PROBLEM LIST 1. CHAIN-RECURRENCE AND HYPERBOLICITY.

SYLVAIN CROVISIER AND RAFAEL POTRIE
SCHOOL ON DYNAMICAL SYSTEMS, ICTP, JULY 2015

(1) Prove the equivalence between the two definitions of the chain-recurrent set.
Hint: for $x \in M$ and $\varepsilon > 0$, consider the set of points y such that there exists a ε-pseudo-orbit $z_0, \dots, z_n$, $n \geq 1$, from x to y.

(2) Prove the equivalence between: (1) each chain-recurrence class is hyperbolic; (2) the chain-recurrent set is a finite union of hyperbolic sets.

(3) Prove that the hyperbolic diffeomorphisms have only finitely many chain-recurrence classes and that each of them is transitive.

(4) Check that the set of hyperbolic diffeomorphism is open in $\mathrm{Diff}^1(M)$.

(5) Prove that the hyperbolic diffeomorphisms are the diffeomorphisms which satisfy the Axiom A and the No-cycle Condition:
  (a) $\Omega(f)$ is a finite union of hyperbolic sets.
  (b) $\mathrm{Per}(f)$ is dense in $\Omega(f)$.
  (c) Consider transitive sets $K_0, K_1, \dots, K_n = K_0$ in $\Omega(f)$ such that for each i, there exists a heteroclinic orbit from K_i to K_{i+1}. Then all the K_i are contained in a same transitive set.

(6) Any continuous linear bundle $E \subset T_K M$ over a compact set $K \subset M$ admits a continuous extension to a neighborhood of K.

(7) Consider a diffeomorphism f, an invariant compact set K and a continuous splitting $T_K M = E \oplus F$ which is preserved by f. Prove that K is hyperbolic if and only if for any invariant probability measure supported on K, the upper Lyapunov exponent along E is negative and the lower Lyapunov exponent along F is positive.