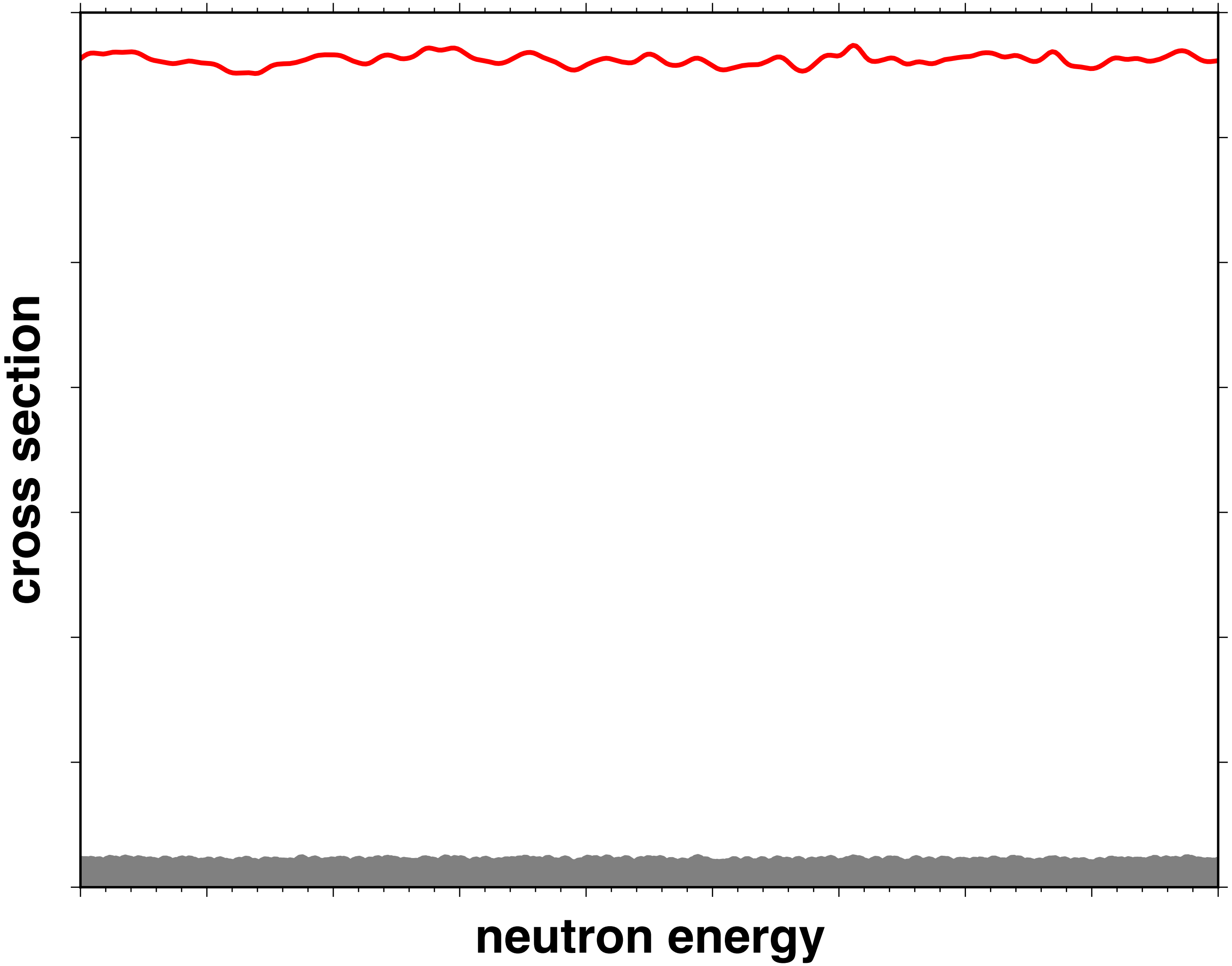
resolved/unresolved resonances



resolved/unresolved resonances



resolved/unresolved resonances



Average cross sections

The relation between the energy averaged collision matrix and energy averaged cross sections:

average scattering:
$$\overline{\sigma_{cc}} = \pi\lambda_c^2 g_c \overline{|1 - U_{cc}|^2}$$

shape elastic (potential)
$$\overline{\sigma_{cc}^{\mathrm{se}}} = \pi\lambda_c^2 g_c |1 - \overline{U_{cc}}|^2$$

compound elastic
$$\overline{\sigma_{cc}^{\mathrm{ce}}} = \pi\lambda_c^2 g_c \left(\overline{|U_{cc}|^2} - |\overline{U_{cc}}|^2\right)$$

average any reaction
$$\overline{\sigma_{cr}} = \pi\lambda_c^2 g_c (1 - \overline{|U_{cc}|^2})$$

average total
$$\overline{\sigma_{c,T}} = 2\pi\lambda_c^2 g_c (1 - \mathrm{Re}\,\overline{U_{cc}})$$

average single reaction
$$\overline{\sigma_{cc'}} = \pi\lambda_c^2 g_c \overline{|\delta_{cc'} - U_{cc'}|^2}$$

average compound nucleus formation
$$\overline{\sigma_c} = \pi\lambda_c^2 g_c (1 - |\overline{U_{cc}}|^2)$$

Average cross sections

- From optical model calculations one can calculate $\overline{U_{cc}}$ but not $\overline{|U_{cc}|^2}$

- Therefore, only $\overline{\sigma_{c,T}}$, $\overline{\sigma_{cc}^{\mathrm{se}}}$, $\overline{\sigma_c}$ can be calculated, of which only the total average cross section can be compared directly with measurements.

- In OMP one uses transmission coefficients $T_c = 1 - |\overline{U_{cc}}|^2$

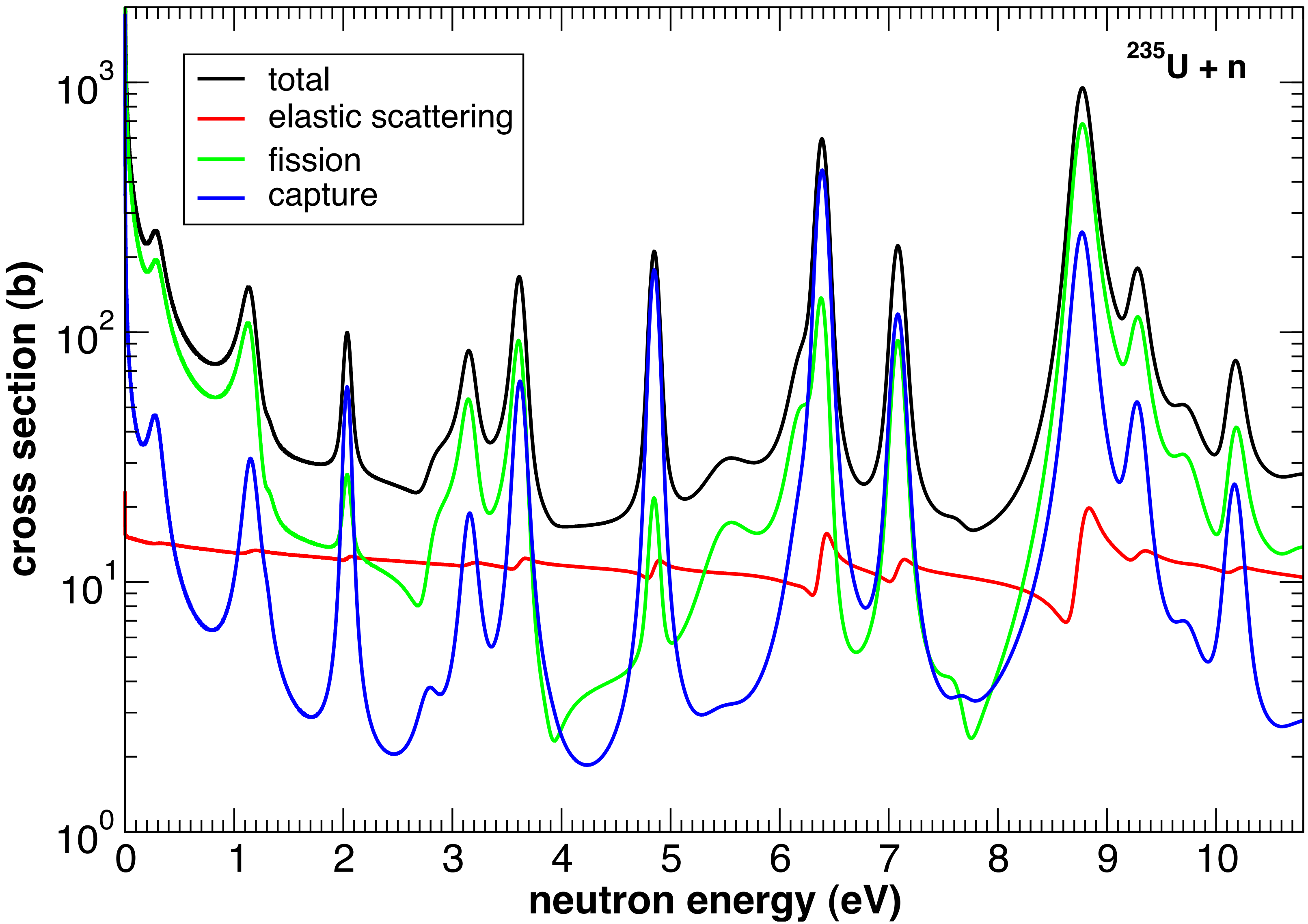
- Average single reaction cross section (Hauser-Feshbach):

$$\overline{\sigma_{cc'}} = \overline{\sigma_{cc}^{\mathrm{se}}}\delta_{cc'} + \pi\lambda_c^2 g_c \frac{T_c T_{c'}}{\Sigma T_i} W_{cc'}$$

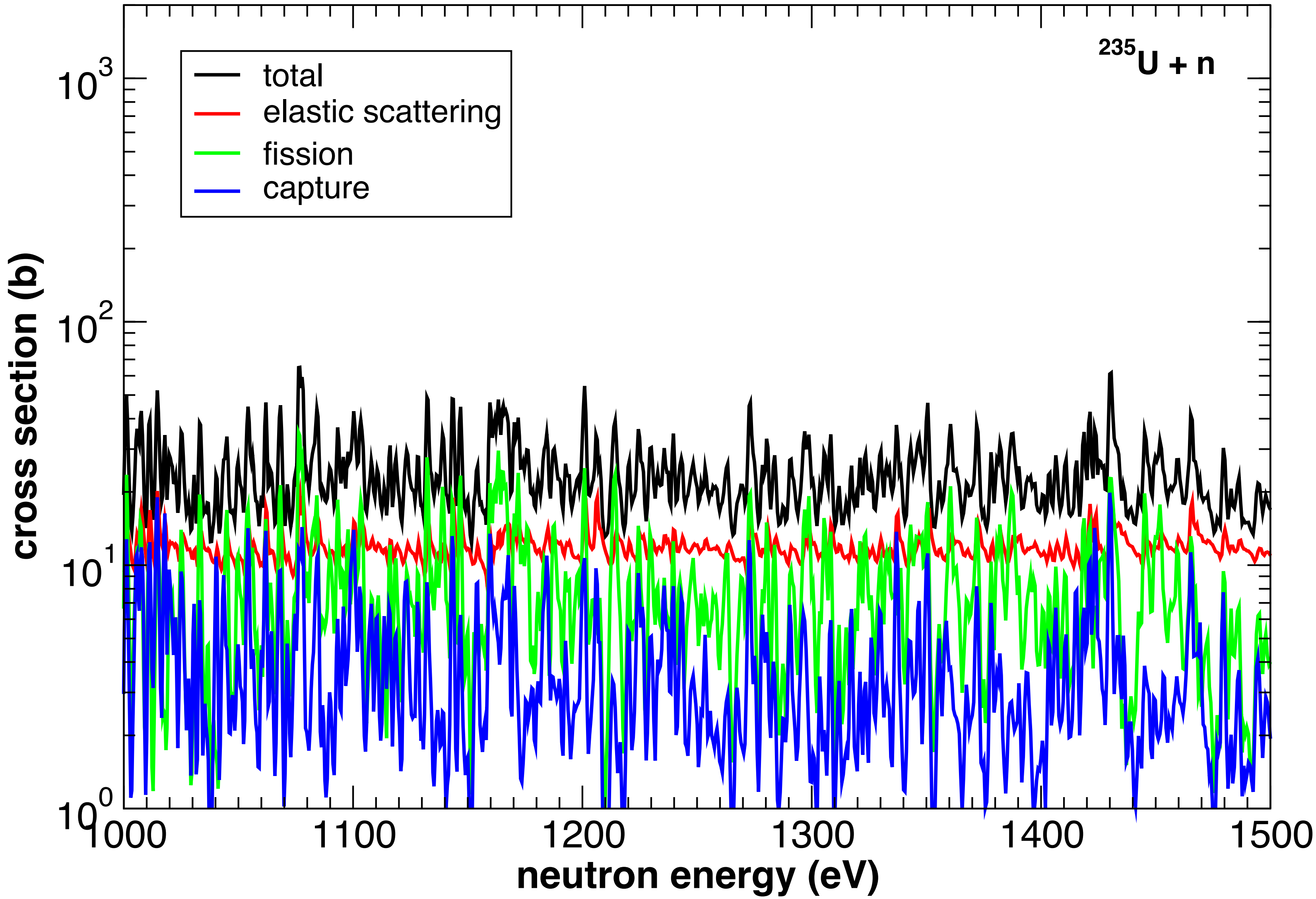
related to average parameters: $T_c = 2\pi\overline{\Gamma_c}/D$

width fluctuations: $W_{cc'} = \overline{\left(\frac{\Gamma_c \Gamma_{c'}}{\Gamma}\right)} \frac{\overline{\Gamma}}{\overline{\Gamma_c}\,\overline{\Gamma_{c'}}}$

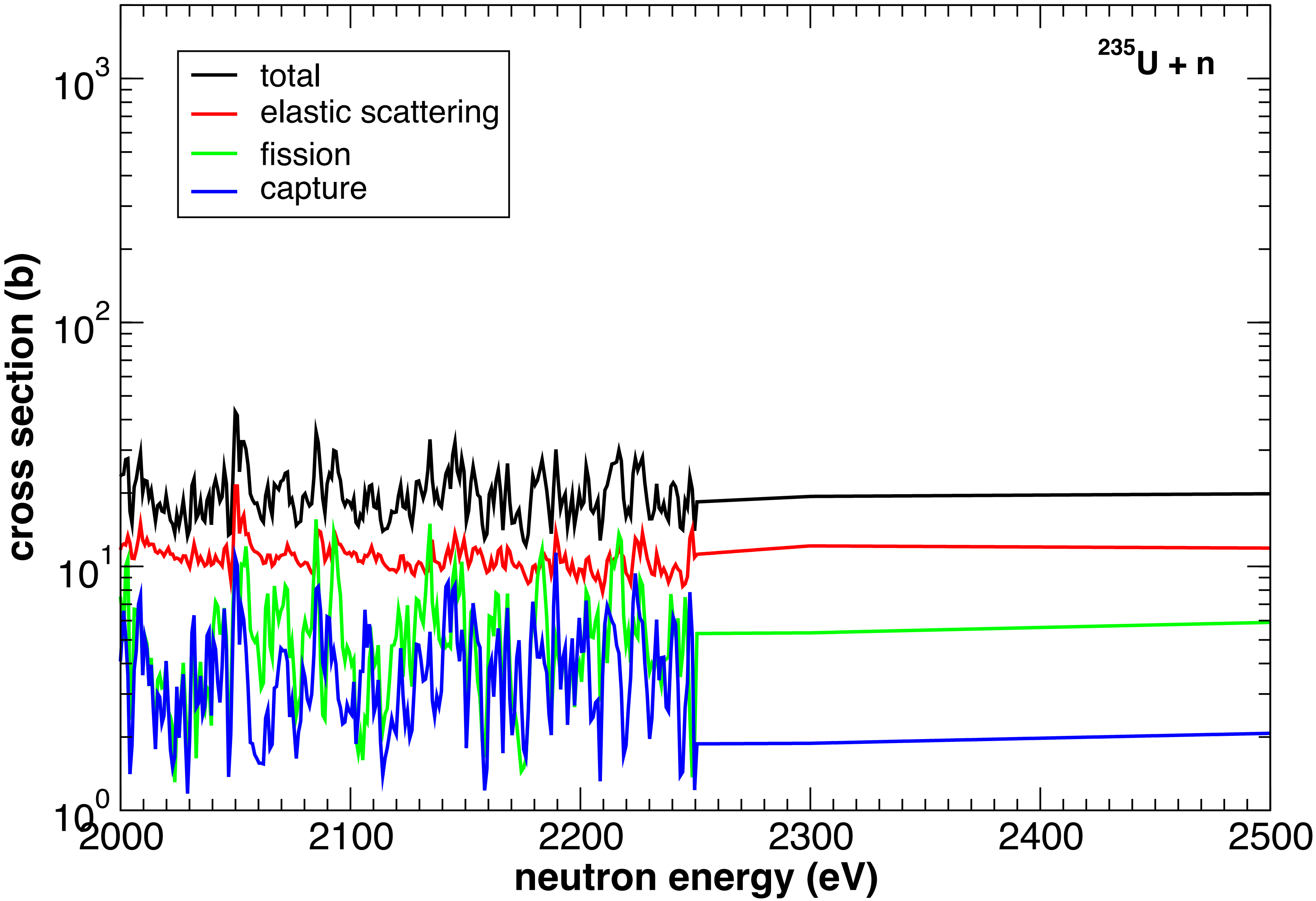
Cross sections σ_T, σ_γ, σ_n et σ_f



Cross sections σ_T, σ_γ, σ_n et σ_f

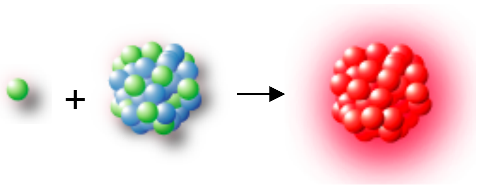


Cross sections σ_T, σ_γ, σ_n et σ_f

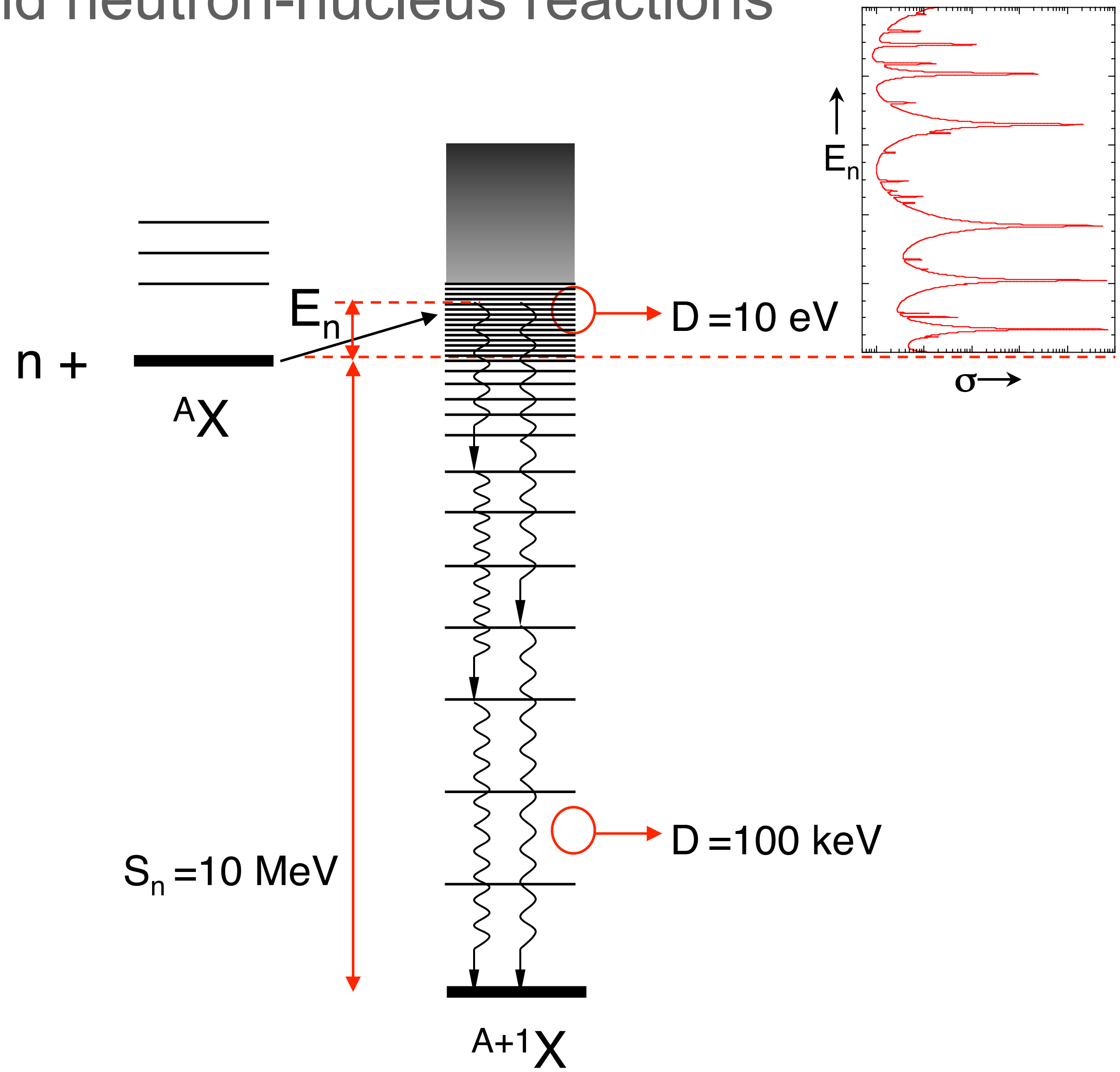


Compound neutron-nucleus reactions

compound
nucleus reaction



$$\tau \sim 10^{-16}$$
$$E_n < 10 \text{ MeV}$$



Orbital momentum

- orbtial momentum of incoming neutron relative to nucleus: ℓ
- Resonance spin and parity:

$$\mathbf{J} = \mathbf{I} + \mathbf{1/2} + \ell$$

$$\pi = \pi_i \times (-1)^\ell$$

- partial waves:

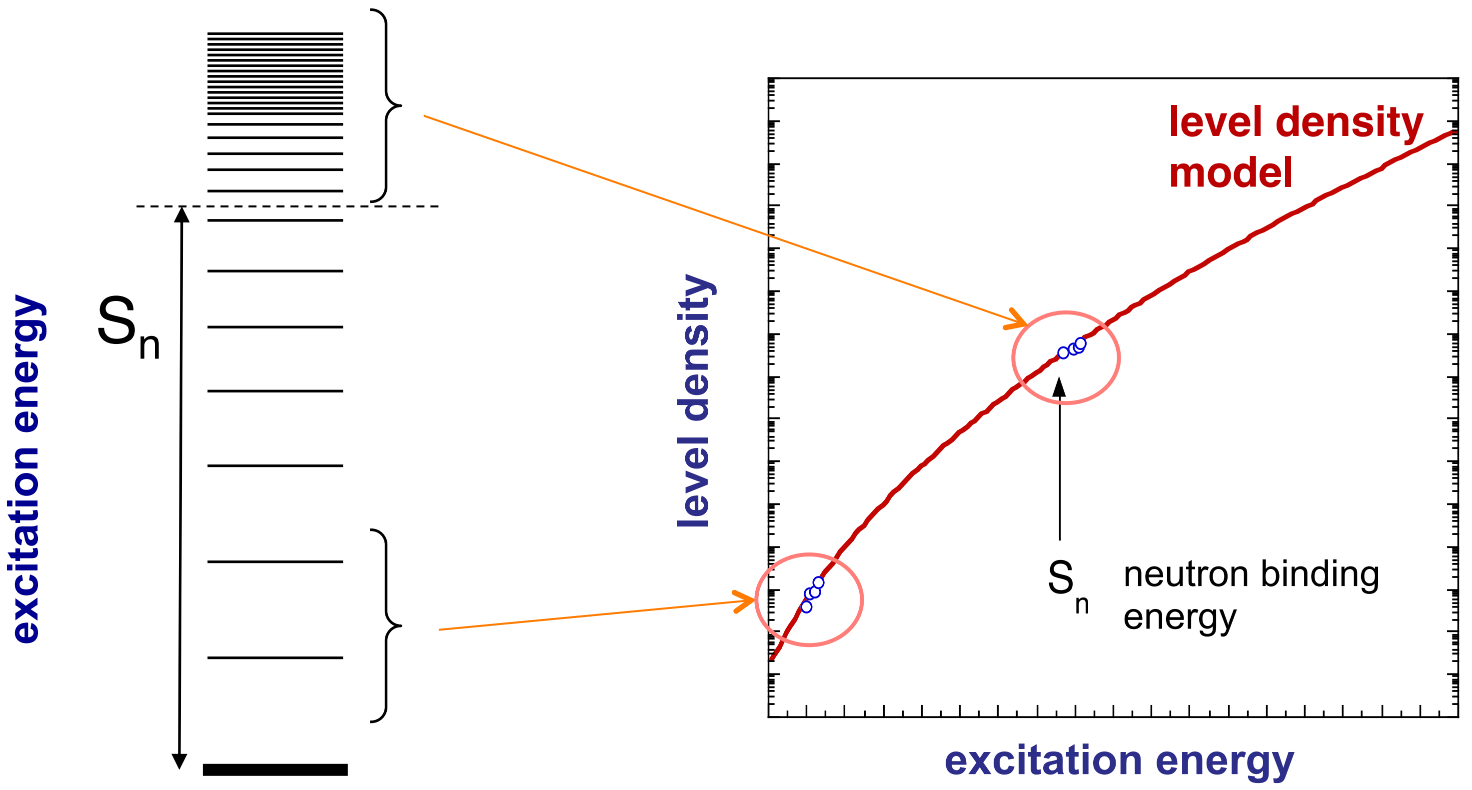
s-wave $\ell = 0$

p-wave $\ell = 1$

d-wave $\ell = 2$

f-wave $\ell = 3$

Nuclear level densities

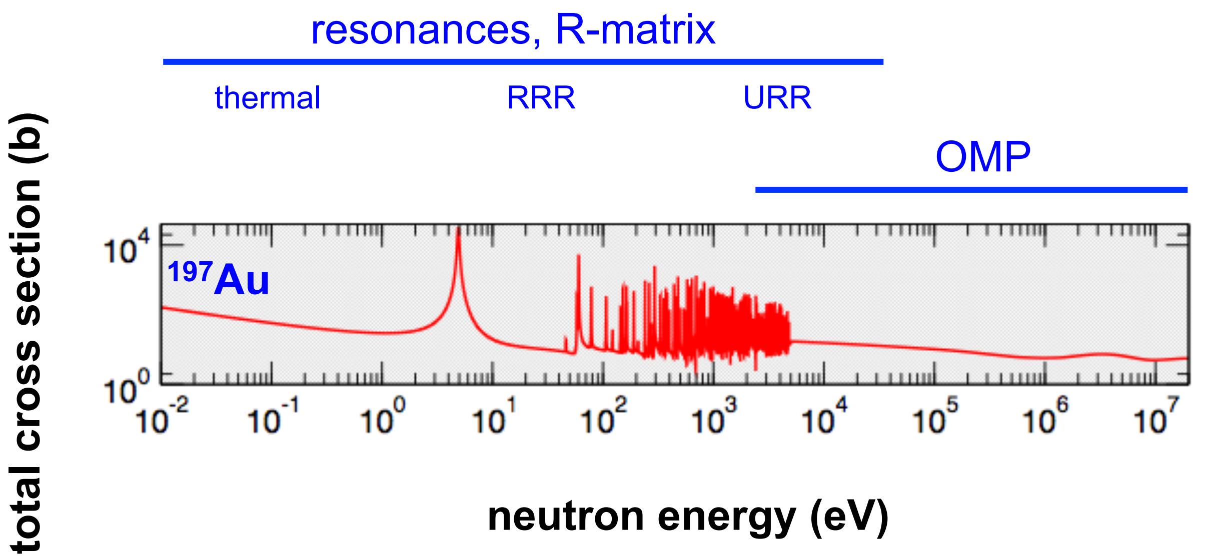


low-lying levels:
Count levels, all J^{π}

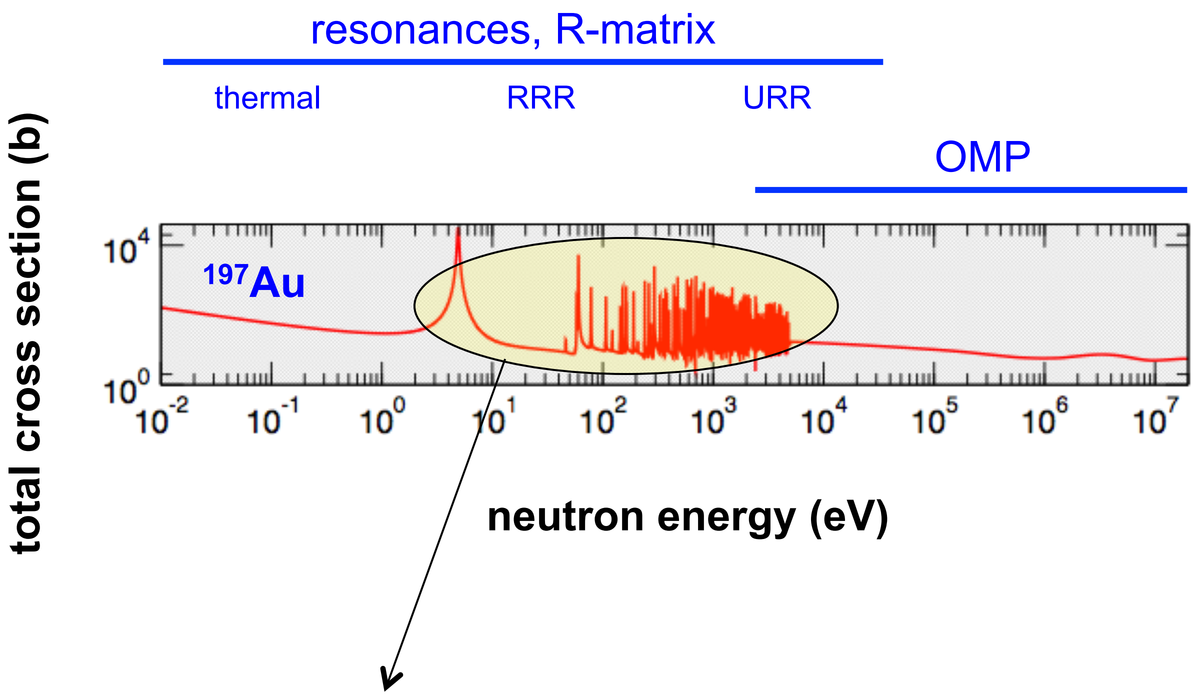
neutron resonances:
Count levels, selected J^{π},
extract $\mathbf{D_0}$

- **All level density models reproduce the low-lying levels and D_0 at S_n**

Compound nucleus reactions

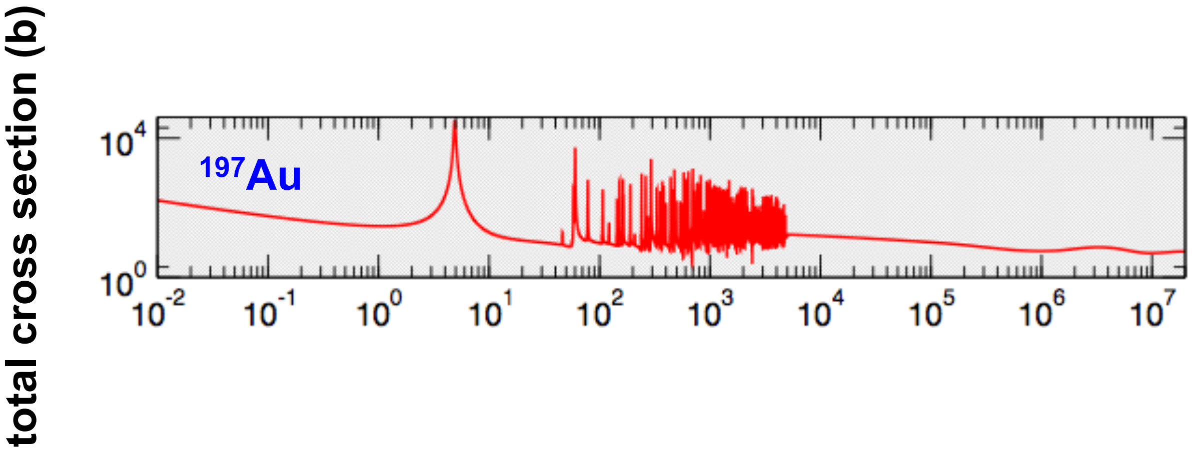


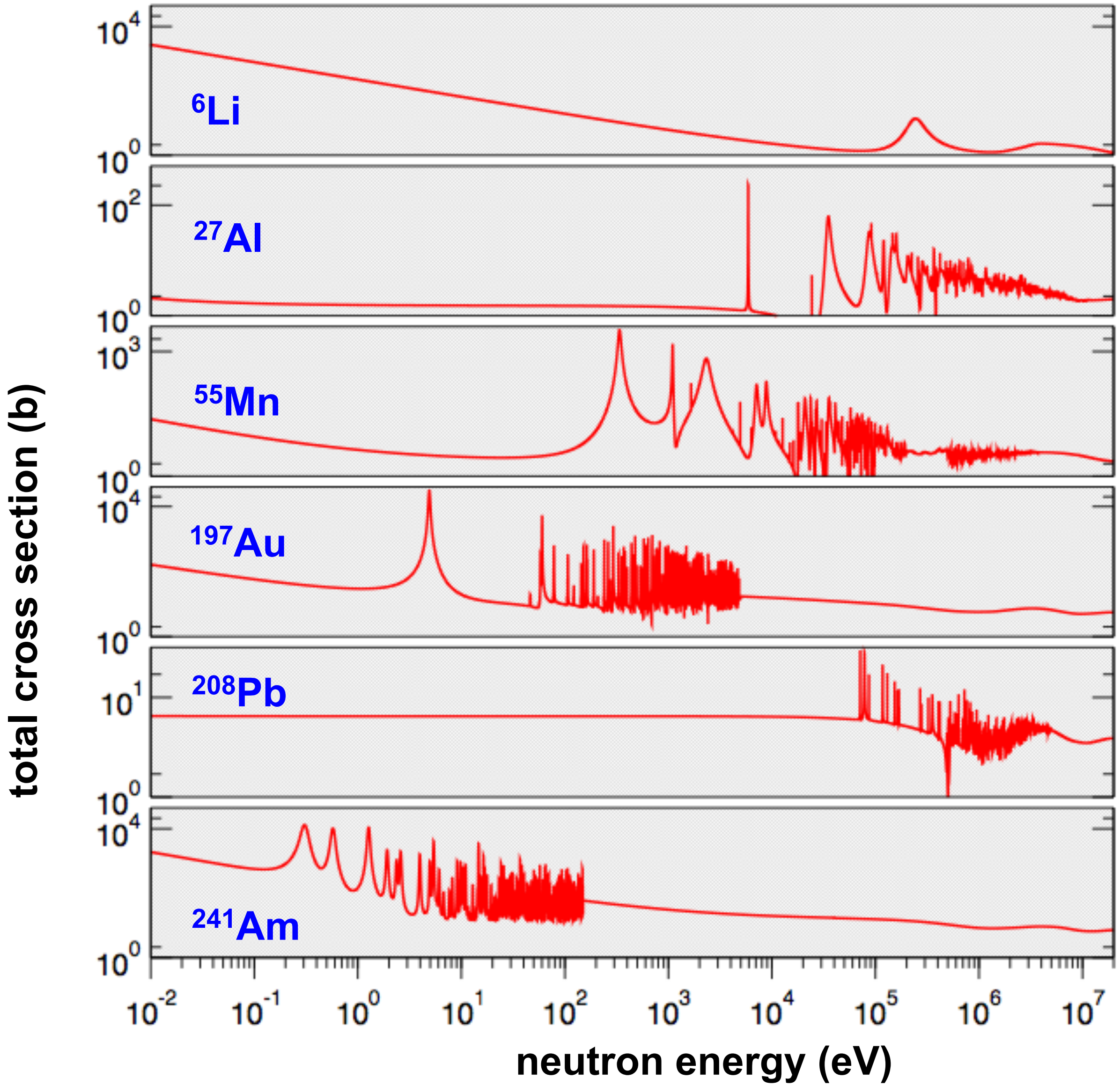
Compound nucleus reactions



Count the number of levels in the energy interval → level density

Compound nucleus reactions





total cross section (b)
neutron energy (eV)
^{6}Li
^{27}Al
^{55}Mn
^{197}Au
^{208}Pb
^{241}Am

Level densities: the level spacing D_0

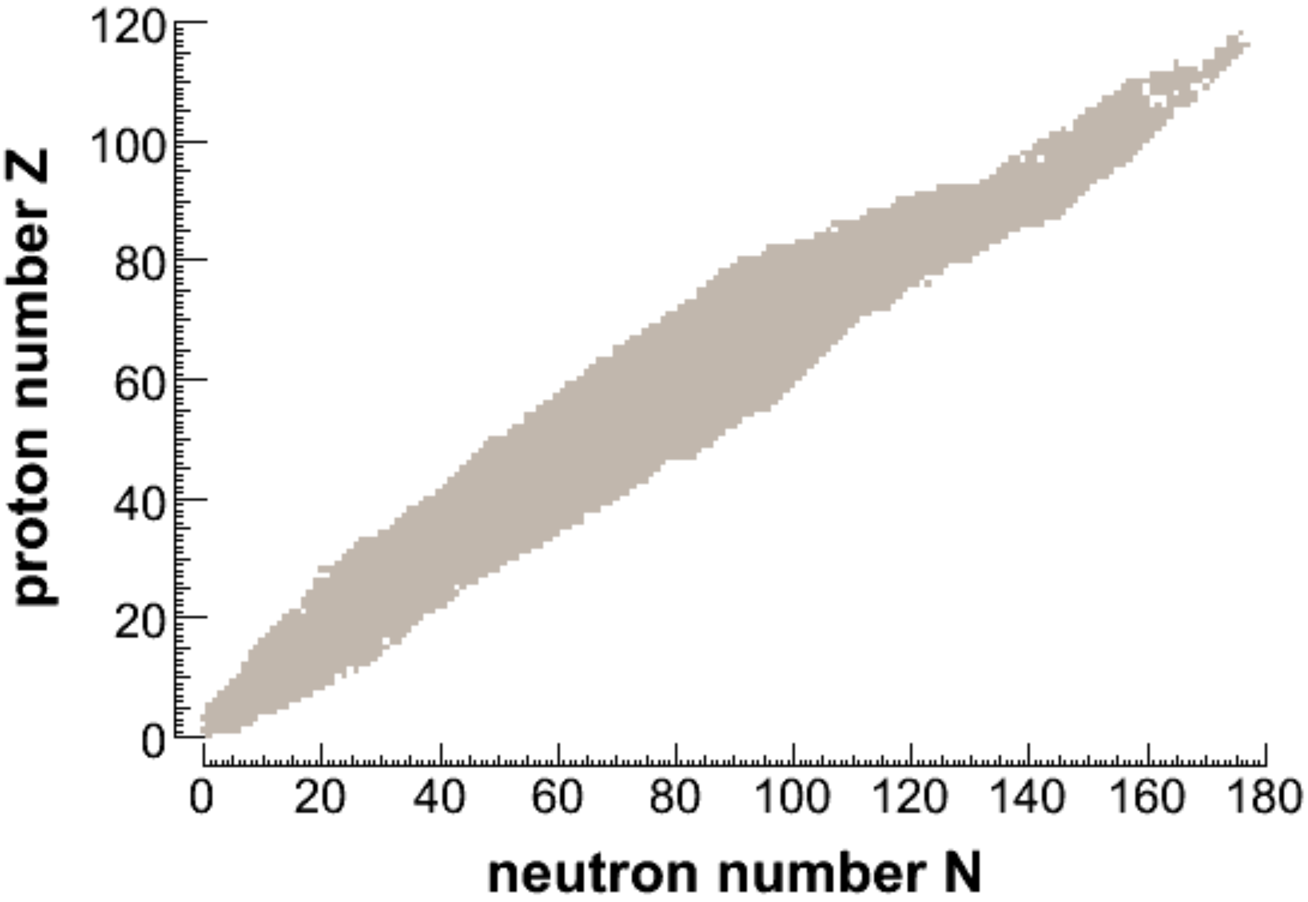
- The level spacing D_0 at the neutron binding energy is a crucial input parameter for calibrating level density models. Level density: $\rho = 1/D$.

- D_0 is the spacing between levels excited by neutrons on nuclei bringing in zero orbital momentum (s-wave resonances).

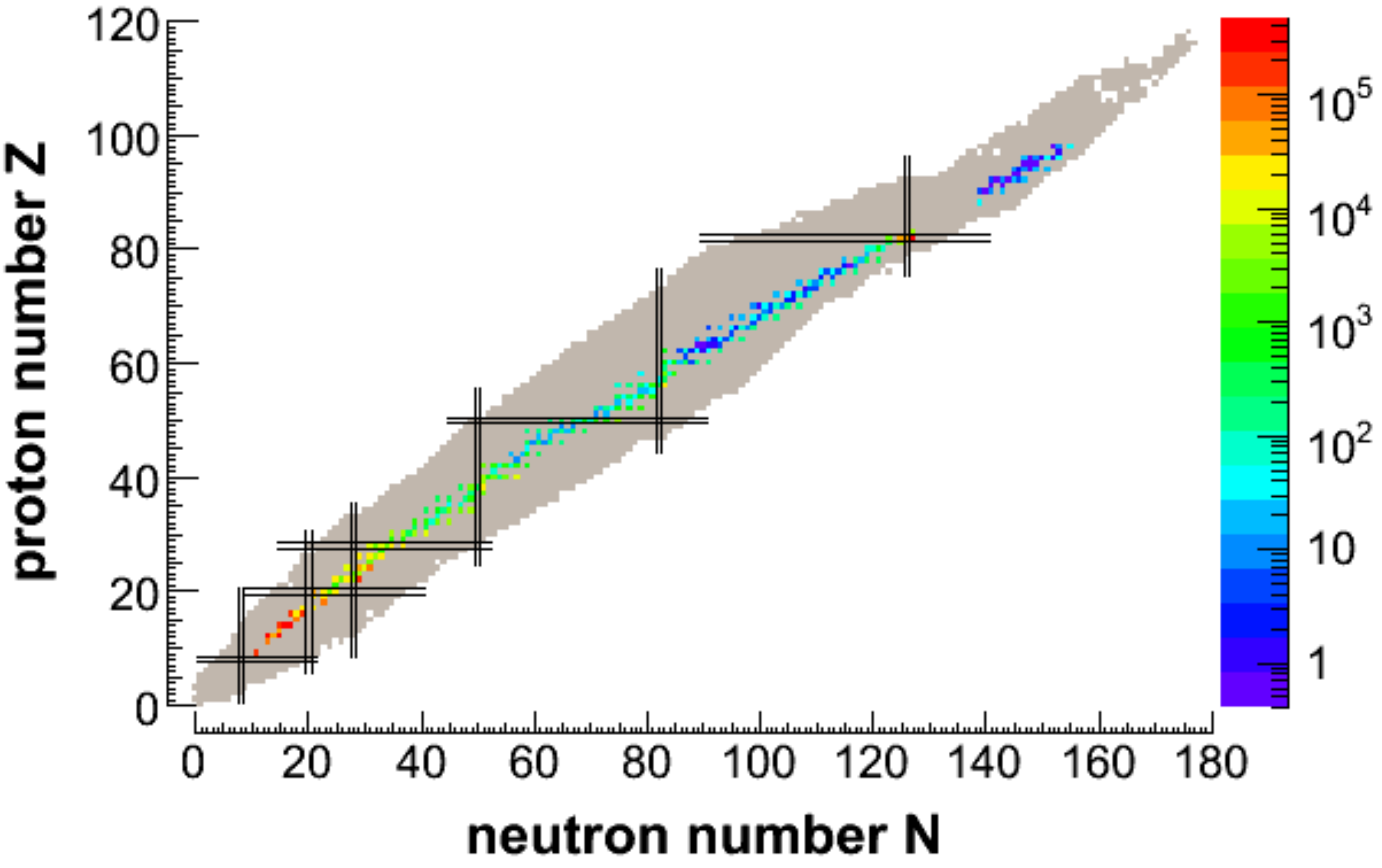
- Spacings from higher orbital momentum are equally important, but in general much more affected by missing levels.

- Problems concerning the determination of D_0:
  - spin and parity assignment of levels
  - corrections for missing levels (which are not observed experimentally)

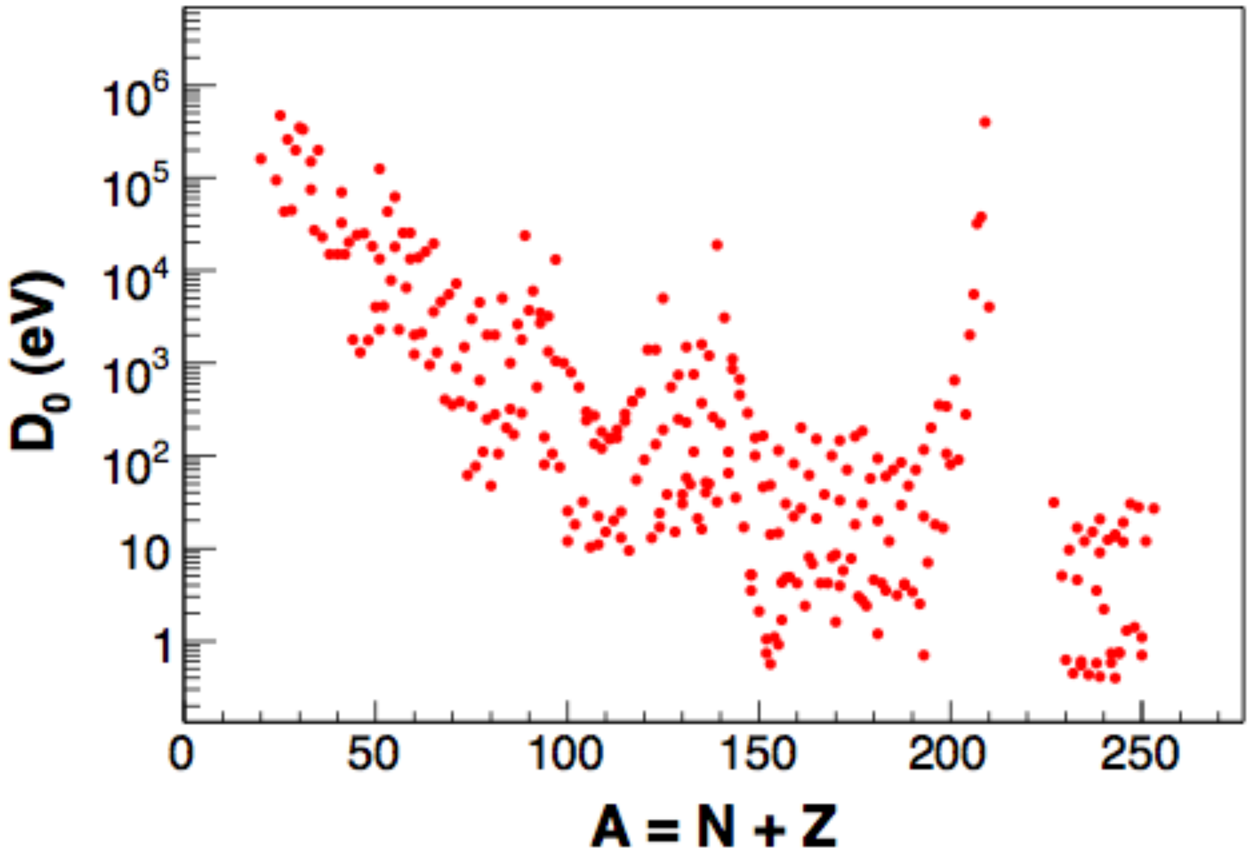
Level spacing D_0



Level spacing D_0



Level spacing D_0



Level density basics

Level density definition: $$\rho(U,J,\pi) = \frac{\partial N(U,J,\pi)}{\partial E}$$

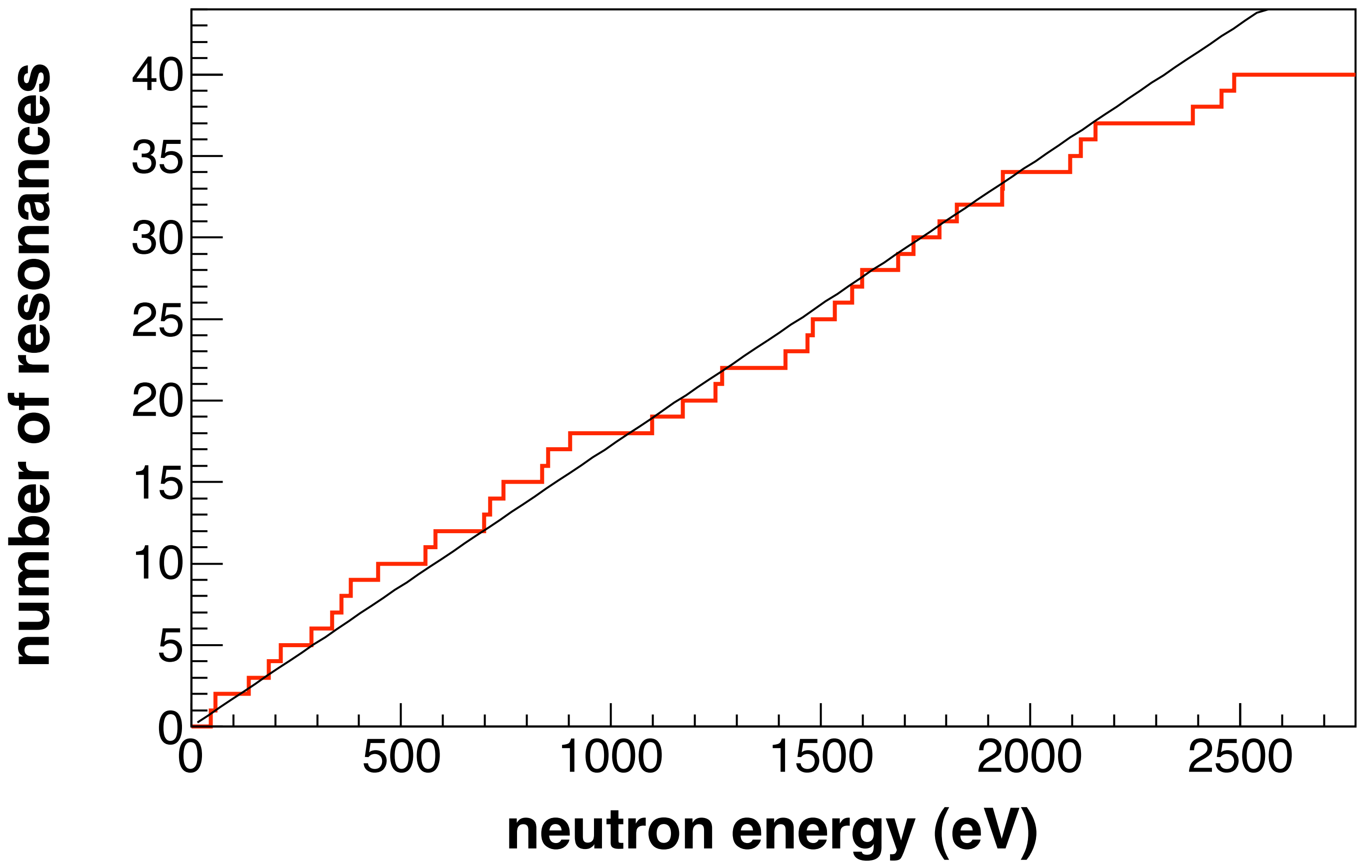
Simplify, use factorization: $$\rho(U,J,\pi) = \rho_U(U) \times \rho_J(J) \times \rho_\pi(\pi)$$

- parity distribution: $$\rho(\pi^+) = \rho(\pi^-) = \frac{1}{2}$$
- spin distribution: $$\rho(J) = \exp\left(-\frac{J^2}{2\sigma_c^2}\right) - \exp\left(-\frac{(J+1)^2}{2\sigma_c^2}\right)$$
- energy distribution: (constant temperature) $$\rho(U) = \frac{1}{T}\exp\left(-\frac{U-U_0}{T}\right)$$

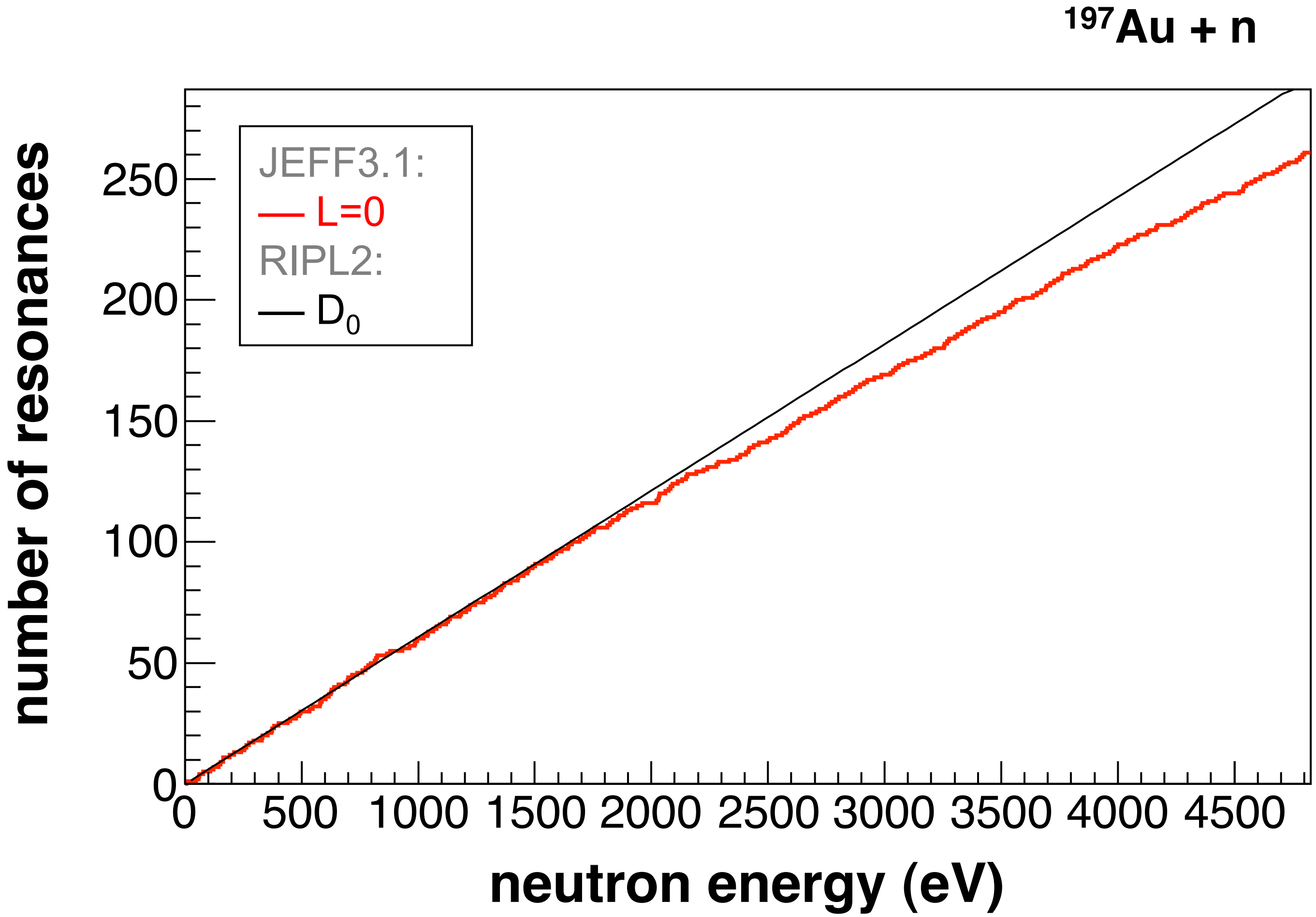
Many more sophisitcated models, especially for ρ(*U*).

Level density by counting levels: staircase plot

$$D(\ell = 0) = D_0 = \Delta E / N \qquad \rho(\ell = 0) = N / \Delta E$$



Level density by counting levels: missing levels



Level density from resonance positions

- Other information needed to estimate the number of missing levels.

- Use the properties of the statistical model of the nucleus to find missing levels.
  Works for medium and heavy nuclei.

What is the statistical model for a nucleus?

• Neutron resonances correspond to states in a compound nucleus, which is a nucleus in a highly excited state above the neutron binding energy.

• The compound nucleus corresponds to a very complex particle-hole configuration.
→ **Gaussian Orthogonal Ensemble (GOE)**

• The transition probability between two levels is related to the matrix elements of the interaction between two levels.

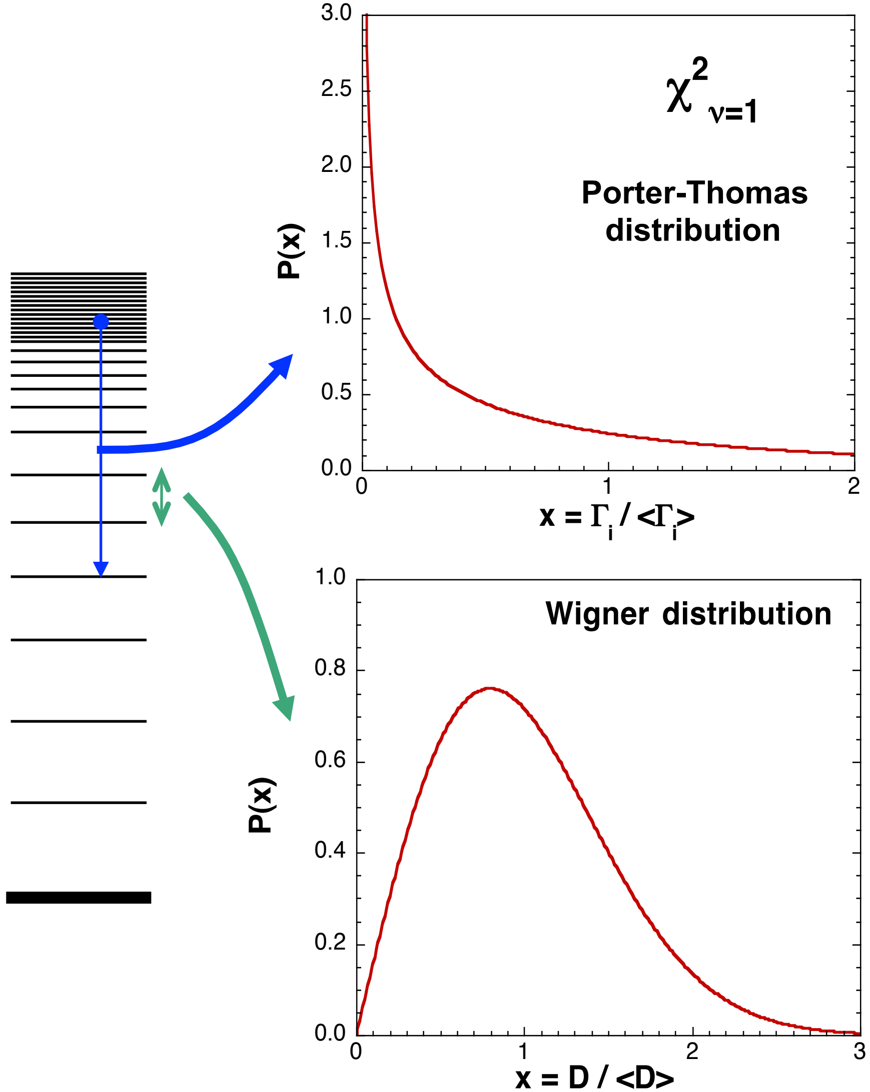
• Matrix elements (amplitudes γ) are Gaussian random variables with zero mean. Observables are widths $\Gamma \sim \gamma^2$.

The statistical model

The nucleus at energies around S_n can be described by the
Gaussian Orthogonal Ensemble (GOE)

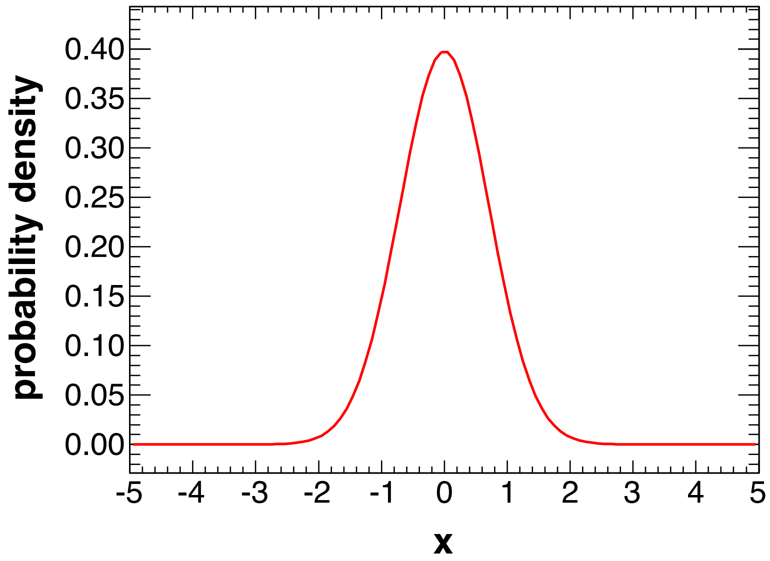
The matrix elements governing the nuclear transitions are random variables with a Gaussian distribution with zero mean.

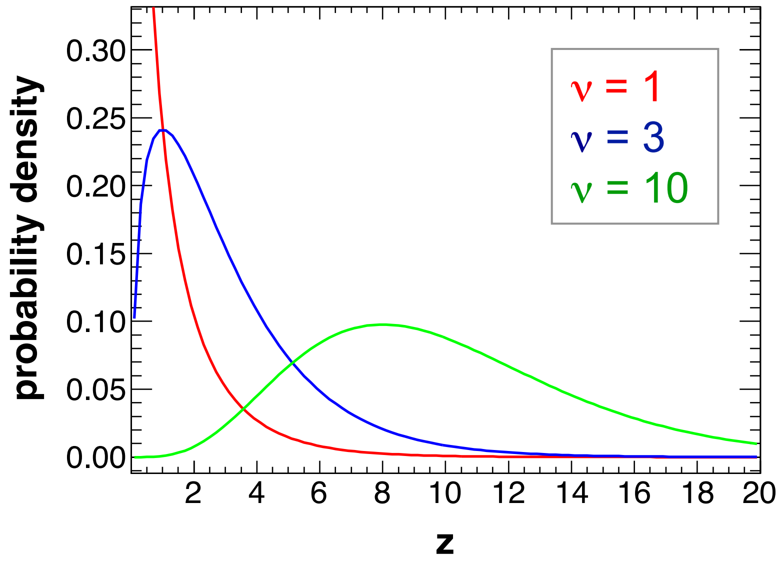
- Consequences:
  - The partial widths have **a** **Porter-Thomas** distribution.
  - The spacing of levels with the same J^{π} have approximately a **Wigner** distribution.



Chi-square distribution

- If ν random variables x_i have independent Gaussian distributions, $z = \Sigma x_i^2$ has a chi-square distribution with ν degrees of freedom.





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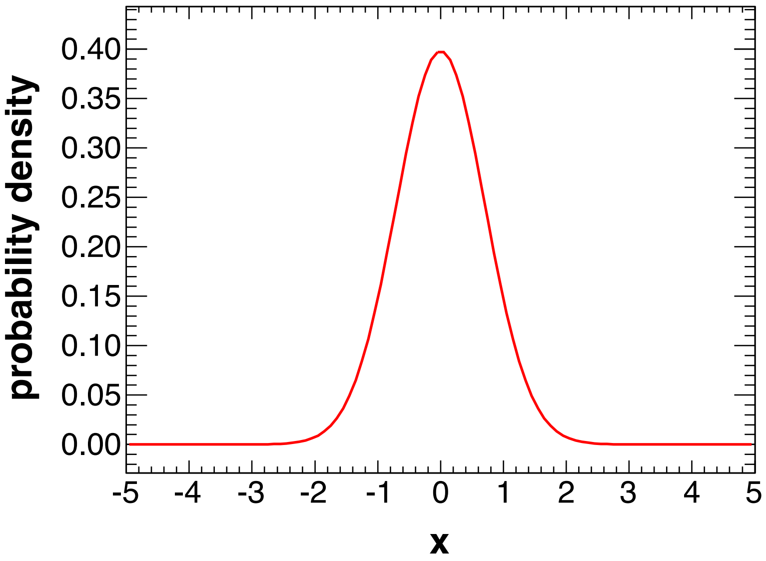

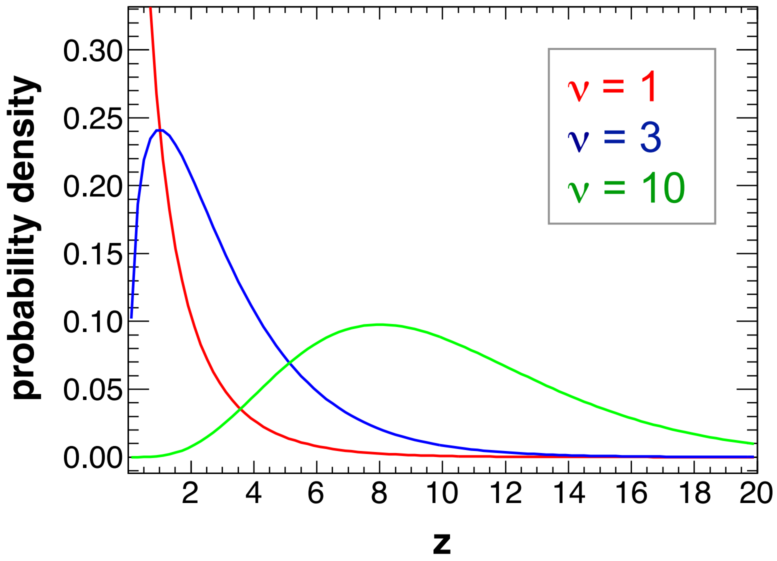

- neutron widths $\nu = 1$
- radiation widths ν = large number
- fission widths $\nu \sim 4$

Chi-square distribution

$$x = \frac{\gamma^2}{< \gamma^2 >} \qquad P_{\mathrm{PT}}(x) = \frac{1}{\sqrt{2\pi x}} \exp\big(- \frac{x}{2} \big)$$

For neutron widths (s-waves), use the effective reduced neutron width

$$\Gamma_n^0 = \Gamma_n / \sqrt(E)$$

and

$$x = \frac{g\Gamma_n^0}{< g\Gamma_n^0 >}$$

Chi-square distribution

$$x = \frac{\gamma^2}{<\gamma^2>} \qquad P_{\mathrm{PT}}(x) = \frac{1}{\sqrt{2\pi x}} \exp\left(-\frac{x}{2}\right)$$

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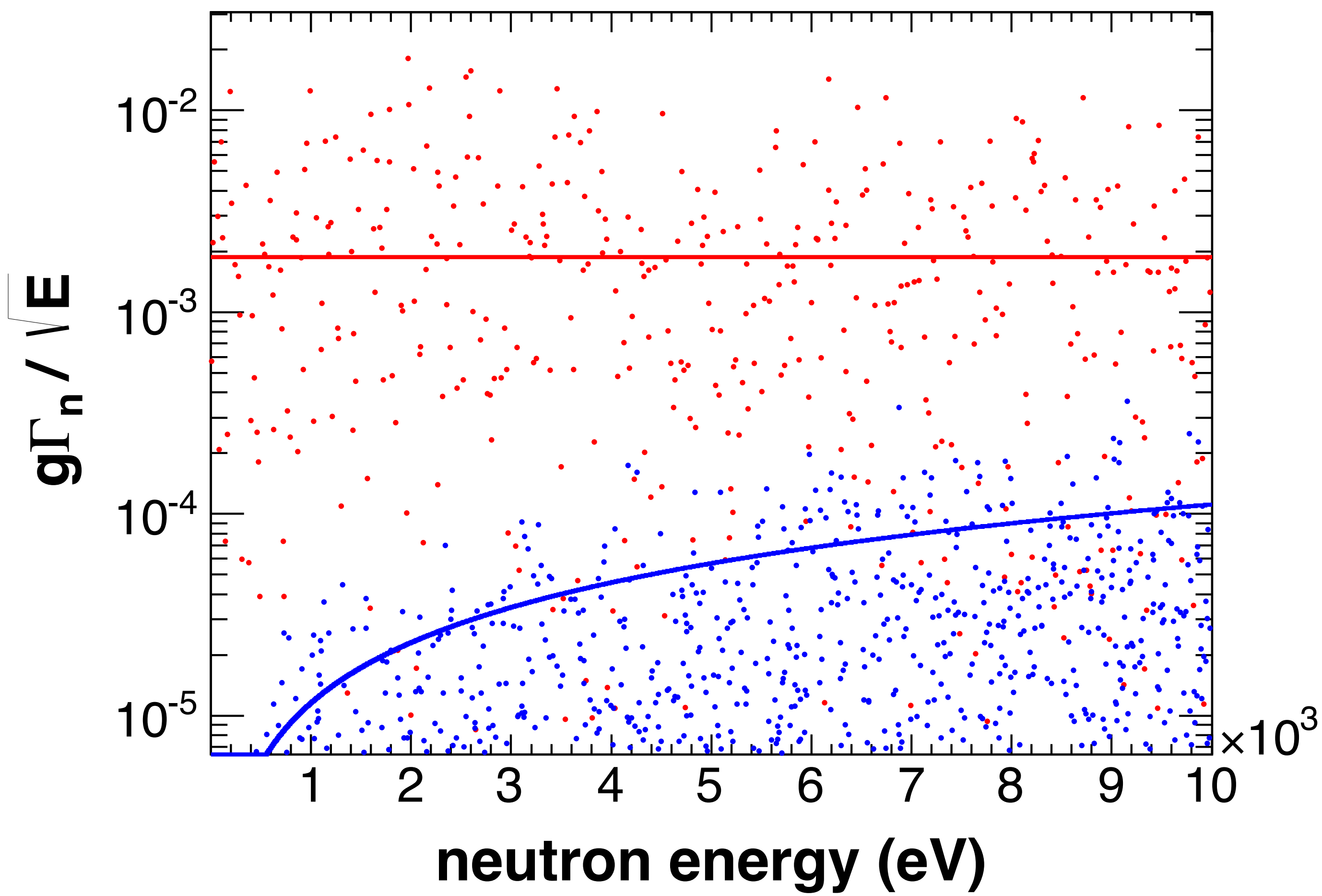
and for easy handling use

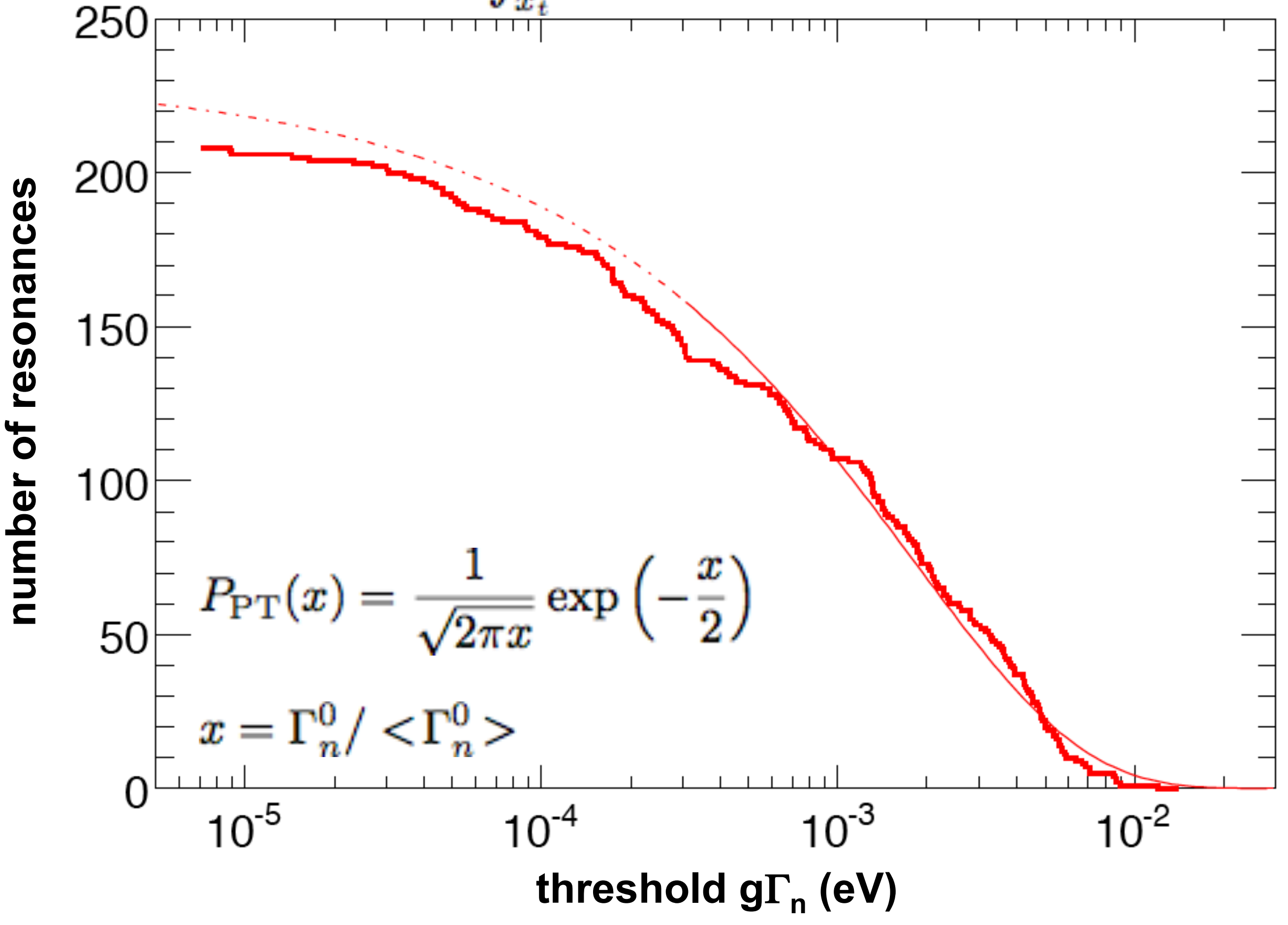
$$\int_{x_t}^{\infty} P_{\mathrm{PT}}(x)$$

Neutron widths

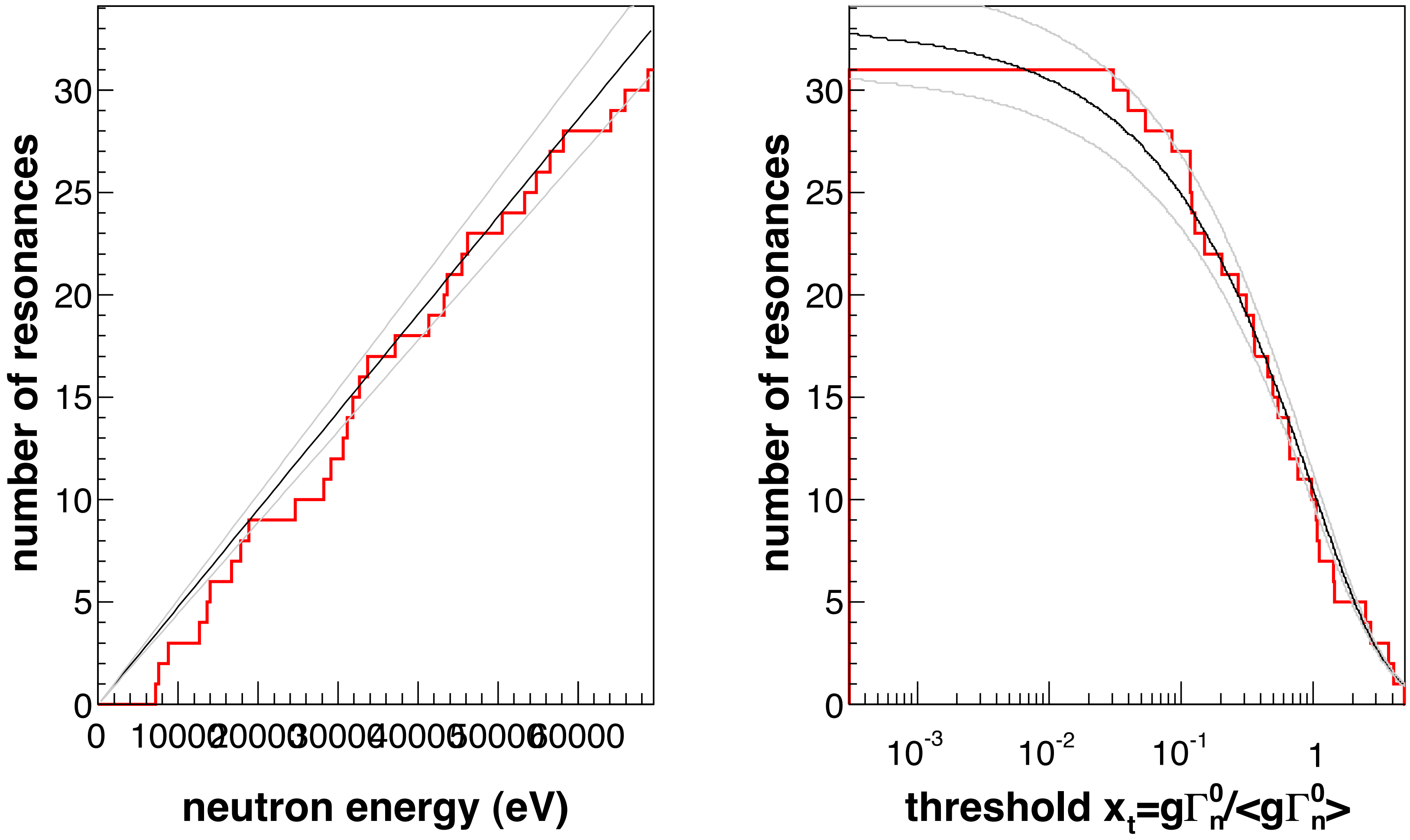


Neutron width distribution

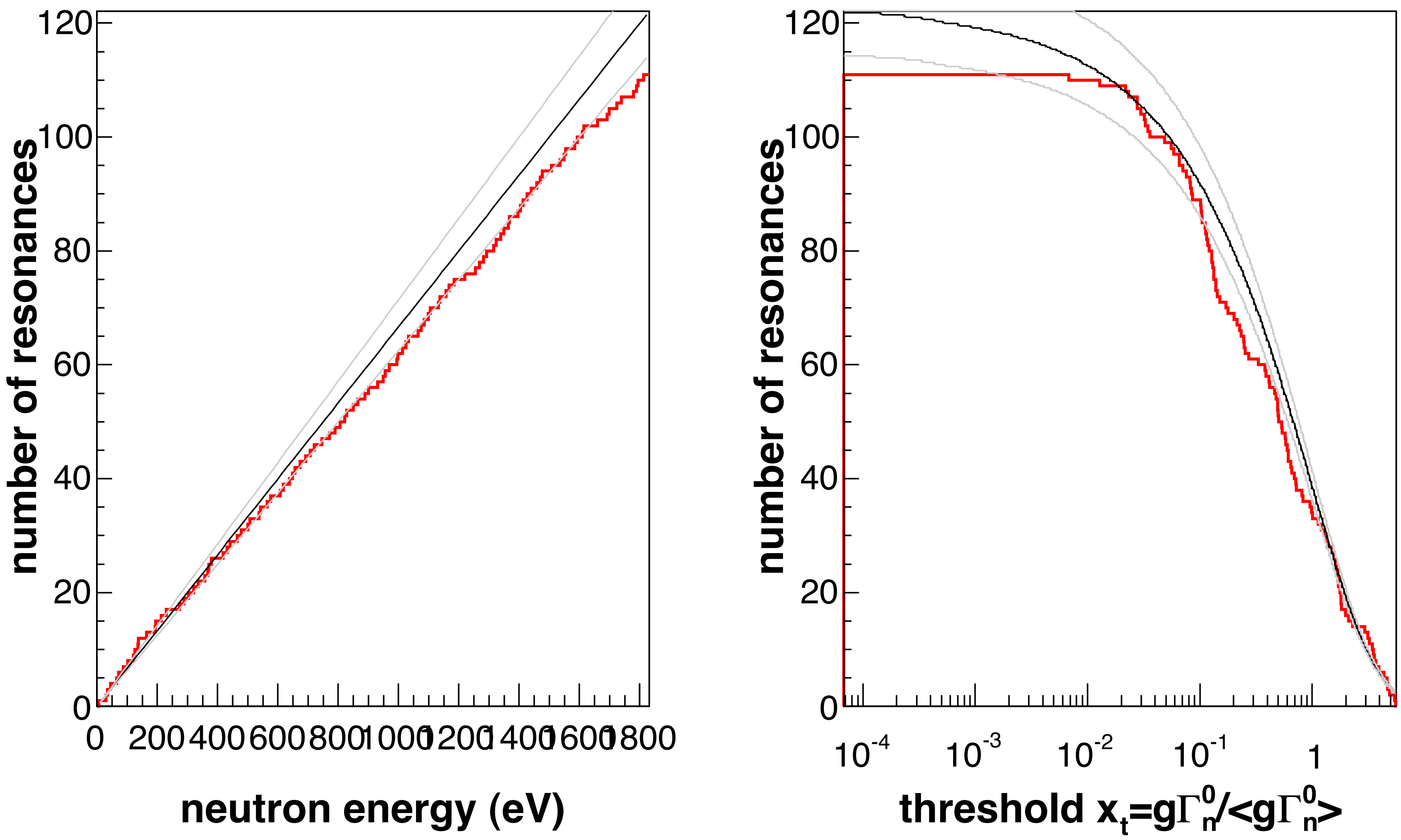
$$N(x_t) = N_0 \int_{x_t}^{\infty} P_{\mathrm{PT}}(x)dx = N_0(1 - \mathrm{erf}\sqrt{x_t/2})$$



Example ^{61}Ni



Example ^{236}U



The statistical model

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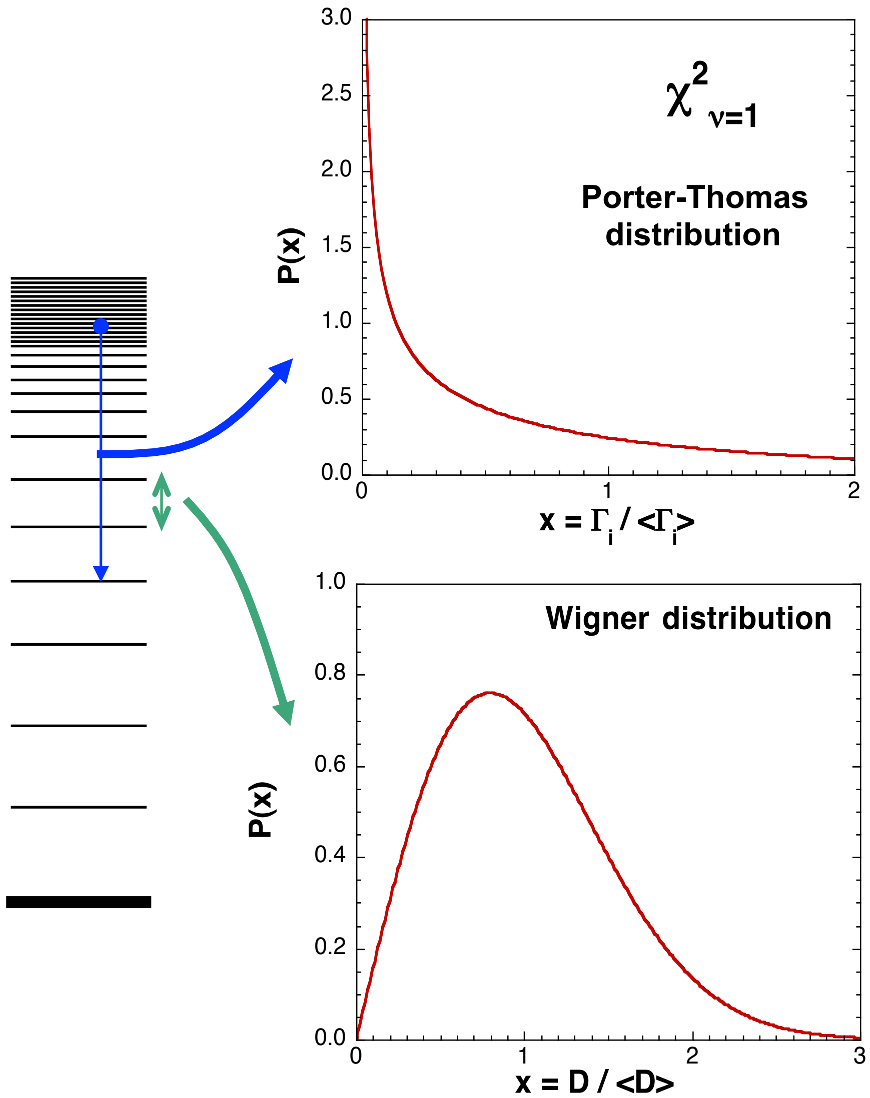
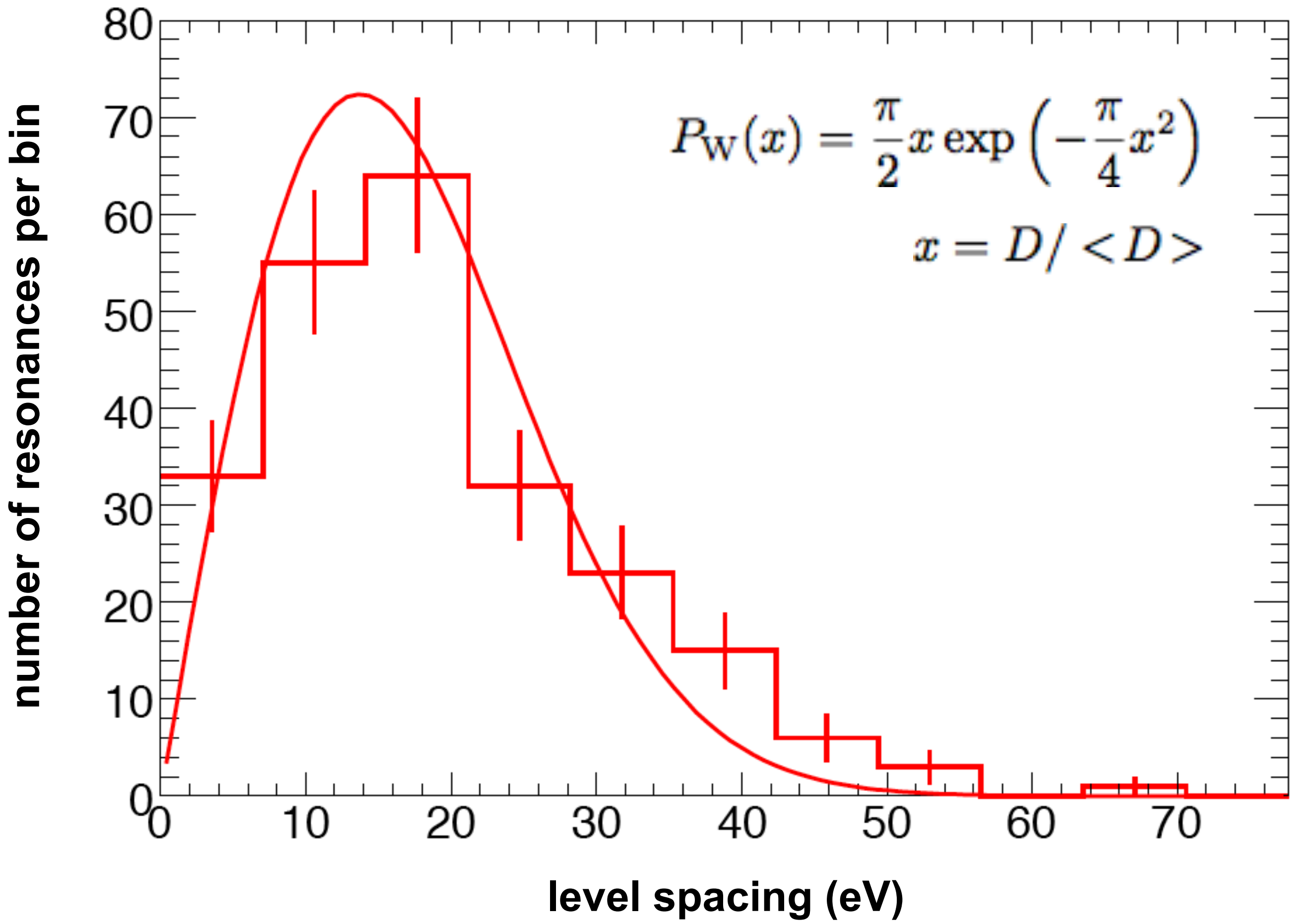


Spacing distribution of two consecutive levels



Evaluated nuclear data libraries

Libraries

- JEFF - Europe
- JENDL - Japon
- ENDF/B - US
- BROND - Russia
- CENDL - China

Common format:

ENDF-6

Contents:

Data for particle-induced reactions (neutrons, protons, gamma, other)
but also radioactive decay data

Data are indentified by “materials”
(isotopes, isomeric states, (compounds))

ex. ^{16}O: mat = 825

^{nat}V: mat = 2300

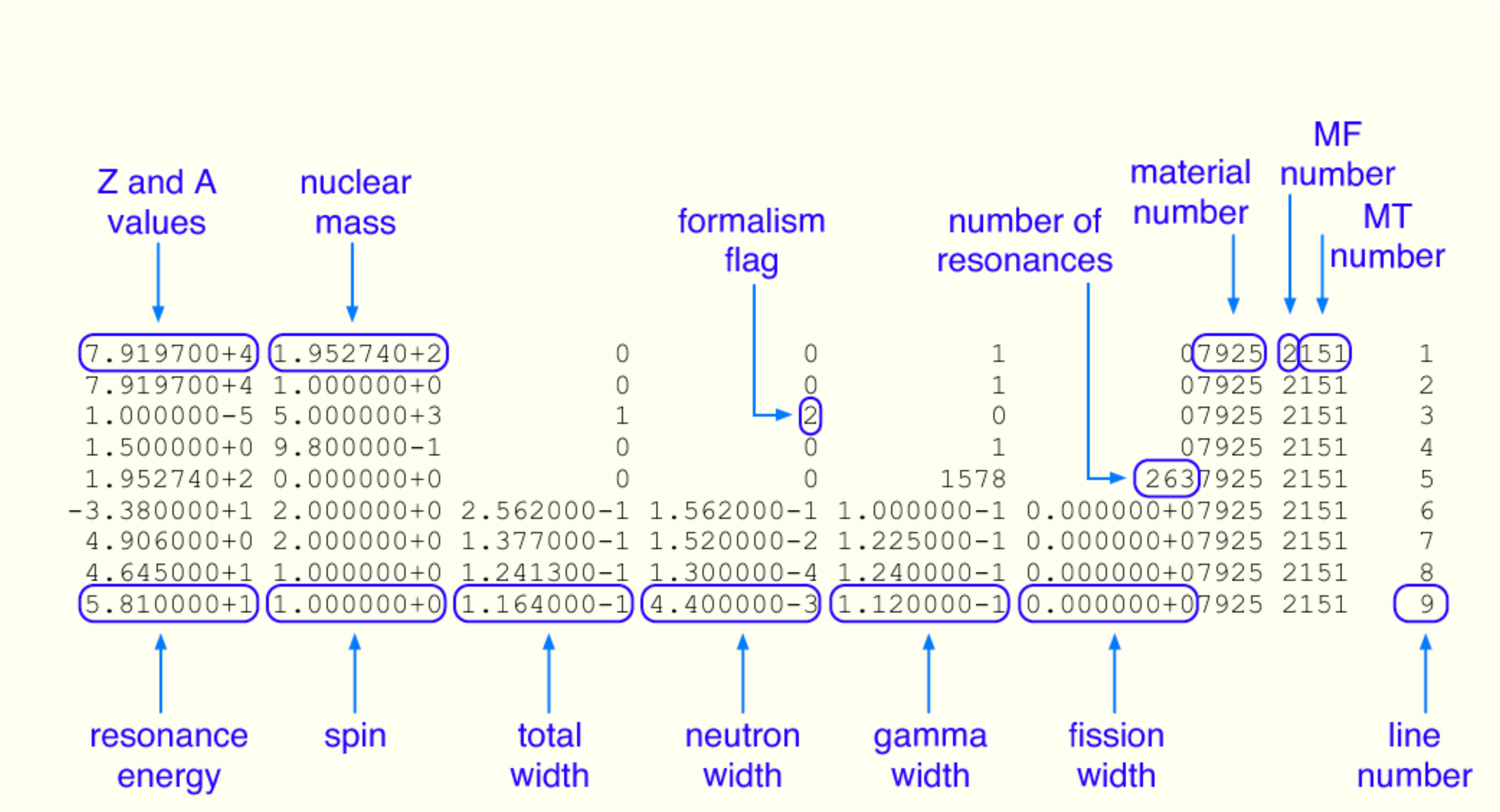
^{242m}Am: mat = 9547

Files for a material

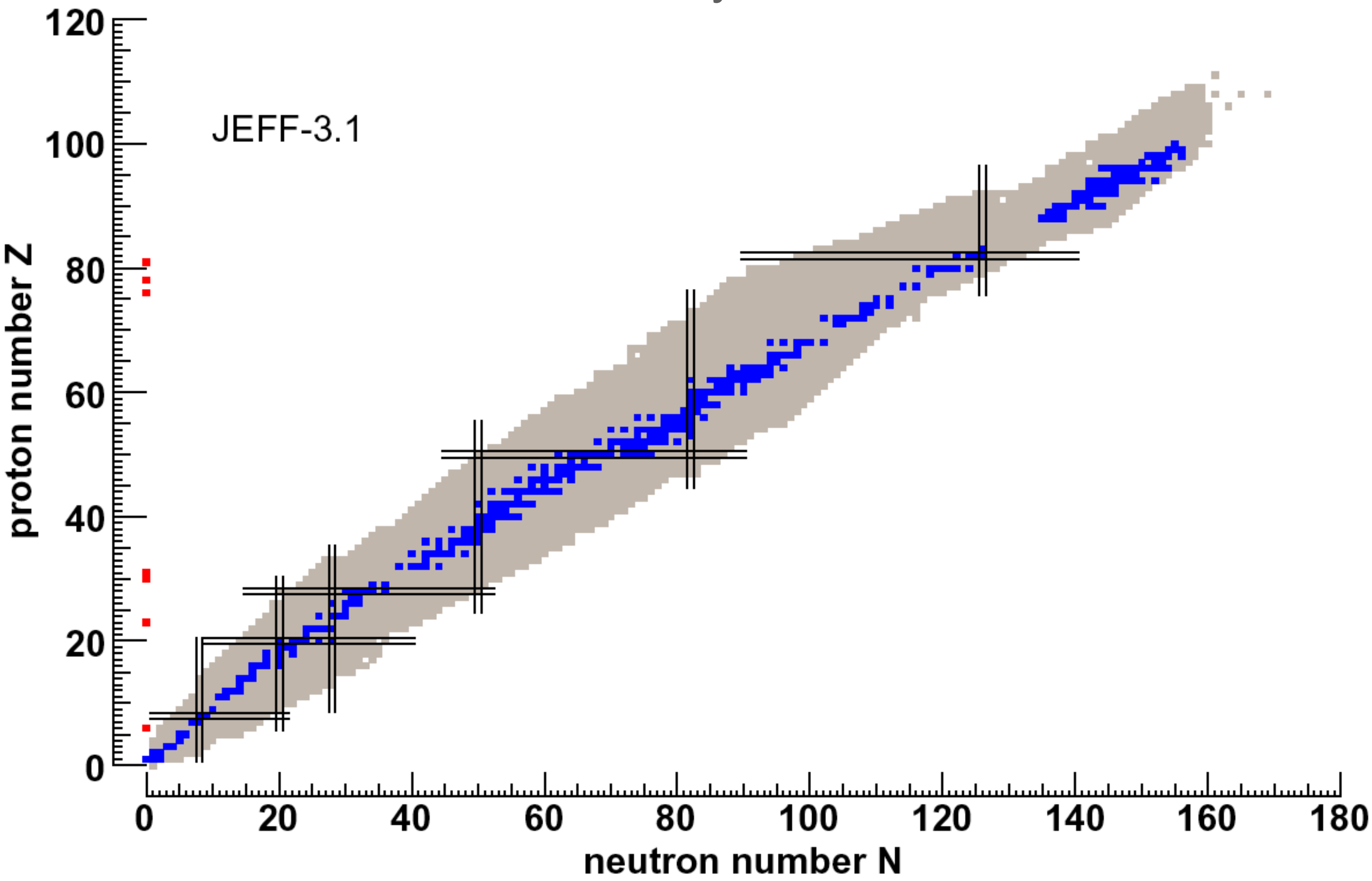
from report ENDF-102

1 General information
2 Resonance parameter data
3 Reaction cross sections
4 Angular distributions for emitted particles
5 Energy distributions for emitted particles
6 Energy-angle distributions for emitted particles
7 Thermal neutron scattering law data
8 Radioactivity and fission-product yield data
9 Multiplicities for radioactive nuclide production
10 Cross sections for photon production
12 Multiplicities for photon production
13 Cross sections for photon production
14 Angular distributions for photon production
15 Energy distributions for photon production
23 Photo-atomic interaction cross sections
27 Atomic form factors or scattering functions for photo-atomic interactions
30 Data Covariances obtained from parameter covariances and sensitivities
31 Data covariances for nubar
32 Data covariances for resonance parameters
33 Data covariances for reaction cross sections
34 Data covariances for angular distributions
35 Data covariances for energy distributions
39 Data covariances for radionuclide production yields
40 Data covariances for radionuclide production cross sections

Example: part of an evaluated data file



The library JEFF-3.1



Further Reading

Books/articles

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- G. F. Knoll, *Radiation Detection and Measurement*, Wiley & Sons, (2000).
- P. Reus, *Précis de neutronique*, EDP Sciences, (2003).
- J. E. Lynn, *The Theory of Neutron Resonance Reactions*, Clarendon Press, Oxford, (1968).
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- A. M. Lane, R. G. Thomas, “R-matrix theory of nuclear reactions”, *Rev. Mod. Phys.* **30** (1958) 257.
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- D. Cacuci (ed.), Handbook of Nuclear Engineering, Springer (2010).

Internet sites

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