This shows the relation between $\mathbb{R}$ and Bifurcation

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Geometry

Thm (Herman)

$f \in C^3$ circle diffeo.

$\rho_f = [a_n]$

$h : S^1 \longrightarrow S^1$ conjugation

between f and $f^*_{\rho_f}$ (rot)

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If a_n does not go to infinity too fast

then h is $C^{1+\alpha}$

(h is cont. diff. ~~and~~ (C^1)

and

Dh is α-Hölder

$(|Dh(x) - Dh(y)| \leq K|x-y|^\alpha)$

Remark: If a conjugation is differentiable the on small scale it looks affine. This implies that on small scale

The geometry of f- orbits is the same as the geometry of f_p^* orbits.

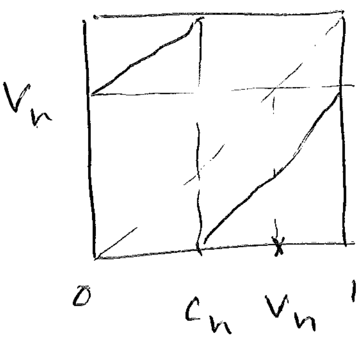
~~This is called: The~~

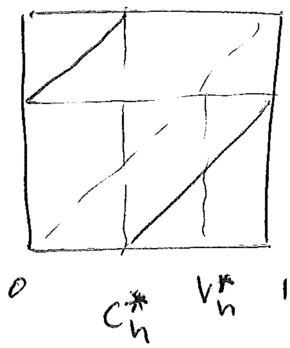
The Dynamics has <u>Rigid</u> geometry.

Proof Herman Theorem (for $a_n \equiv 1$)

Let $f^* = f_{pf}$.

$R^n f$





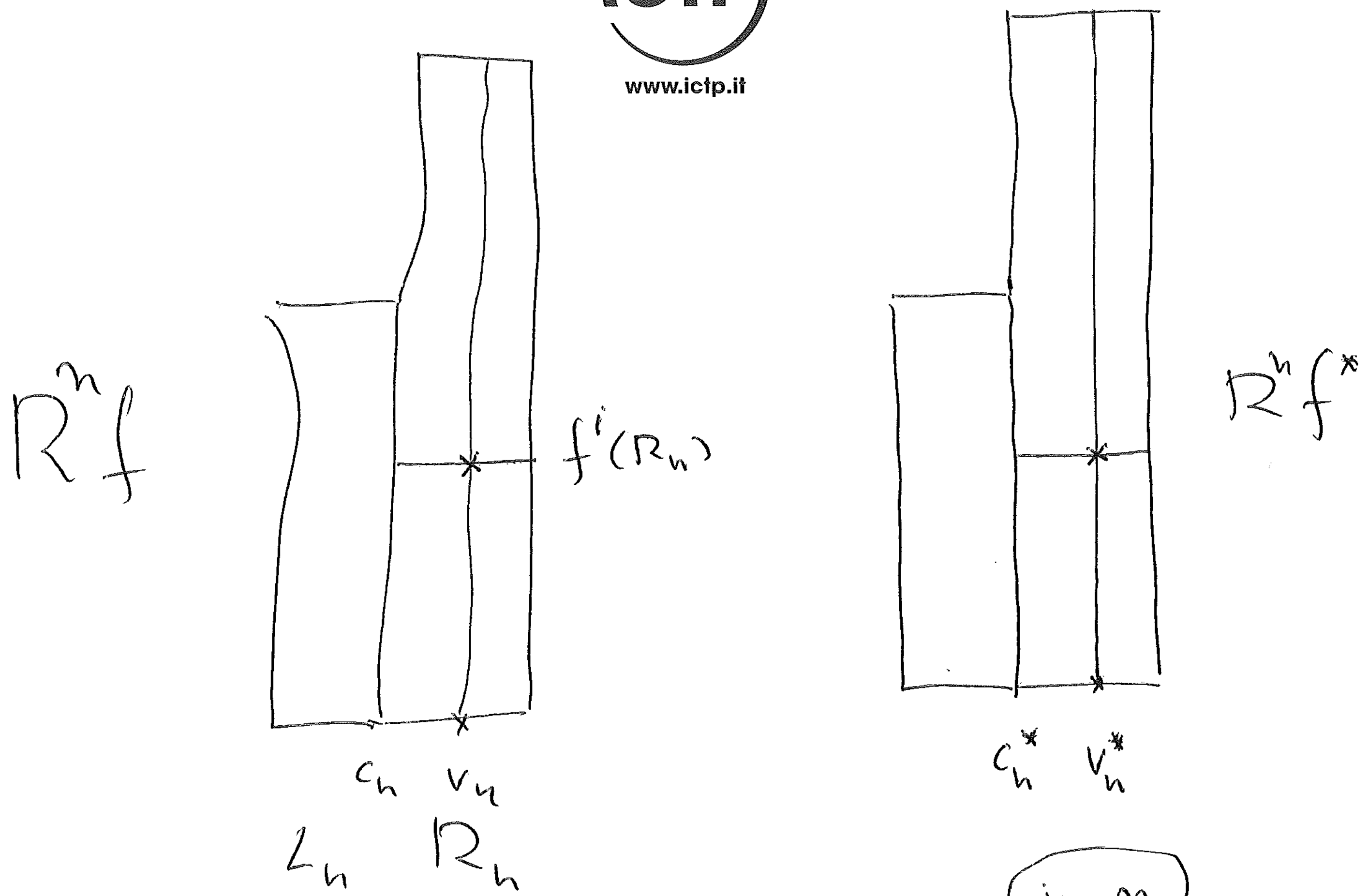
$R^n f^*$

$$|R^n f - R^n f^*| \leq \lambda^n \quad , \lambda < 1$$

So

$$|c_n - c_n^*| \lesssim \lambda^n$$

$$|v_n - v_n^*| \lesssim \lambda^n$$



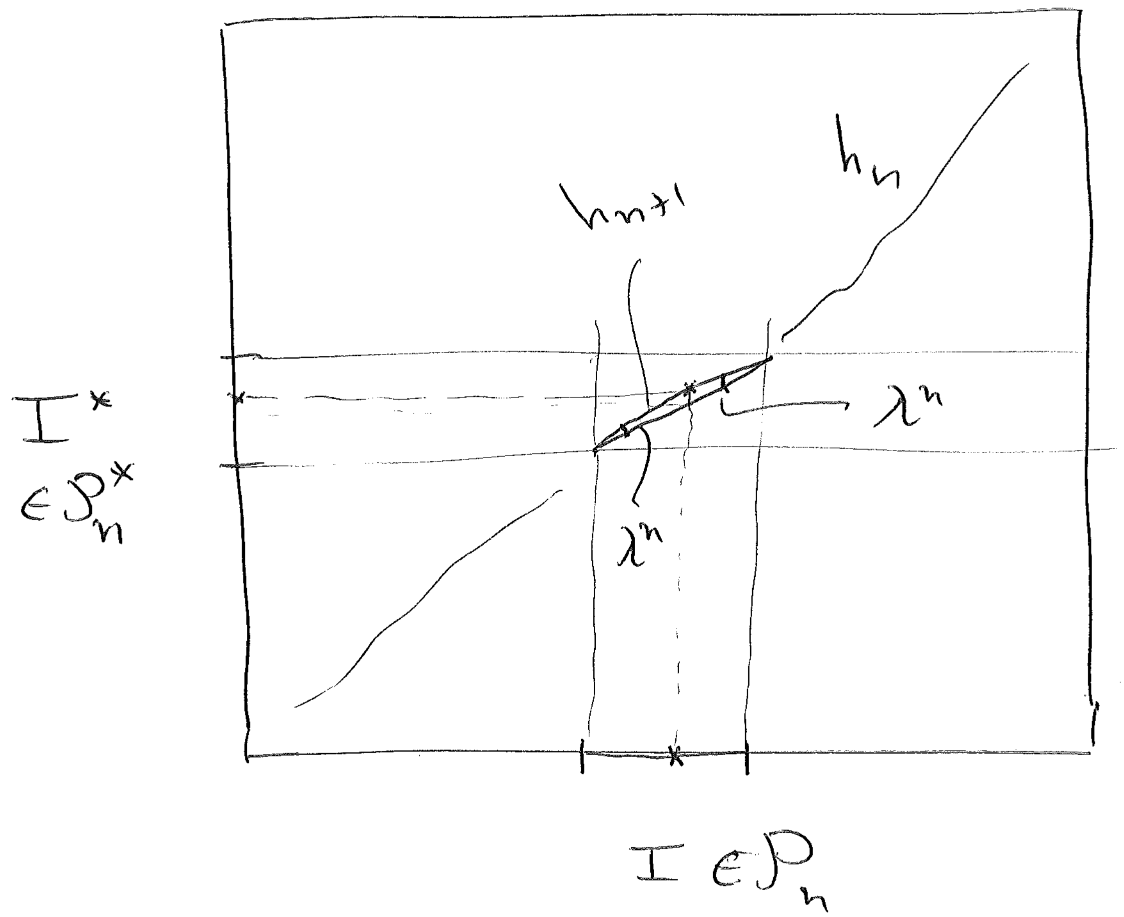
$f^i : R_n \longrightarrow f^i(R_n)$ is exp. close (in n) to affine. $\Longrightarrow$

The cutting of $f^i(R_n)$ happen at the same place as R_n is cut (exp. close to v_n)

But $|v_n - v_n^*| = O(\lambda^n)$

So $f^i(R_n)$ and $(f^*)^i(R_n^*)$ are cut at the same place up to exponential small error.

Draw h_n:



So $\left| Dh_n - Dh_{n+1} \right| = O(\lambda^n)$

$\xrightarrow{\text{Exercise}}$

$Dh_n \longrightarrow Dh \in C^1$

$\square$

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Observe, in the example above, that the invariant measure: $\nu \ll \lambda$

($h_* \nu = \lambda$

or $\nu = (h^{-1})_* \lambda$

where h^{-1} is a diffeo.)

Thm (Arnold):
There exists an analytic circle diffeo. $f: S^1 \circlearrowleft$ with $\rho_f \notin \mathbb{Q}$ (∞-veno) such that

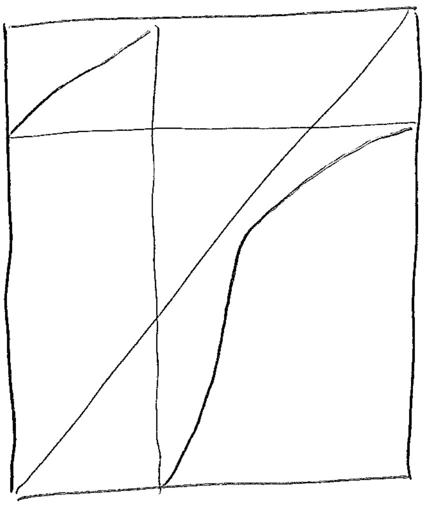
$$\nu \perp \lambda.$$

In particular, the conjugation $h: S^1 \to S^1$ between f and f^*_ρ is not abs. cont.
It sends a set of measure zero to a full measure set.

Idea of the construction:

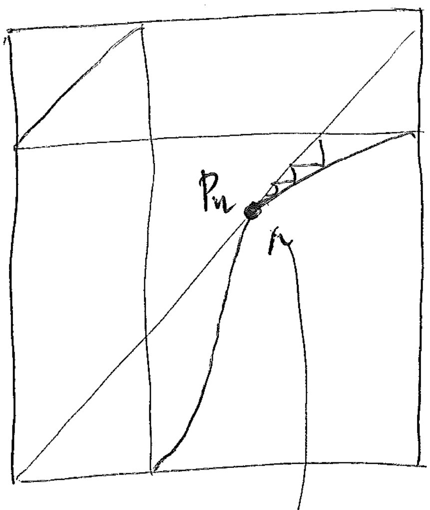
Consider the family $f_t = f + t$ where f is analytic. The construction is by induction: each time one adjusts t a little.

Suppose f_{t_n} is the resulting map after n-steps:



$R^n f_{t_n}$

$\xrightarrow{1^{st}}$ adjustment $t_n \to t'_{n+1}$

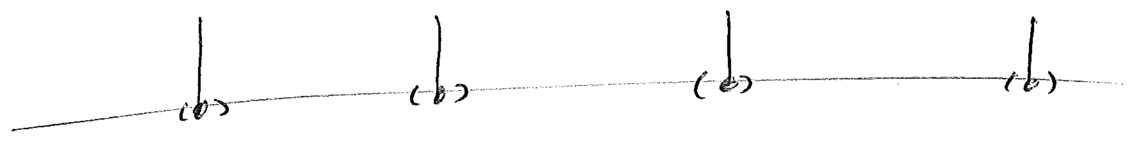


a periodic point is created.

Consider the dynamics of $f_{t'_{n+1}}$:

Choose an arbitrarily small ngh. $\mathcal{U}_n$ of the periodic orbit $Orb(p_n)$. After a high enough number of iterates of $f_{t'_{n+1}}$ 99% of the points will end up in this very small neighborhood $\mathcal{U}_n$.

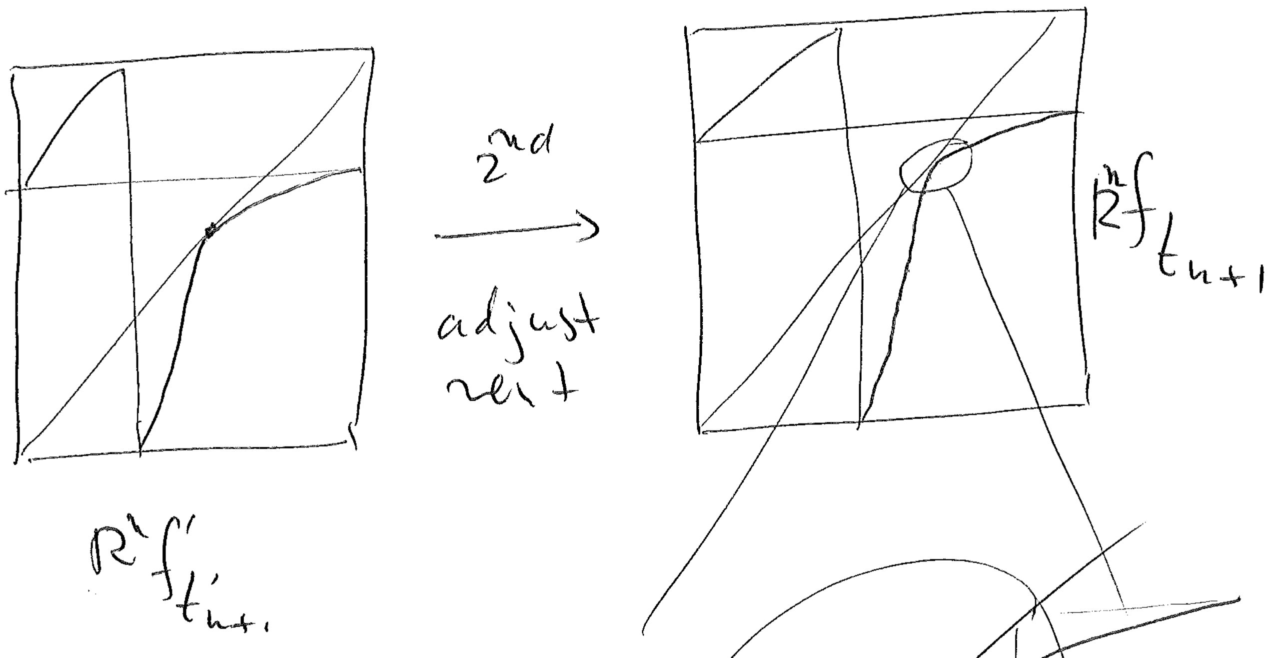
The invariant measure describing the statistics of orbits is the discrete measure on $Orb(p_n)$.

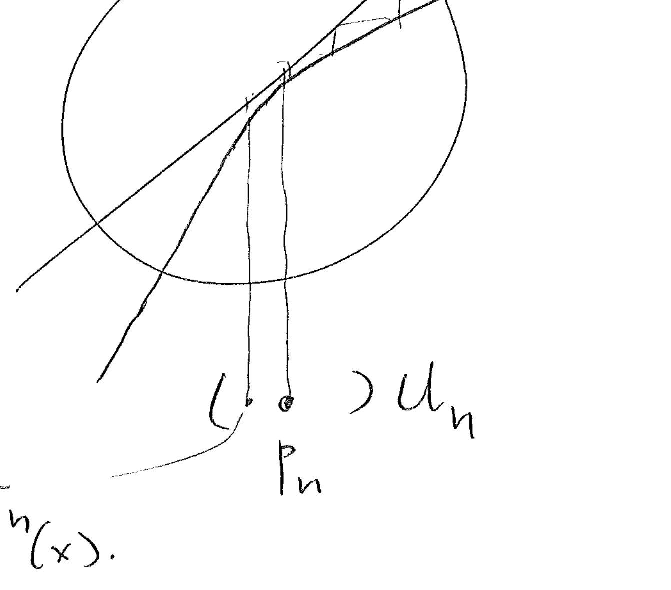
$\delta_{Orb(p_n)}$ 

There is $T_n \gg 1$ such that the distribution of orbits of $f_{t'_{n+1}}$ up to T_n

is essentially given by the discrete measure $\delta_{\mathrm{Orb}(p_n)}$. $\circledast_n$

<u>2^{nd} adjustment</u>: Open up very little such that $\circledast_n$ still holds.





Repeat