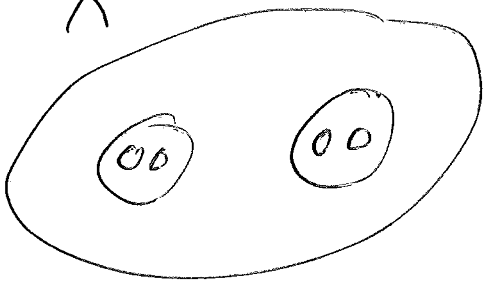
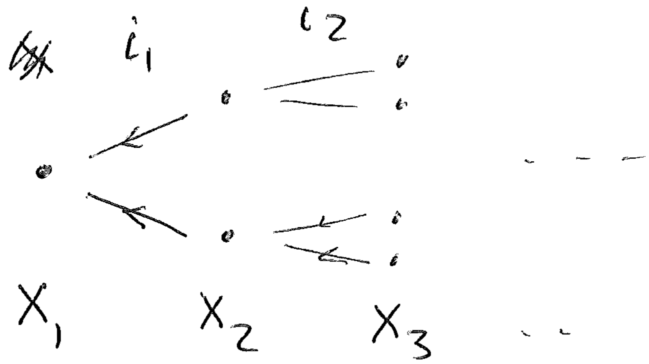
Minimal Cantor sets

Cantor set : compact, totally disconnected.
perfect: ($\forall x$: $x = \overline{C \setminus x}$)

X





$$X = \varprojlim X_n = \{\text{backward path}\}.$$

$$= \{\text{all ways of zooming in}\}.$$

X Cantor set

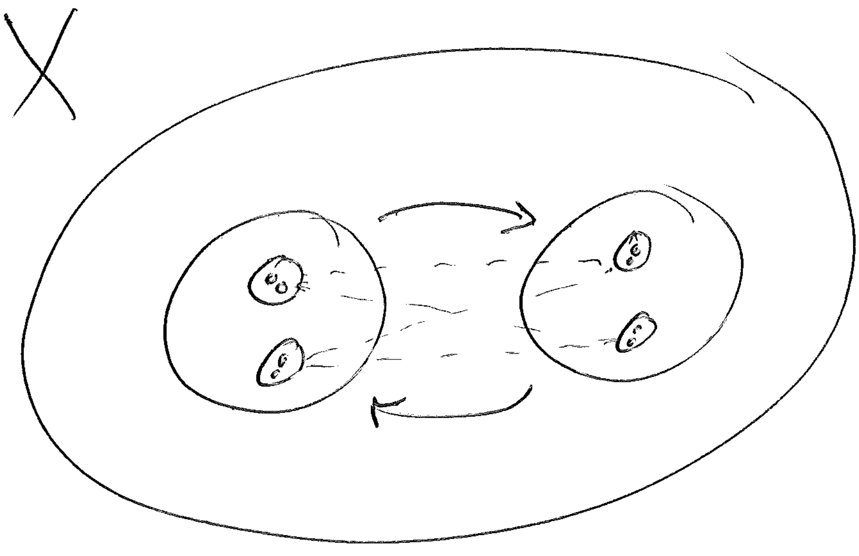
$f : X \longrightarrow X$ Continuous

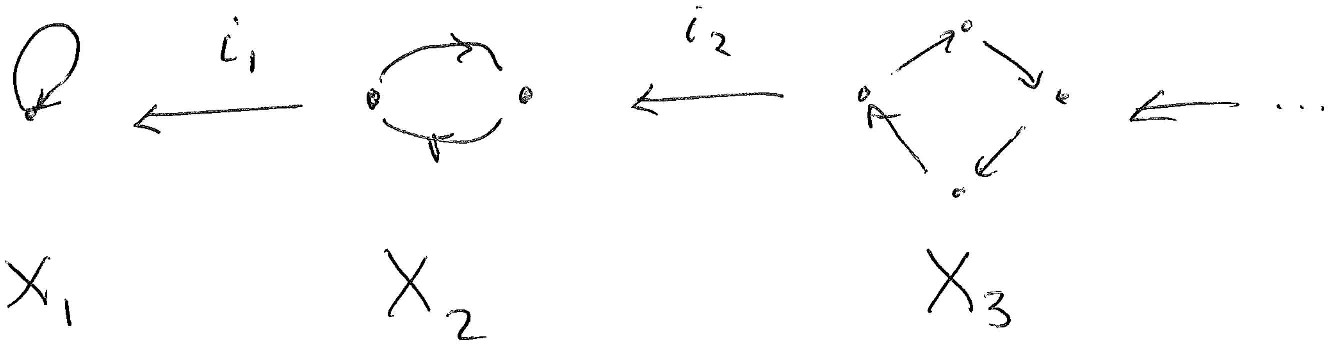
f is minimal if $\forall x$

$$\mathrm{Orb}(x) = \{f^i(x)\}$$

is dense in X ($\overline{\mathrm{Orb}(x)} = X$)

Example : adding machine / solenoid.

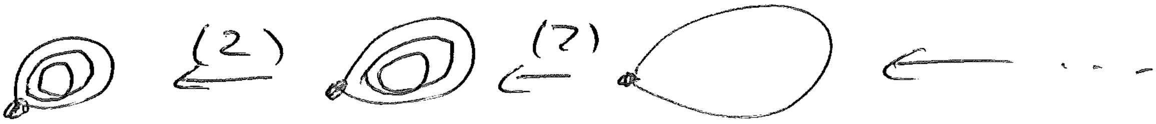




f make each X_n into a directed graph.

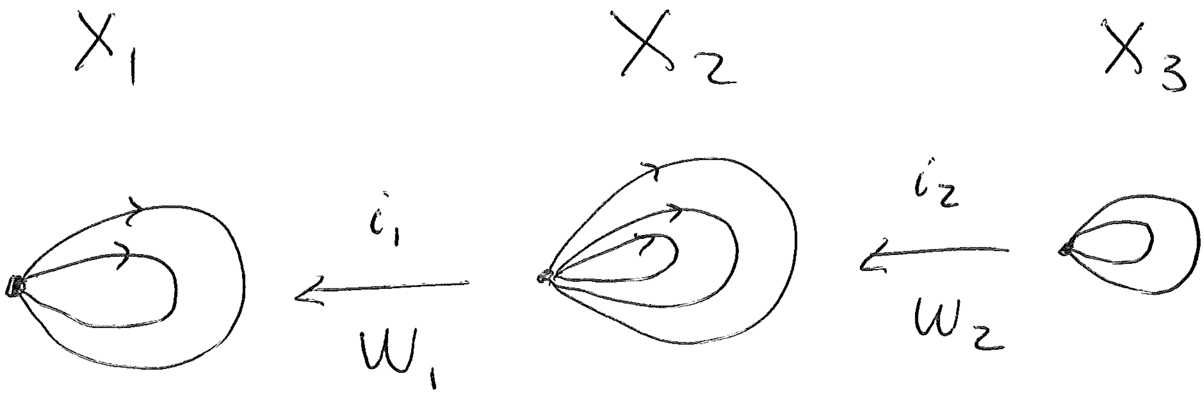
$\varprojlim X_n = X$ is a directed "graph" describe by the map f.

<u>schematically.</u>



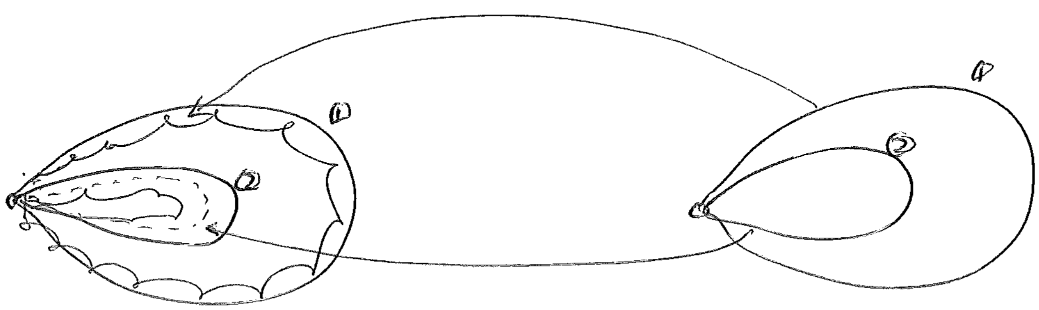
<u>Lemma</u>: prove the directed graphs indeed define a map on X which is the original f.

Generalize



The inclusion maps contain winding information: <u>Winding Matrix</u>

Example

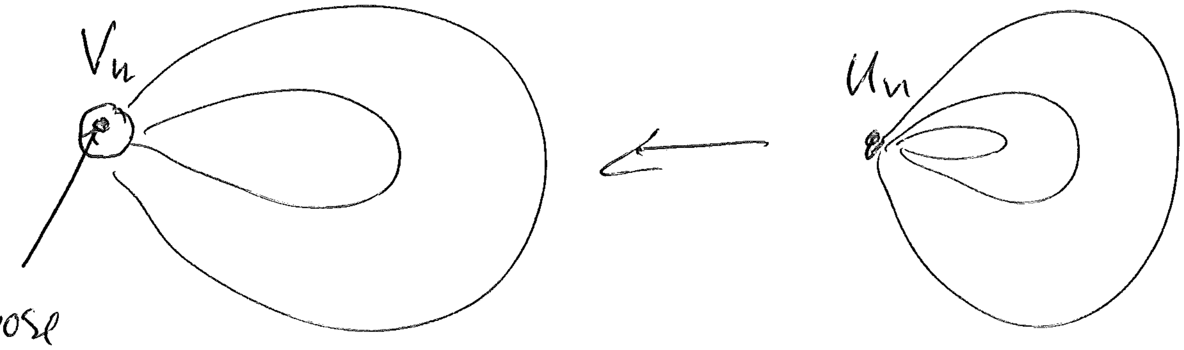


$$W = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \end{array}$$

Thm: Every minimal Cantor set is ~~be~~ conjugated to some

$$X = \varprojlim_{(\omega_n)} X_n$$

sketch Proof: by Induction



choose $U_n \subset V_n$ clopen.

X_n

↳ the $Rf|U_n$ defines X_{n+1} □.

A measure on $\underset{f}{X \circlearrowleft}$ (minimal Cantor set)

is called invariant if

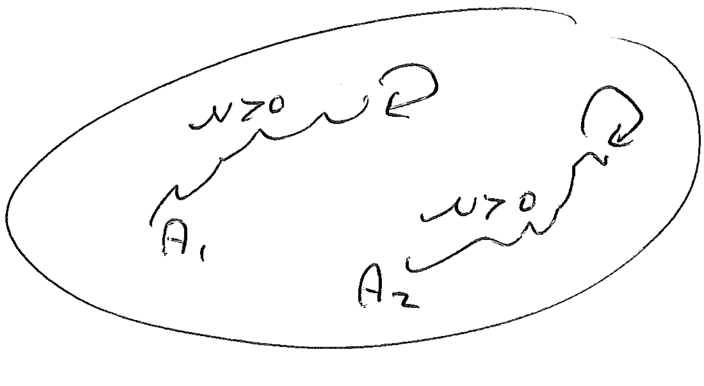
$$\mu(f^{-1}(A)) = \mu(A)$$

$\forall A \subset X$ measurable.

μ is called ergodic if

$$\mu(A) = 0 \quad \text{or} \quad 1$$

for all invariant sets $\underset{f}{A \circlearrowleft} \subset X$.



Not ergodic: more pieces of support

Birkhof Ergodic Theorem

$f \circlearrowright X$ minimal Cantor set

ν ergodic measure (with $\nu(X)=1$)

then

$$\boxed{\frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) \longrightarrow \int \varphi \, d\nu}$$

$\forall \ \varphi: X \to \mathbb{R}$ continuous.

A central question in dynamics is to find ~~relevant~~ physical measures:

$$\frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) \longrightarrow \int \varphi \, d\nu.$$

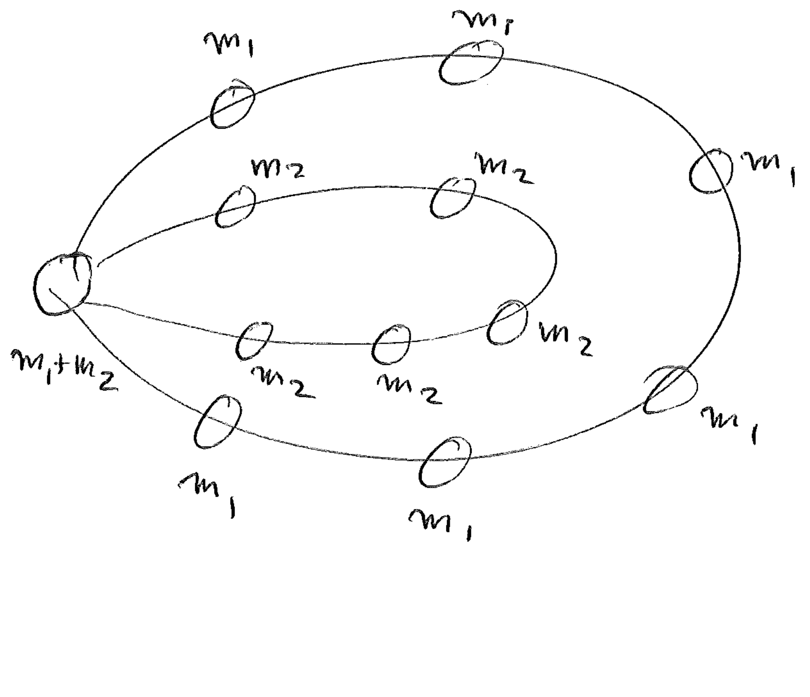
for "almost every" x

How to make invariant measures for minimal Cantor sets?

Take $X = \varprojlim X_n$ and let $\circlearrowleft f$

ν be an invariant measure:

$\nu|X_n$:

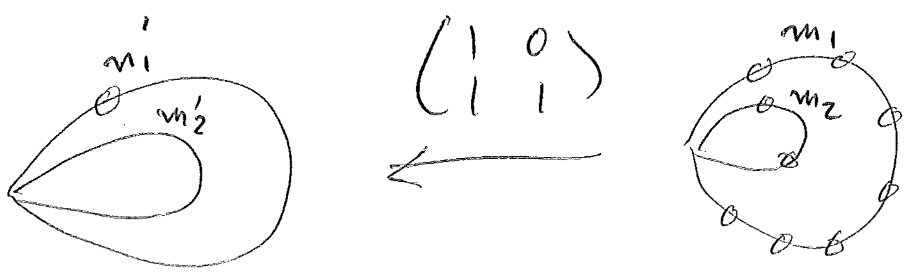


Let M be the space of all invariant measures. Let $M_n = \mathbb{R}^{\ell_n}$

be the invariant measures on scale n.

$$\begin{tikzcd}
 & M \arrow[dl, "\pi_n"'] \arrow[dr, "\pi_{n+1}"] \arrow[d, phantom, "\circlearrowright"] & \\
M_n = \mathbb{R}^{\ell_n} & & M_{n+1} = \mathbb{R}^{\ell_{n+1}} \arrow[ll, "W_n"']
\end{tikzcd}$$

The winding induces a map from M_{n+1} to M_n:



$$\begin{cases} m_1' = m_1 \\ m_2' = m_1 + m_2 \end{cases}$$

Remark: $M_n = H_1(X_n)$

"measures $\sim$ homology."

Thm: $M = \varprojlim_{w_n} M_n$

Proof: exercise $\square$.

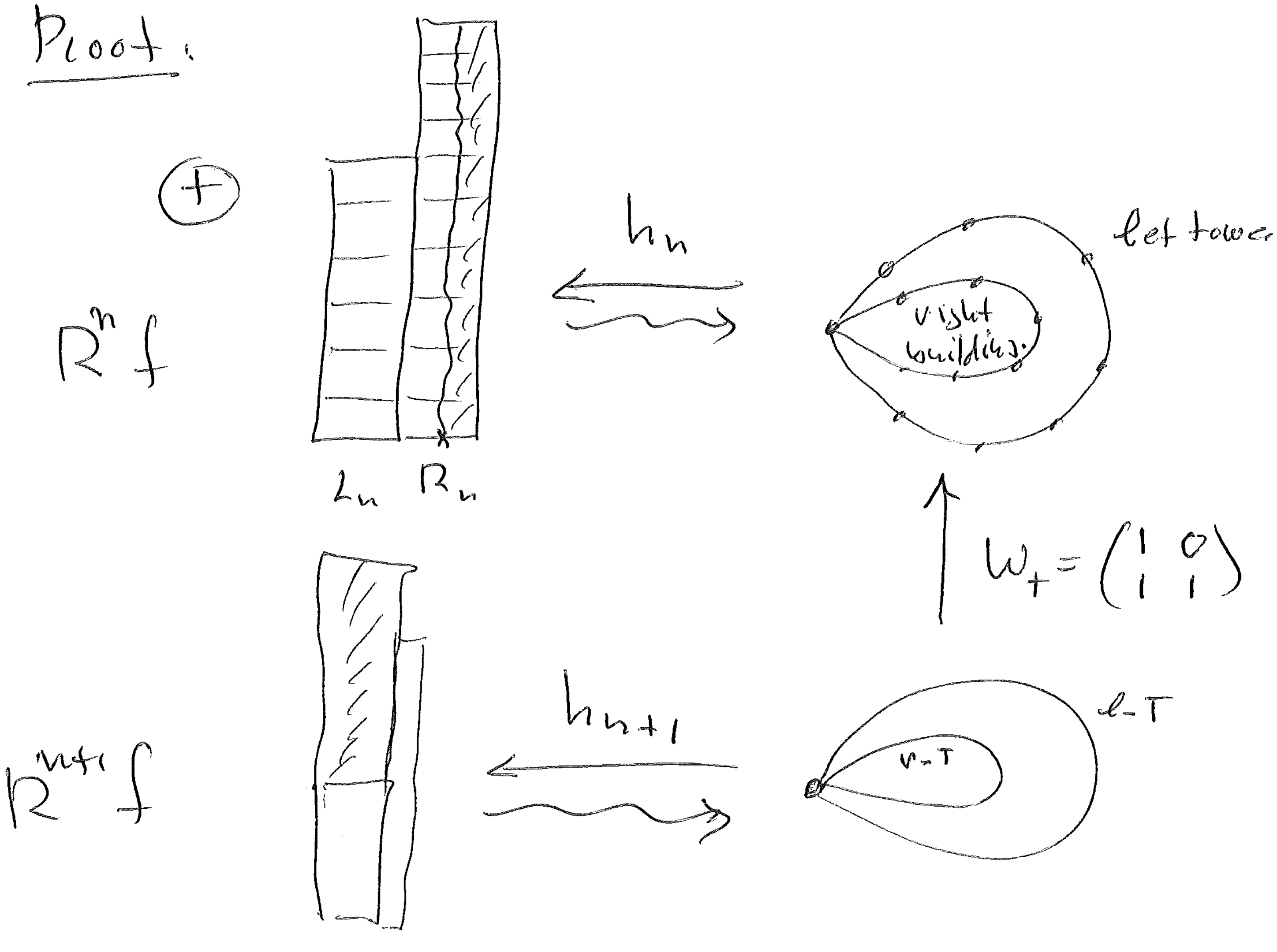
Lemma: The number of __ergodic__ measures $= \dim M$.

* Proof: exercise. $\square$

Thm: $S^1 \circlearrowleft f$ ∞-renormalizable circle map.

f has only 1 invariant probability measure ($\dim M = 1$).

Proof:



$$W_- = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

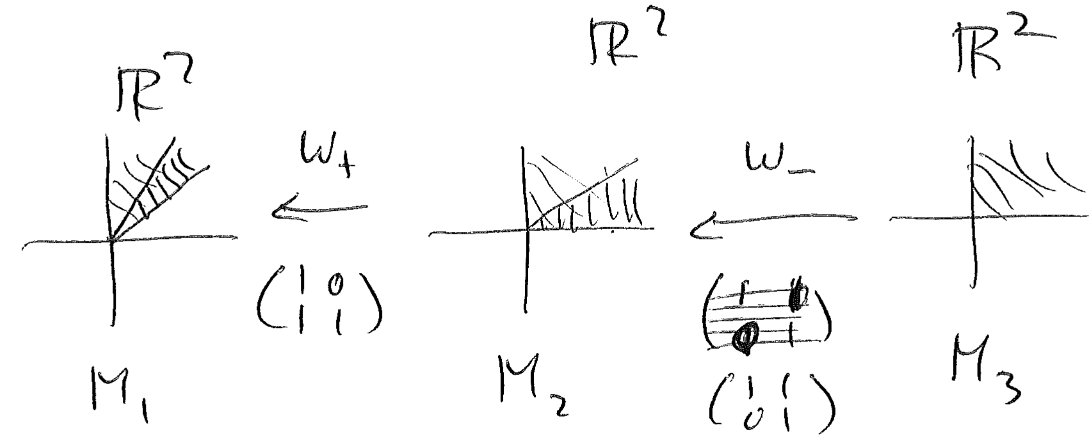
<u>Lemma</u>: $h_n \to h$ continuous map

$$\begin{tikzcd} S^1 \arrow[loop below, "f"] & X = \varprojlim_{w_n} X_n \arrow[l, "h"'] \end{tikzcd}$$

h is 1-1 except on ~~countab~~ the orbit of c.

<u>Proof</u>: exercise $\square$.

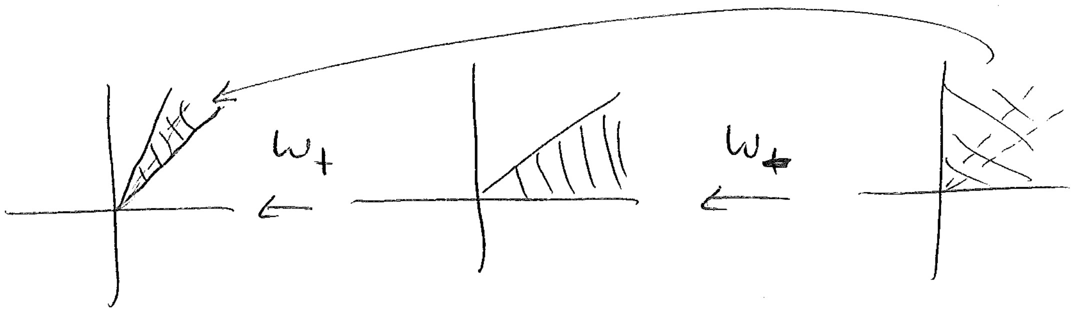
M :



$$W_+ W_- = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$W_- W_+ = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$M = \bigcap_{n \geq 1} w_1 w_2 \cdots w_n (\mathbb{R}^2_+)$$



$w_+ w_-$ (and $w_- w_+$) contracts the cone. In the limit there will be only a line: $\dim M = 1$.*

X is uniquely ergodic

Hence $f: S^1 \circlearrowleft$ has a unique probability measure.

* Use PF-Theorem.

<u>Example</u> $f = R_\rho$: $\nu = \lambda$. (Lebesgue)

$$h_*(\nu) = \lambda.$$

There are cases when

$$\nu \ll \lambda$$

and

$$\nu \perp \lambda \quad \text{(Arnold Example)}$$

This depends on the "quality" of h.