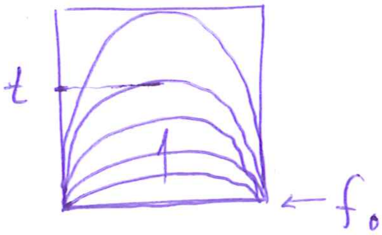
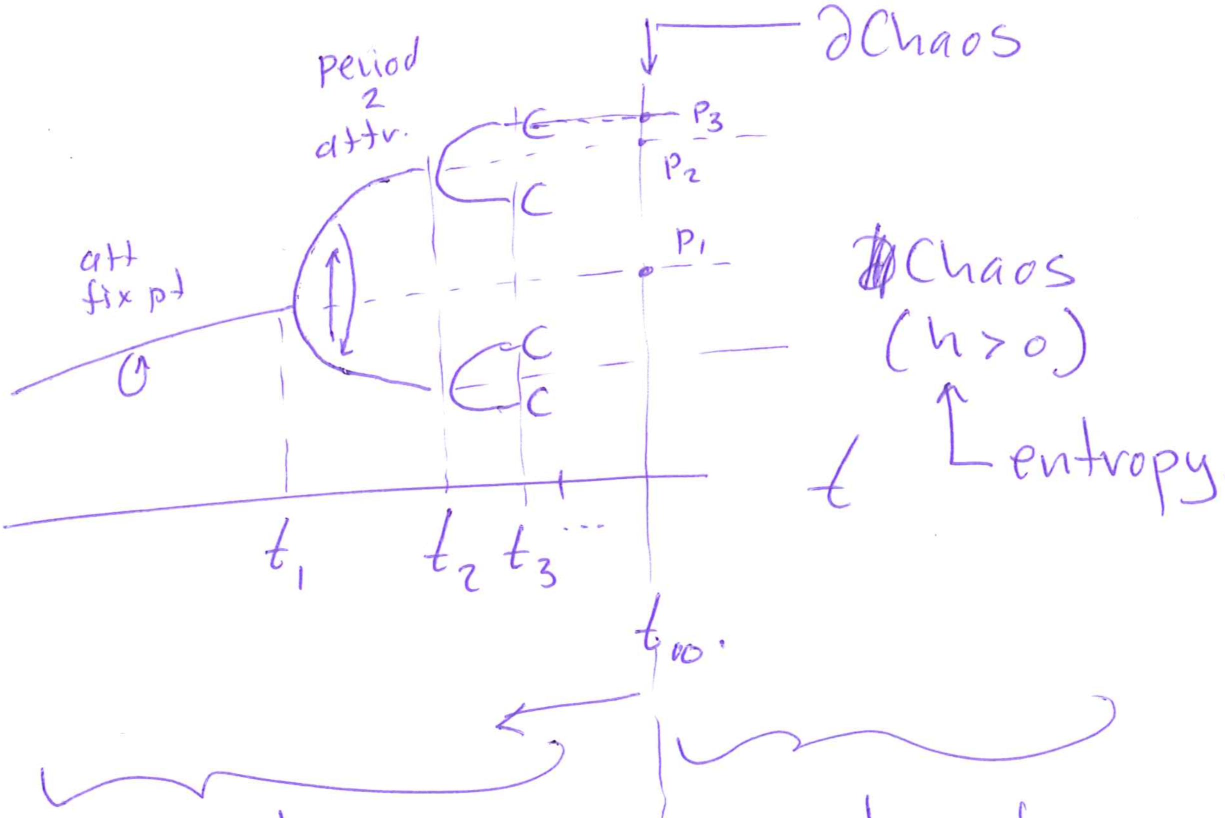
II Unimodal Maps

$f_t : [0,1] \circlearrowleft$



Period Doubling Cascade



number of periodic pts finite

number of periodic pts of period n $\gtrsim e^{hn}$, $h>0$

The accumulation pt of a period doubling cascade in in the Boundary of Chaos

periodic attractors

$h > 0: \#P_n \geq e^{hn}$

number of period pts of period n

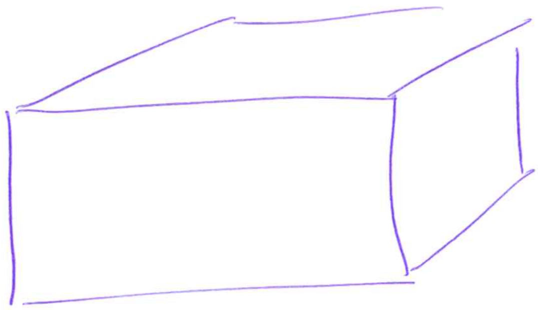
t_{∞}

no Chaos

Chaos

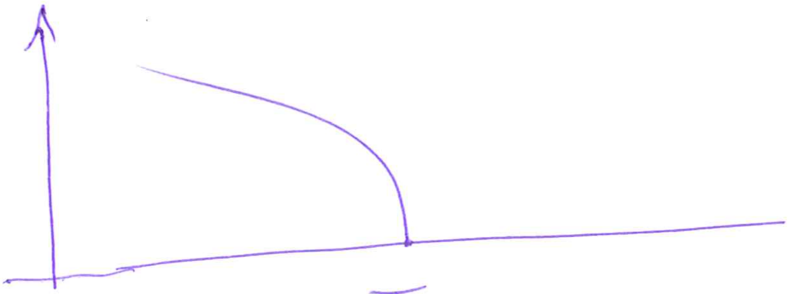
"Phase Transition"

Phase Transition is Statistical Mechanics



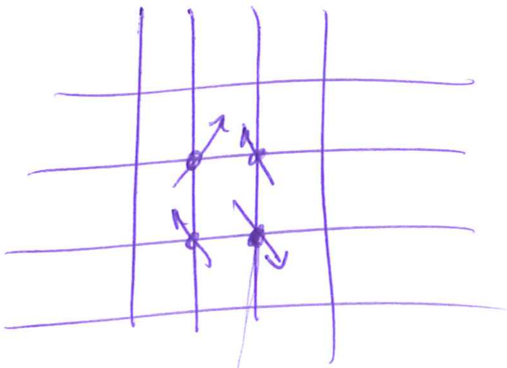
Fe

Magnetisation



$$M_T \approx |T - T_c|^{0.3} \qquad T < T_c.$$

"Explanation": interplay between spins and ~~[illegible]~~ kinetic vibration of the atoms



each atom has a spin and vibrates.

when the temperature goes down the vibration slows down and at some point T_c the spins align suddenly to form a magnet.

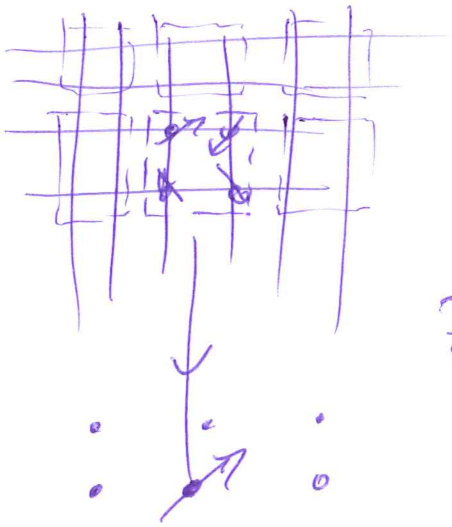
Other metal : from large class

0.3 is again the same

Universality of the critical exponent

Renormalization is used to explain this Universality Phenomenon.

R is Stat. Mech. is "Blocking"



Zooming Out.

Four random spins are replace by one random spin on a larger scale.

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Coullet-Tresser / Feigenbaum:

"let's look at the phase transition from non-chaos to chaos. The we should see Universality"

Observations they made.

$$t_n \xrightarrow{\text{exp.}} t_* \quad \text{rate} = \frac{1}{4.669\ldots}$$

parameter universality

(independent of the family as long as $D^2 f(c) < 0$).

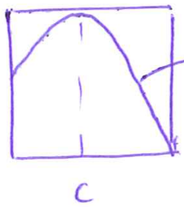
Also

$$p_n \xrightarrow{\text{exp}} p_* \quad \text{rate} \ \frac{1}{(2.66)^2}$$

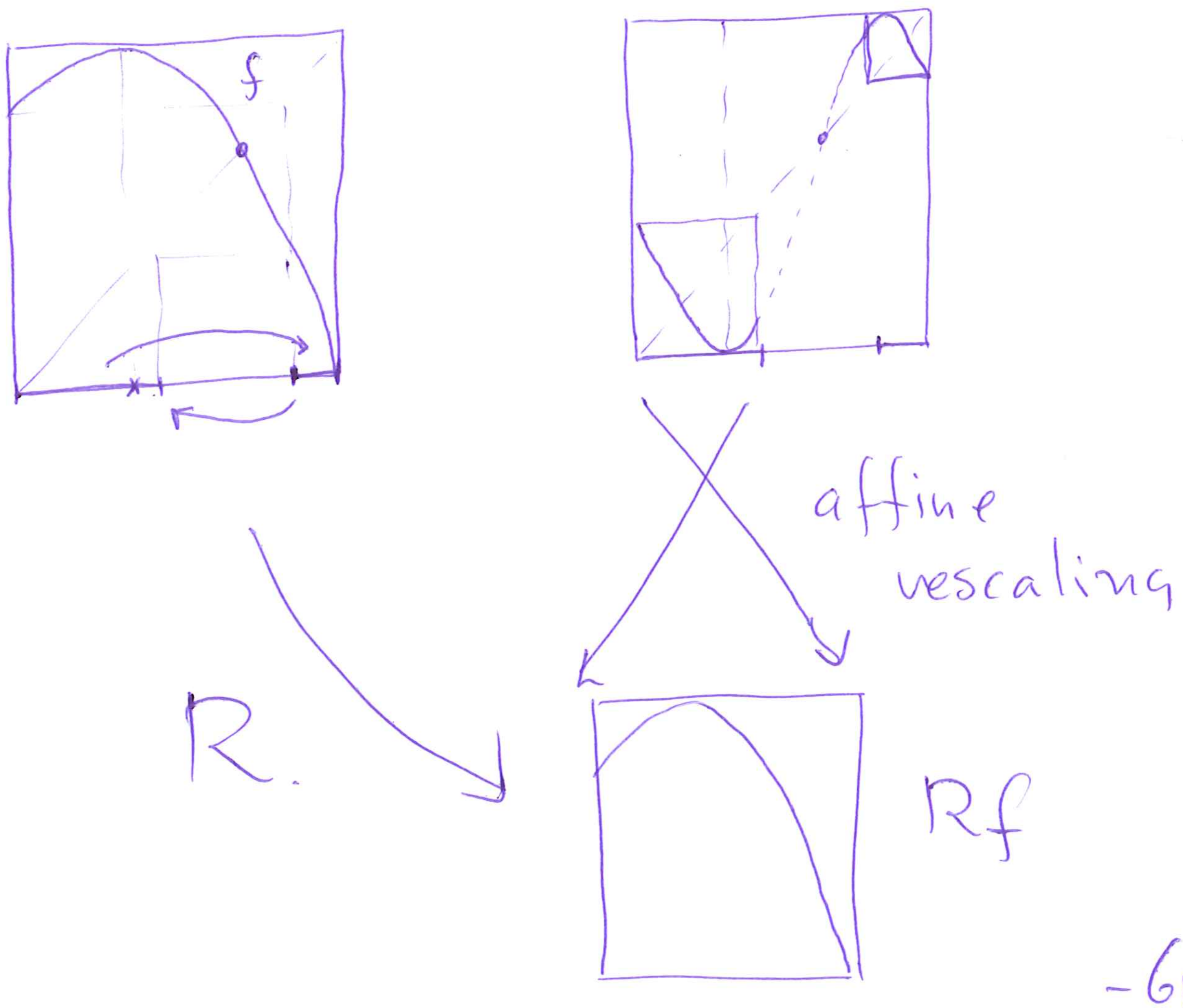
Universality in Phase Space

(independent of the family as long as $D^2 f(c) < 0$).

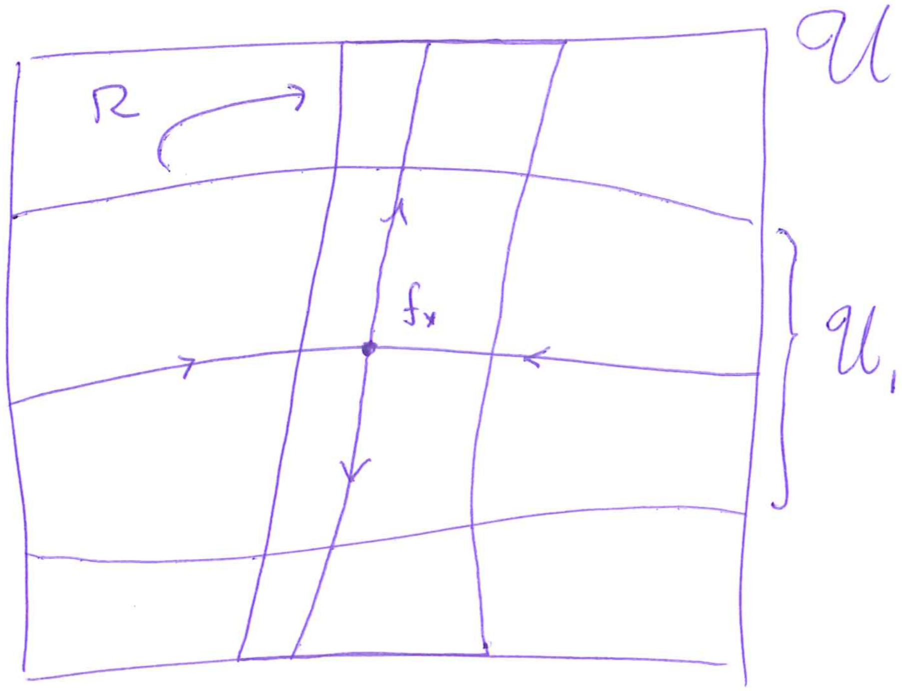
Renormalization for Unimodal maps (of Period Doubling type)

$\mathcal{U} = \{ f : [0,1] \circlearrowleft \mid$  $C^3,\ D^2 f(c) < 0 \}$.

$\mathcal{U}_1 = \{ f \in \mathcal{U} \mid \text{renormalizable} \}$:

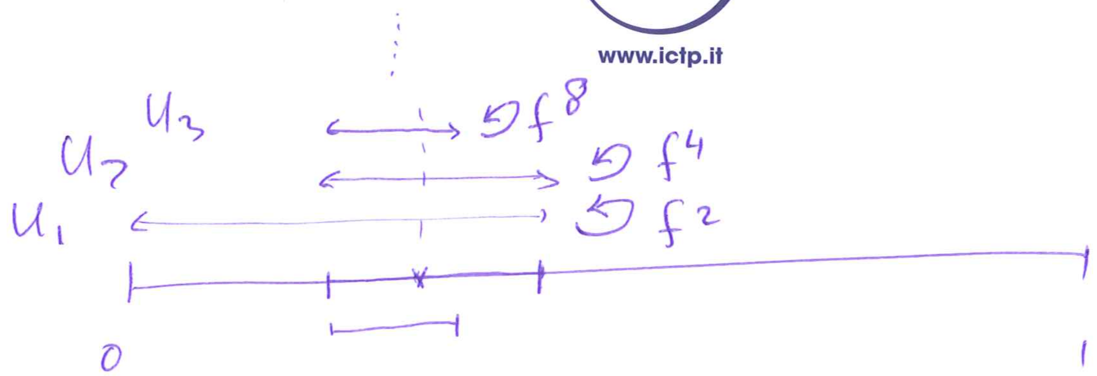


$R : \mathcal{U}_1 \longrightarrow \mathcal{U}$



<u>Thm</u>: R has a unique fixed pt f_*.

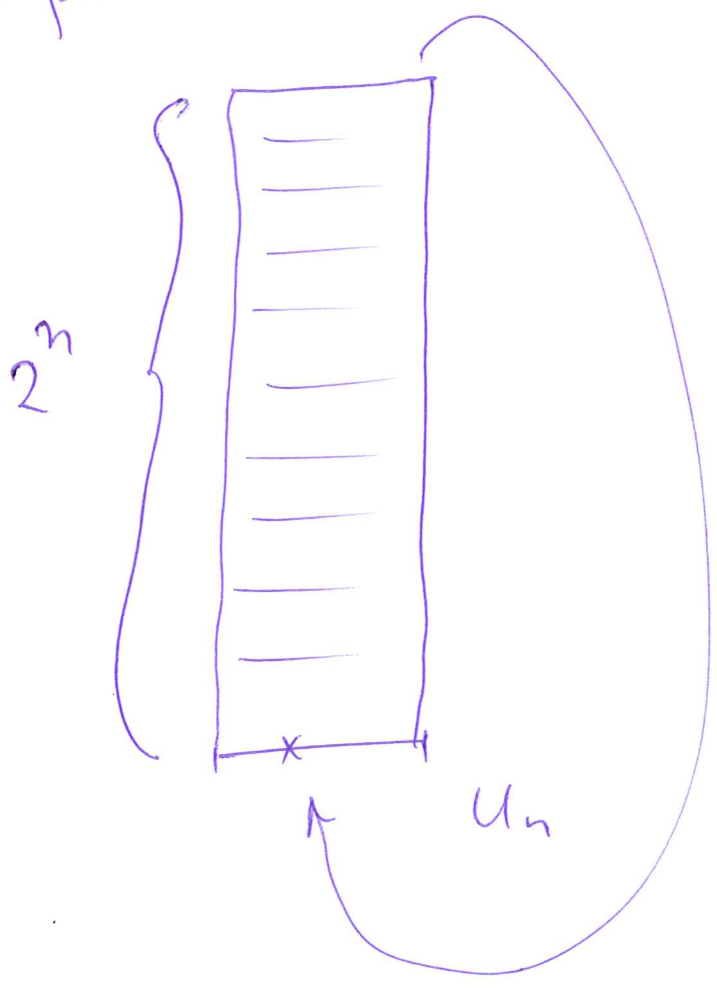
- f_* is hyperbolic.
- $\dim W^u_{f_*} = 1$ $\qquad \delta = 4.669\ldots$
- $\operatorname{codim} W^s_{f_*} = 1$
- $W^s_{f_*} = \{\infty\text{-reno}\} = \partial\,\text{Chaos}.$



So $R^n f$ is a rescaled version of

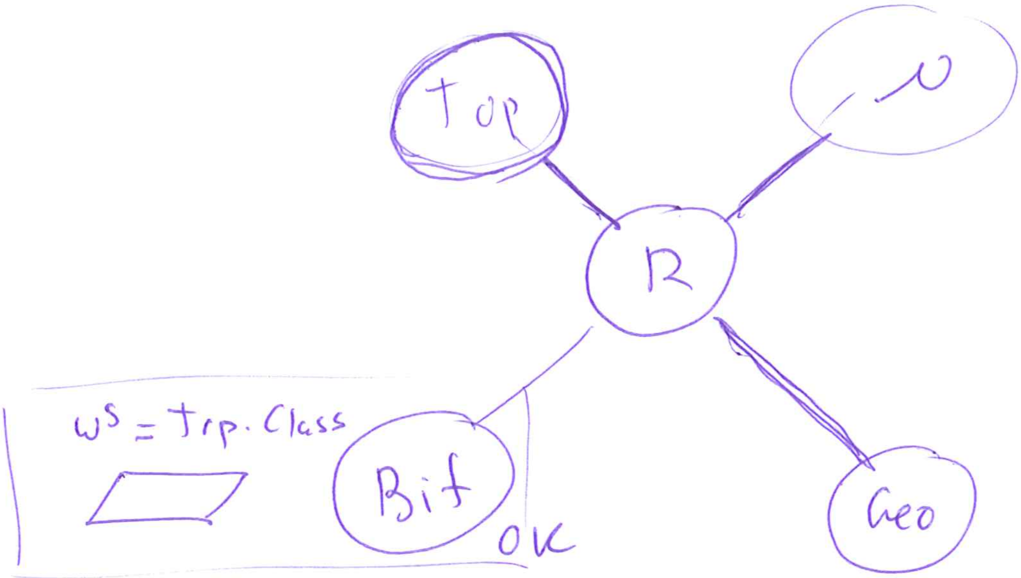
$$f^{2^n} : U_n \circlearrowleft$$

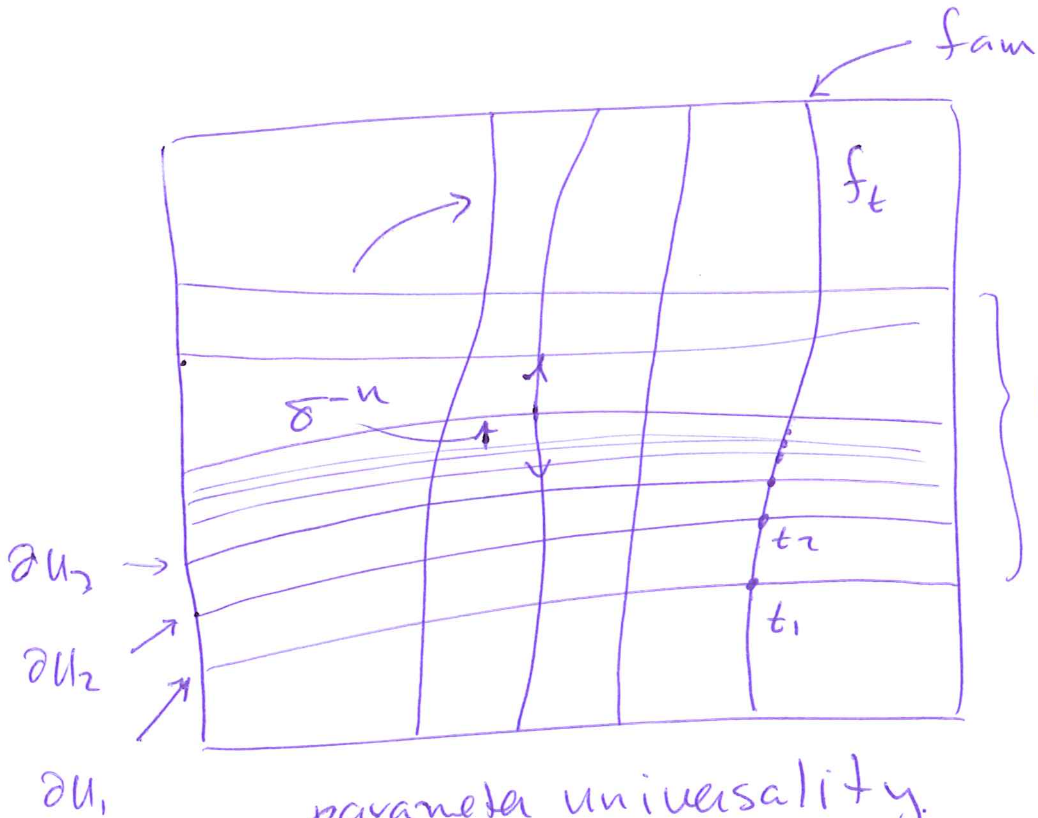
Simple Tower



$\mathcal{P}_n$: Dynamical partition.

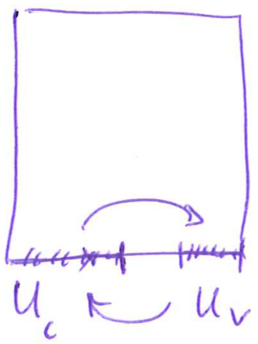
This Theorem has a long history:
Coullet Tresser & Feigenbaum, Epstein & Lanford,
Sullivan, McMullen, M, Avila & Lyubich

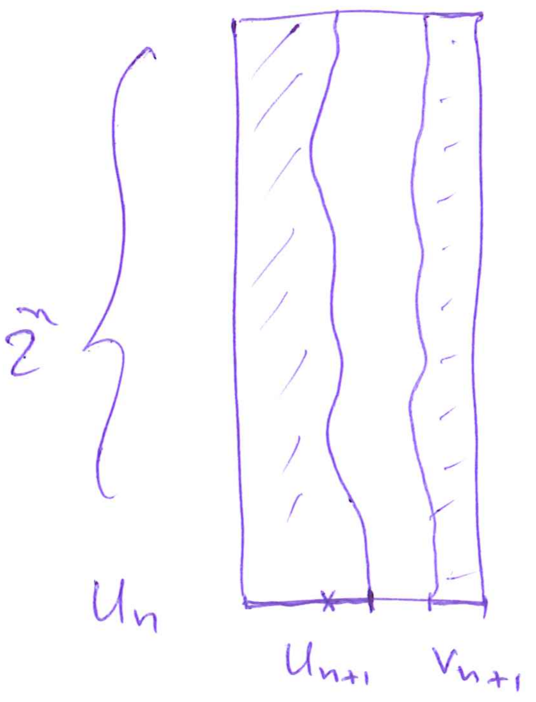


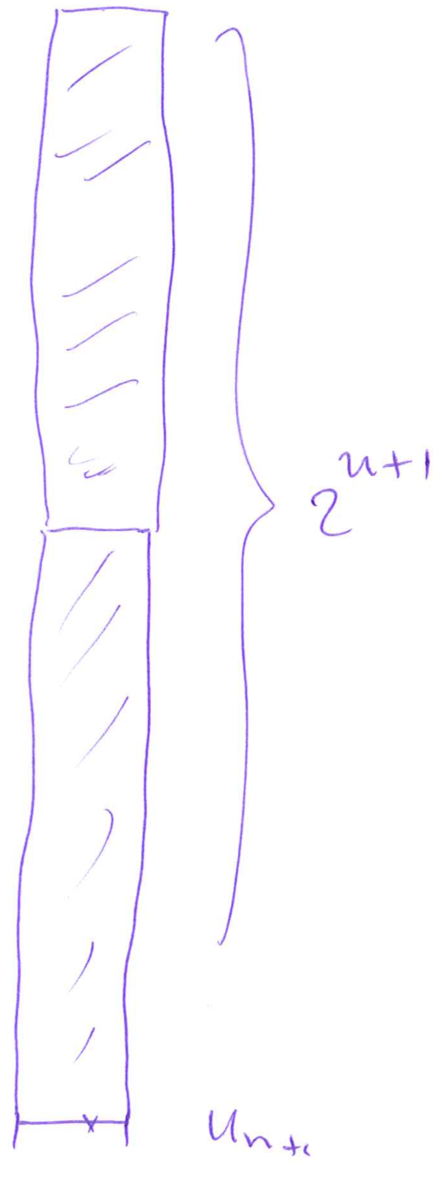


parameter universality

How does
Renormalization acts on Towers.

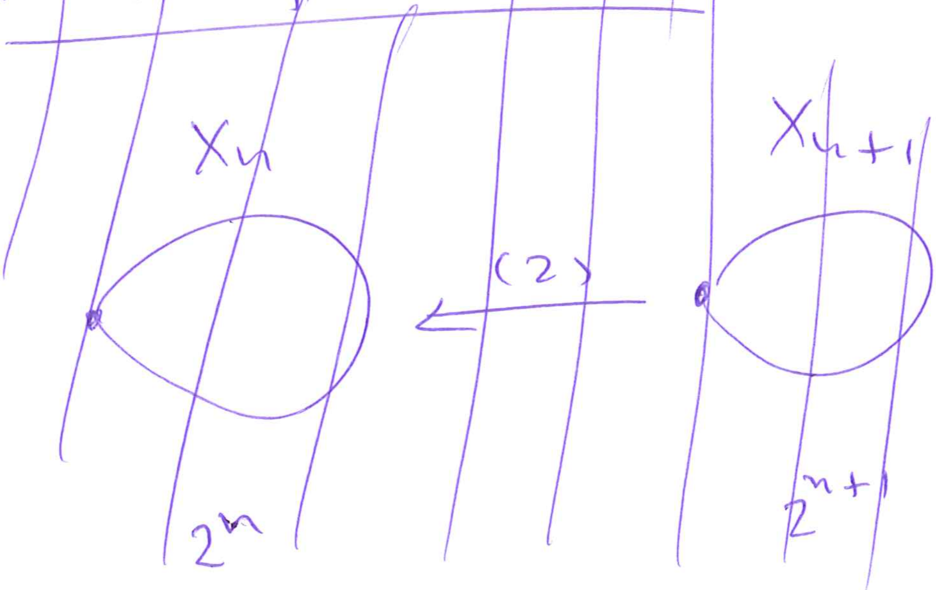






Remark 1: in the case of circle diffeos: the limits of renormalization are rotation. and the limiting towers are affine-towers. What makes unimodal renormalization difficult is that the limiting maps are not affine or simple.

Measure Theory



Remark 2: The "dynamical partition" $\mathcal{P}_n$ is not a partition of $[0,1]$. It consists of disjoint intervals but they don't fill up $[0,1]$. For obvious reasons they are called cycles $\mathcal{C}_n$ $(= \mathcal{P}_n)$.

Thm: f is ∞-renormalizable.

- $$\mathcal{C} = \bigcap \mathcal{C}_n$$

  Cantor set.
- $\forall n \ \exists!$ periodic orbit of period 2^n. P_n
- $x \notin \bigcup P_n$: $\omega(x) = \mathcal{C}$.

Topology

Thm: if f, g are ∞-reno.

then $f \sim_{top} g$:

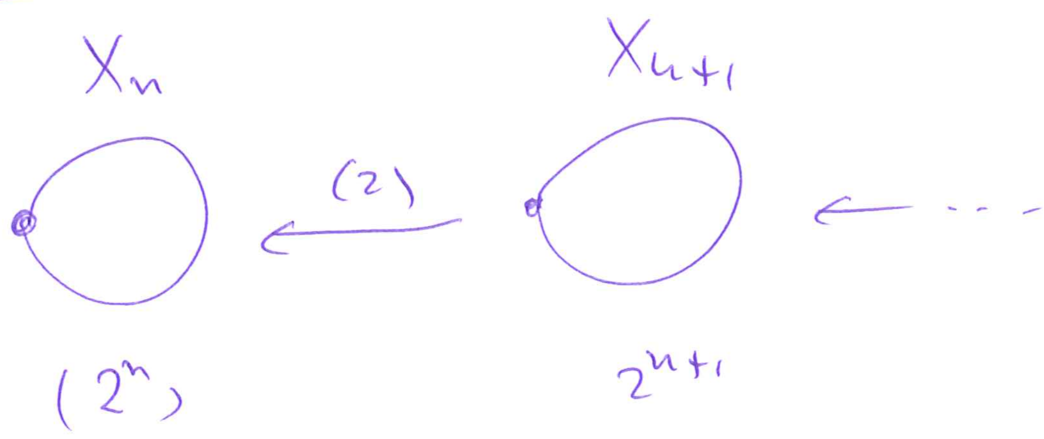
$$\begin{tikzcd} {[0,1]} \arrow[r, "h"] \arrow[loop below, "f"] & {[0,1]} \arrow[loop below, "g"] \end{tikzcd}$$

$$h \circ f = g \circ h.$$

Measure Theory

Thm: $\mathcal{C}$ is uniquely ergodic.

Pf



$\square$.