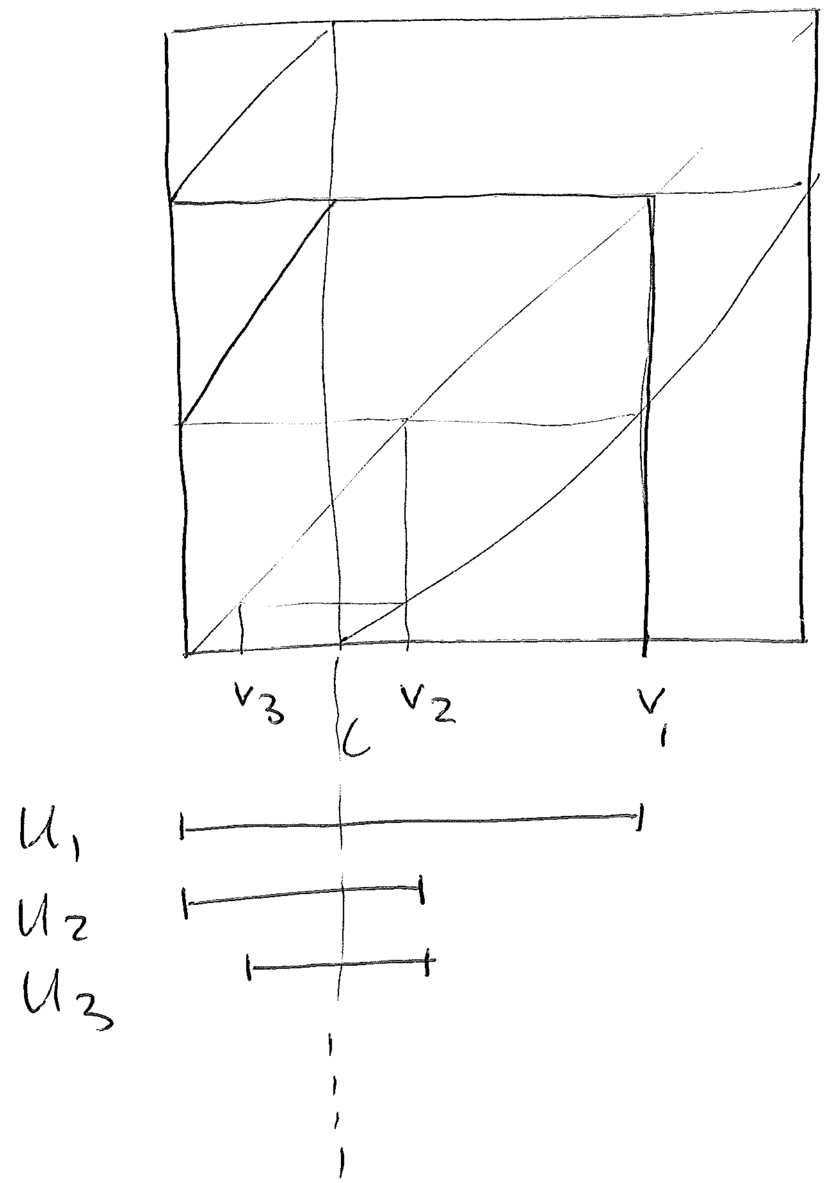
③ Dynamical Partitions

Observe, $R^n f$ is the first return map to an interval U_n up to scaling.



$U_n = L_n \cup R_n$

$F_n : U_n \circlearrowleft$ first return map.

$$F_n : \begin{cases} f^{\ell_n} & \text{on } L_n \\ f^{r_n} & \text{on } R_n \end{cases}$$

$$\mathcal{L}_n = \{ L_n, f(L_n), \dots, f^{\ell_n - 1}(L_n) \}.$$

$$\mathcal{R}_n = \{ R_n, f(R_n), \dots, f^{r_n - 1}(R_n) \}.$$

$$\mathcal{P}_n = \mathcal{L}_n \cup \mathcal{R}_n.$$

Lemma:

1. $\mathcal{P}_n$ pairwise disjoint

2. $\displaystyle\bigcup_{I \in \mathcal{P}_n} \overline{I} = [0,1]$

Lemma:

$c < v$ (D_+)

$$\ell_{n+1} = \ell_n + r_n$$

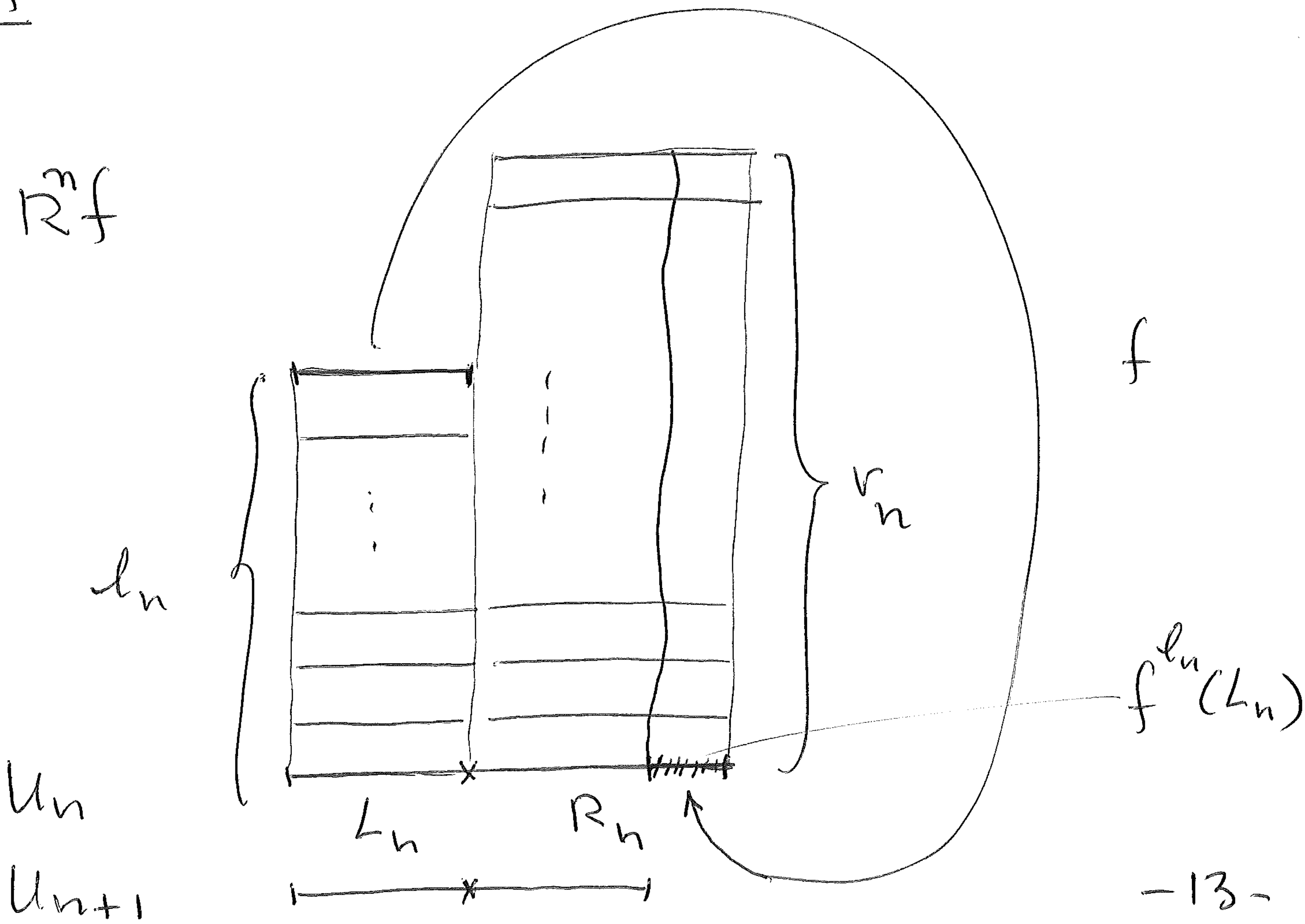
$$r_{n+1} = r_n$$

$$L_{n+1} = L_n$$

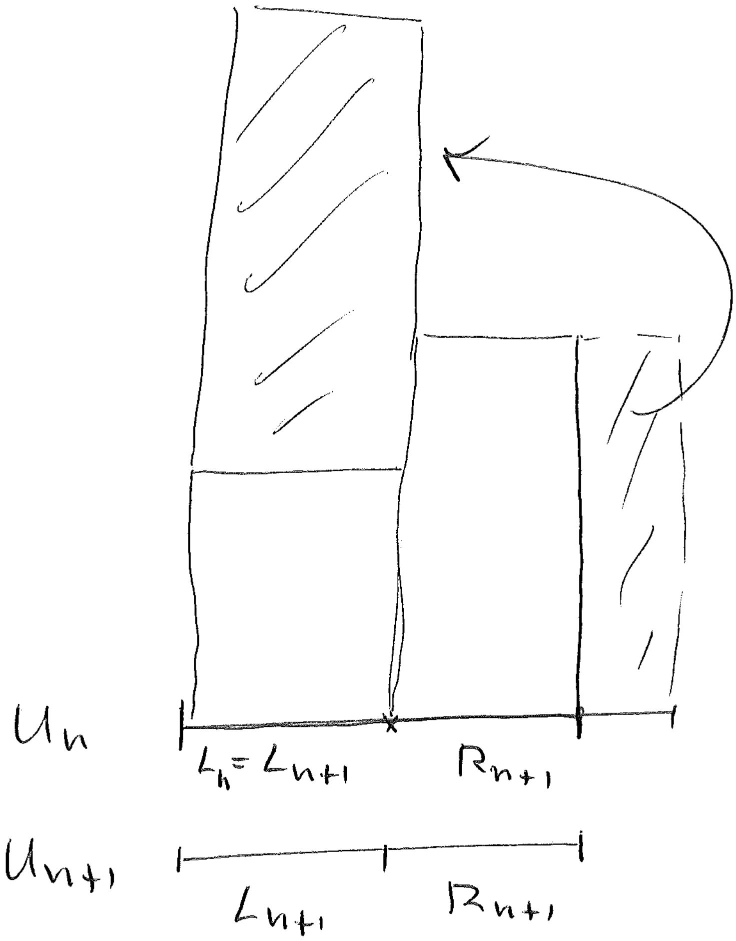
$$R_{n+1} = R_n \setminus f^{\ell_n}(L_n)$$

$$f^i(R_n) = f^i(R_{n+1}) \cup f^i\left(f^{\ell_n}(L_n)\right).$$

Pf: Tower.



$R^{n+1}f$



$\square$

<u>Lemma</u>: $c > v$ (D_-)

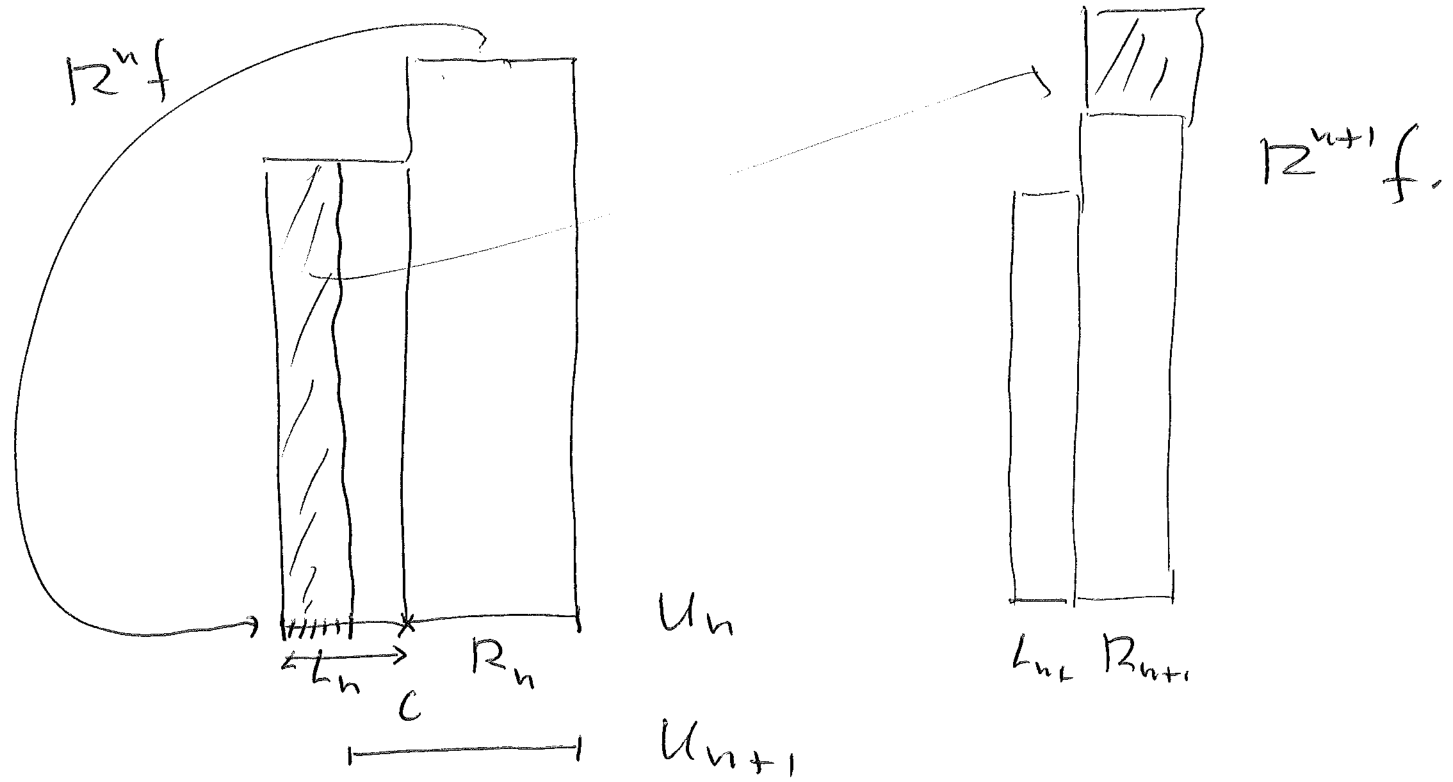
$$\ell_{n+1} = \ell_n$$

$$r_{n+1} = r_n + \ell_n$$

$$L_{n+1} = L_n \setminus f^{r_n}(R_n)$$

$$R_{n+1} = R_n$$

$$f^i(L_n) = f^i(L_{n+1} \cup f^{r_n}(R_n))$$



<u>Note</u>:

<u>$c<v$</u>: $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$ obtained by cutting the intervals $f^i(R_n)$

<u>$c>v$</u>: $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$. obtained by cutting $f^i(L_n)$.

Prop: $f, \tilde{f} \in \mathcal{I}$ with $a(f) = a(\tilde{f})$

$\Longrightarrow \exists^{C^0} h_n : [0,1] \to [0,1]$ s.t.

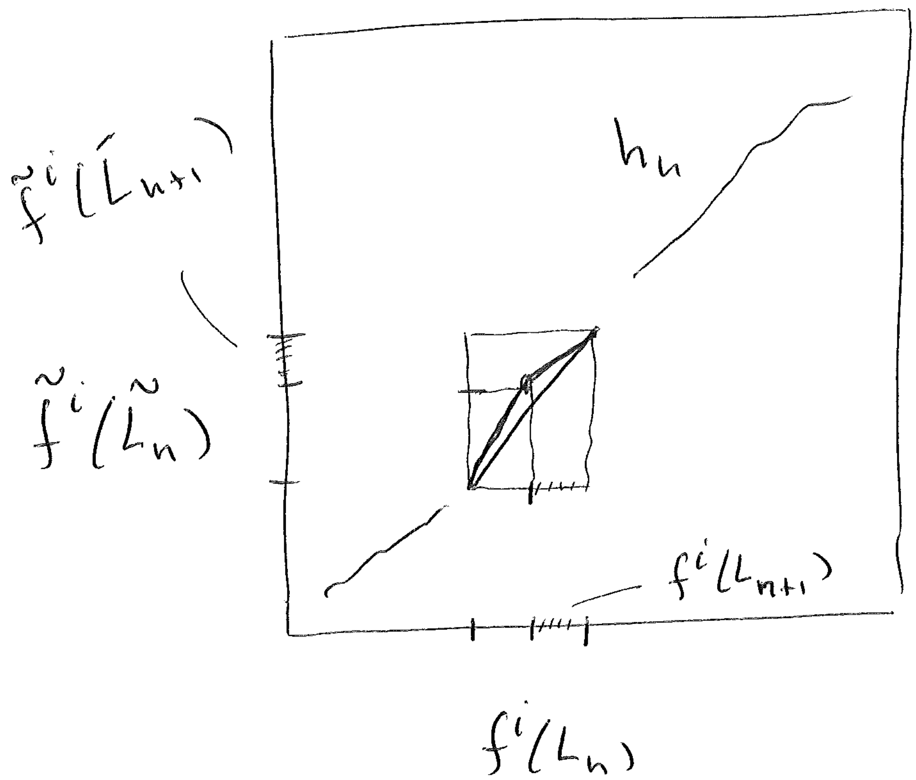
① $h_n(\mathcal{P}_n) \Longrightarrow \tilde{\mathcal{P}}_n$

② $h_n : f^i(L_n) \to \tilde{f}^i(\tilde{L}_n)$ affine onto

$h_n : f^i(R_n) \to \tilde{f}^i(R_n)$ affine onto

③ $I \in \mathcal{P}_n$: $h_{n+1}(I) = h_n(I)$

Pf: Inductively



Prop: mesh (P_n), mesh $(\tilde{P}_n) \longrightarrow 0$

the $h_n \xrightarrow{C^0} h \in \cancel{C^0([0,1])}$ Homeo$([0,1])$

and

$$h \circ f = \tilde{f} \circ h$$

$f, \tilde{f}$ are top. conjugated.

Remark: If $f, \tilde{f}$ are topologically conjugated then $\underline{a}(f) = \underline{a}(\tilde{f})$

. Hence ρ_f is a topological invariant.

Apriori bounds

A central question in Renormalization is whether

$$\sup_n |R^n f|_{C^1} \leq K,$$

whether the renormalizations stay in a bounded domain. Such a bound is called the a priori bound.

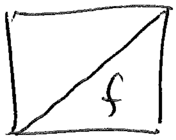
Non-linearity

$$f : [0,1] \longrightarrow [0,1] \quad C^2 \text{ diffeo.}$$

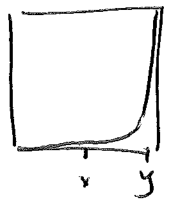
$$\eta = \eta_f = \frac{D^2 f}{Df} : [0,1] \longrightarrow \mathbb{R}.$$

$$\ln \frac{Df(y)}{Df(x)} = \int_x^y \eta \qquad \text{Distortion}$$

Example

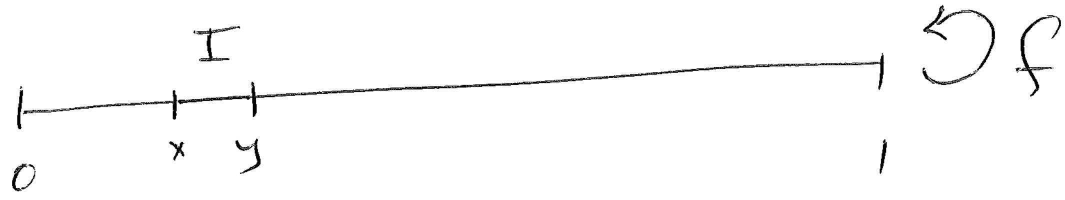
affine f: 

dist $= 0$



dist $\gg 1$

Distortion of iterates



Let $f : [0,1] \longrightarrow [0,1]$ Diffeo with

$$\eta \in L^1([0,1])$$

Let $I \subset [0,1]$ s.t.

- $f^n : I \longrightarrow f^n(I)$ diffeo.

- $f^i(I) \cap f^j(I) = \emptyset \quad i,j < n$

$$Df^n(x) = \prod_{i=0}^{n-1} Df(f^i(x))$$

$$\text{Distortion} = \max_{s,t \in I} \left| \ln \frac{Df^n(s)}{Df^n(t)} \right|$$

$$= \max \left| \sum \ln \frac{Df(f^i(s))}{Df(f^i(t))} \right|$$

$$\leq \max \sum \left| \int_{f^i(s)}^{f^i(t)} \eta \right|$$

$$\leq \sum \int_{I_i = f^i(I)} |\eta| \leq \int |\eta| < \infty.$$

<u>Thm:</u> Renormalization of C^2-circle map has a priori bounds.

In particular, given $f \in D_2$,

$\exists K$.

$$\operatorname{dist}\left(f^t : f^i(L_n) \to f^{i+t}(L_n)\right) \le K$$

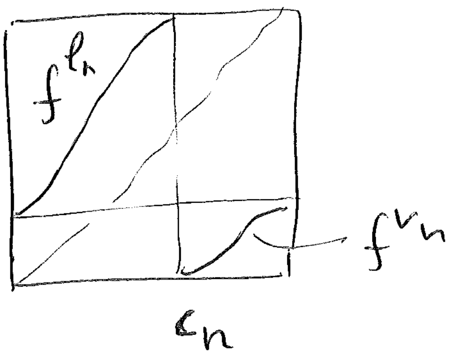
$$i+t \le l_n$$

and

$$\operatorname{dist}\left(f^t : f^i(R_n) \to f^{i+t}(R_n)\right) \le K.$$

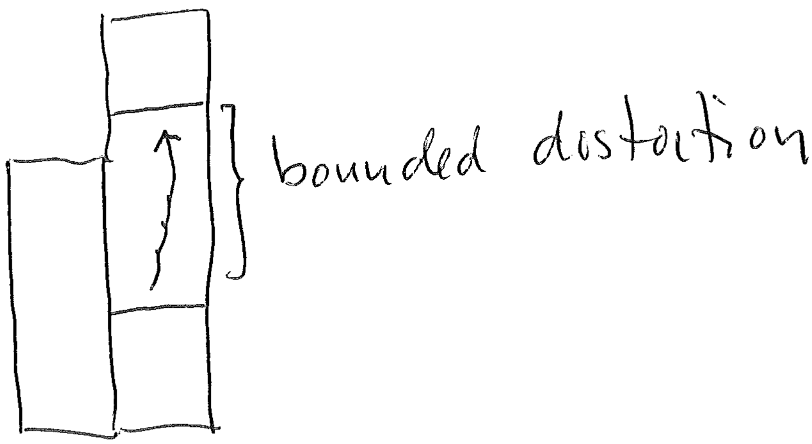
So

$R^n f$:



f^{l_n}, f^{r_n} have bounded distortion

R^n



Thm: If $f, g \in D_2$ with

$$a_f = a_g$$

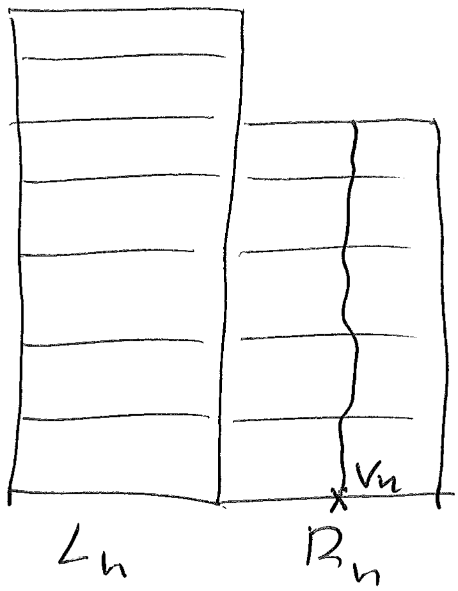
Then f and g are topologically conjugated; $\exists\, h : [0,1]^{\circlearrowleft}$ homeo

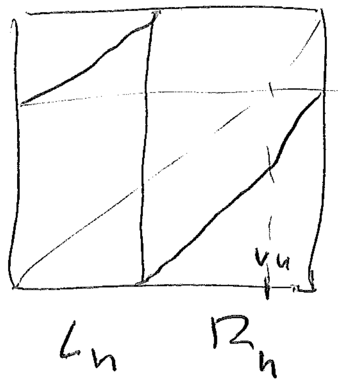
$$h f = g h$$

Proof: we need to show that

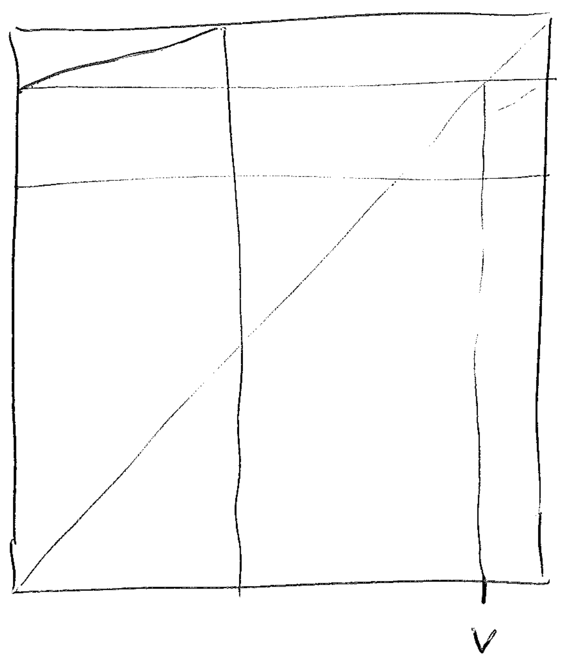
$$\max_n \{ |f^i(L_n)|, |f^i(R_n)| \} \longrightarrow 0.$$

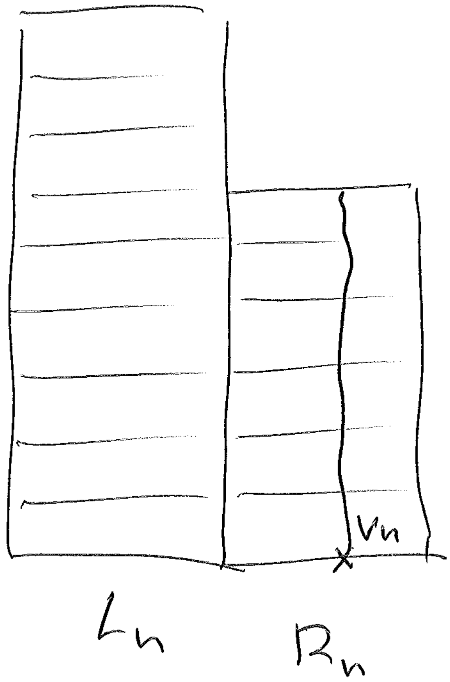
Each renormalization "cuts" either the R_n-orbit or the L_n-orbit producing $\mathcal{P}_{n+1}$ out of $\mathcal{P}_n$

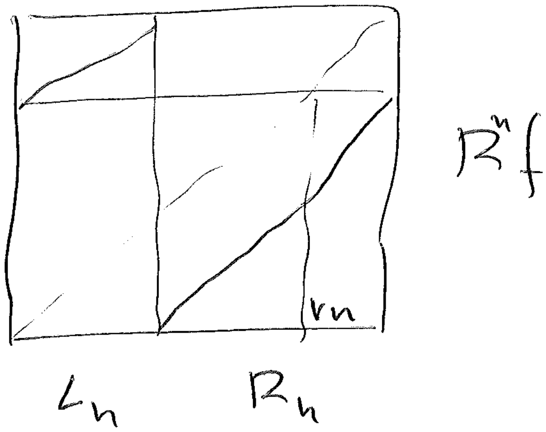




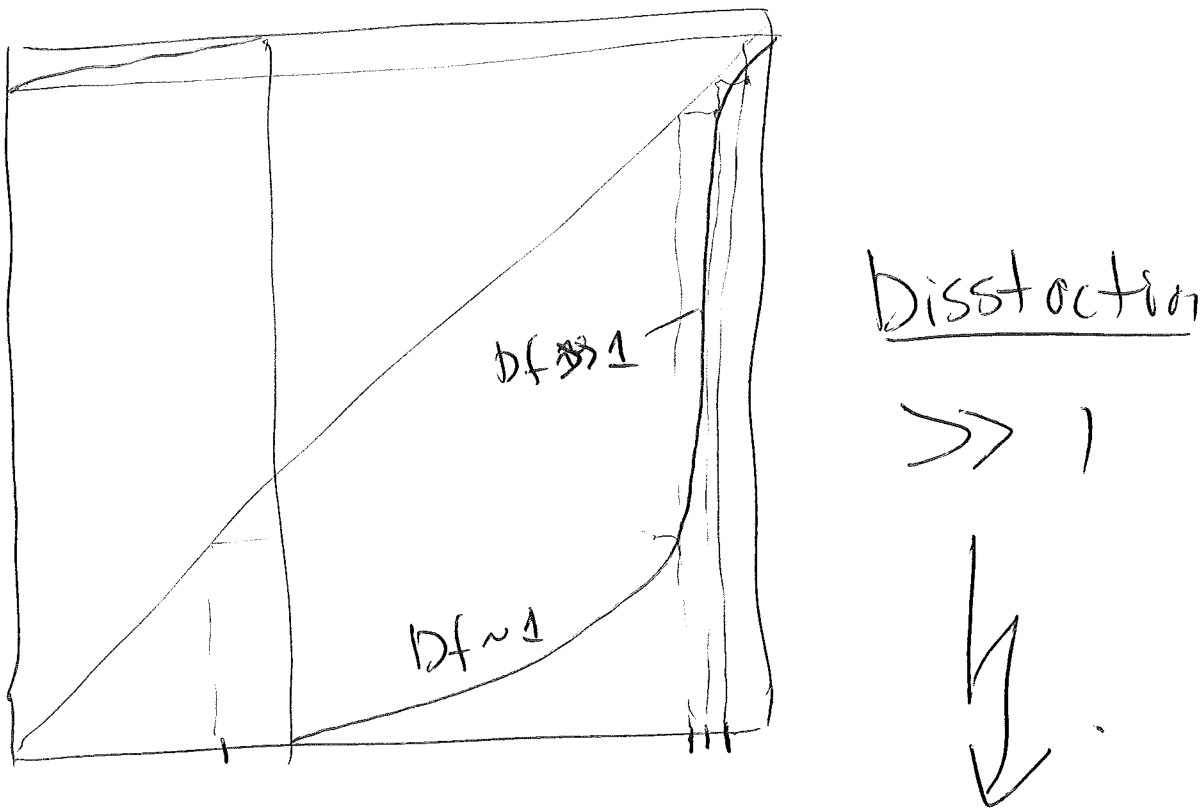
The orbit of v_n will end up on the left. Suppose it will never pass through the middle of R_n



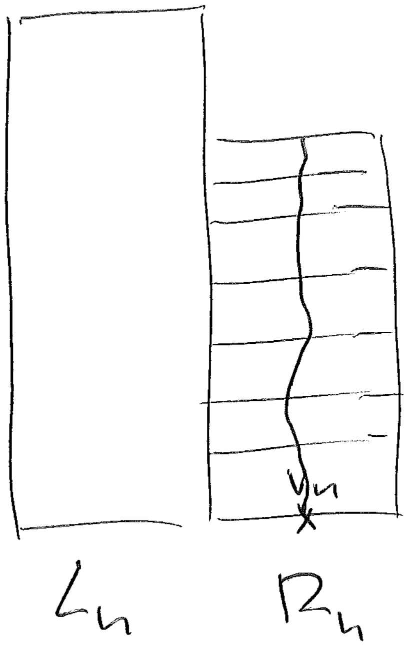




~~So~~ The orbit of v_n will eventually end up in L_n. Suppose it doesn't pass through the middle.



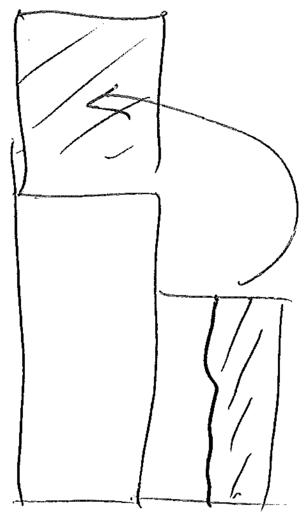
So at some moment we will see



v_n in the middle of R_n.

The the bounded distortion of all $f^i : R_n \to f^i(R_n)$ imply that all $f^i(R_n)$ are nicely cut in definitely smaller parts.

$R^{n+1}f$



$\Longrightarrow$

$\max\{|f^i(L_n)|, |f^i(R_n)|\}$

$\longrightarrow 0$

$\square$.

Thm. Let $f \in D_2$ (C^2 circle diffeo).

and $\rho_f = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$

the f is topologically conjugated to the rigid rotation $R_{\rho_f} : S^1 \to S^1$

Thm. (Denjoy Counter Example).

$\forall \rho \notin \mathbb{Q} \ \exists f \in D_1$ (C^1- circle diffeo)

such that $h : S^1 \to S^1$ with

$$h \circ f = R_\rho \circ h$$

is not a homeo.

Thm: Let $f \in D_2$. (C^2- circle differ).

If f is only finitely many times renormalizable:

$$\rho_f \in \mathbb{Q} \quad \text{or}$$

$$\exists n \quad R^n f \notin D_- \cup D_+$$

then $\forall x$

$\omega(x) =$ periodic orbit.

with period independent of x.

If f is ∞-finitely renormalizable

$$\rho_f \in \mathbb{P} = \mathbb{R}/\mathbb{Q}$$

then $\forall x$

$$\omega(x) = S^1.$$

Proof: exercise. $\square$.