Lyapunov Exponents of Linear Cocycles*

Continuity via Large Deviations

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The definition of a linear cocycle

Given

- an ergodic system (X, μ, T), where
  - (X, μ) is a probability space and
  - $T\colon X \to X$ is an ergodic map
- a measurable function $A : X \to \mathrm{Mat}(m, \mathbb{R})$

we call linear cocycle over T the skew-product map $F : X \times \mathbb{R}^m \to X \times \mathbb{R}^m$ defined by

$$F(x, v) = (Tx,\, A(x)v)$$

This map defines a new dynamical system on the bundled space $X \times \mathbb{R}^m$, and its iterates are $F^n(x, v) = (T^n x,\, A^{(n)}(x)v)$, where

$$A^{(n)}(x) := A(T^{n-1}x) \cdot \ldots \cdot A(Tx) \cdot A(x)$$

We will fix the base dynamics T and identify the cocycle F with the function A defining its fiber action.

Example: random (i.i.d.) cocycles

- Base dynamics: (X, μ, T) is a Bernoulli shift i.e. given a probability space of symbols (Σ, ν), we put

  $X = \Sigma^{\mathbb{Z}}$,
  $\mu = \nu^{\mathbb{Z}}$ and
  if $x = \{x_k\}_{k\in\mathbb{Z}} \in X$ then $Tx = \{x_{k+1}\}_{k\in\mathbb{Z}}$.

- Fiber action: $A\colon X \to \mathrm{GL}(m, \mathbb{R})$ locally constant i.e. it depends on a (fixed) finite number of coordinates:
  If $x = \{x_k\}_{k\in\mathbb{Z}}$, then $A(x) = \widehat{A}(x_0)$ for some measurable map $\widehat{A}\colon \Sigma \to \mathrm{GL}(m, R)$.
  More generally,

$$A(x) = \widehat{A}(x_0, x_1, \ldots, x_{l-1})$$

  for some measurable map $\widehat{A}\colon \Sigma^l \to \mathrm{GL}(m, R)$.

Related example: Markov cocycles.

Example: quasi-periodic cocycles

- Base dynamics: (X, μ, T) is a torus translation
  $X = \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ ($d = 1$ or $d > 1$),
  $\mu =$ Haar measure and
  $Tx = x + \omega$ for some ergodic translation $\omega \in \mathbb{T}^d$.

- Fiber action: $A \colon \mathbb{T}^d \to \mathrm{Mat}(m, \mathbb{R})$ real analytic
  hence holomorphic in a neighborhood of the torus.

Other examples: cocycles over a skew-translation or over a hyperbolic toral automorphism.

The Lyapunov exponents of a linear cocycle

Consider a base dynamical system (X, μ, T) and a linear cocycle $A\colon X \to \mathrm{Mat}(m, \mathbb{R})$ s.t. $\log^+ \|A\| \in L^1(d\mu)$. Let $A^{(n)}(x) := A(T^{n-1}x) \dots A(Tx)\, A(x)$ be its iterates.

By Kingman's ergodic theorem, for every $1 \leqslant k \leqslant m$, the following limit exists:

$$L_k(A) := \lim_{n\to\infty} \frac{1}{n} \log s_k(A^{(n)}(x)) \quad \mu \text{ a.e. } x \in X$$

$$= \lim_{n\to\infty} \int_X \frac{1}{n} \log s_k(A^{(n)}(x))\, \mu(dx)$$

and it is called the k-th Lyapunov exponent of A.

We have

$$\infty > L_1(A) \geqslant L_2(A) \geqslant \dots \geqslant L_m(A) \geqslant -\infty$$

The continuity problem

Fix a base ergodic system (X, μ, T).
Consider a metric space $\mathcal{C}_m$ of measurable cocycles $A\colon X \to \mathrm{Mat}(m, \mathbb{R})$.

Establish continuity properties (or lack thereof) for the Lyapunov exponents, i.e. for the maps

$$\mathcal{C}_m \ni A \to L_k(A) \in [-\infty, \infty)$$

Establish quantitative continuity properties, i.e. a modulus of continuity (e.g. Hölder, weak-Hölder) under further appropriate assumptions.

Nota bene: The continuity of the Lyapunov exponents depends on the space of cocycles and its topology, and in some sense is rare (J. Bochi), unless strong regularity assumptions on the cocycles are made.

People that have worked on this problem

Random cocycles:

- H. Furstenberg and Y. Kifer
- E. Le Page
- C. Bocker-Neto and M. Viana
- E. Malheiro and M. Viana
- A. Ávila, A. Eskin and M. Viana
- P. Duarte and S. K.

Quasi-periodic cocycles:

- M. Goldstein and W. Schlag
- J. Bourgain and S. Jitomirskaya
- C. Marx and S. Jitomirskaya
- A. Ávila, S. Jitomirskaya and C. Sadel
- P. Duarte and S. K.

An abstract, general approach to proving continuity properties of Lyapunov exponents

We devise an abstract scheme to prove quantitative continuity of the Lyapunov exponents, one that is applicable to different types of base dynamics, e.g.

- Quasi-periodic models
- Random models
- Markov models

  . . .

The main assumption required by our scheme is that some 'appropriate' large deviation type (LDT) estimates are available.

This scheme relies upon ideas introduced in the context of quasi-periodic Schrödinger cocycles in

Reference: M. Goldstein and W. Schlag, Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions, Annals of Math, 2001.

LDT (i.e. concentration of measure) estimates in classical probability theory

Let $X_0, X_1, X_2, \ldots$ be a real valued independent random process, and let

$S_n = \sum_{j=0}^{n-1} X_j$ be the corresponding sum process.

Hoeffding's inequality: If $|X_i| \leqslant C$ a.s. then

$$\mathbb{P}\left[\left| \frac{1}{n} S_n - \mathbb{E}(\frac{1}{n} S_n) \right| > \varepsilon \right] \lesssim e^{-c(\varepsilon)\, n}$$

holds for all n, where $c(\varepsilon) \sim C^{-2}\, \varepsilon^2$.

Compared to the classical large deviation principle of Cramér, this statement is less sharp / precise, but it is more general and it is uniform in the data.

Introducing our 'appropriate' version of large deviations

Let (X, μ, T) be an ergodic base dynamical system.

An observable is any measurable map $\xi \colon X \to \mathbb{R}$. Denote

$$S_n\, \xi(x) := \sum_{j=0}^{n-1} \xi(T^j x) \quad \text{and} \quad \langle \xi \rangle := \int_X \xi(x) \mu(dx)$$

We say that the observable ξ satisfies a **base LDT** if for all $\varepsilon > 0$ there is $n_{00} \in \mathbb{N}$ such that for all $n \geqslant n_{00}$

$$\mu\{x \in X \colon \left| \frac{1}{n} S_n\, \xi\,(x) - \langle \xi \rangle \right| > \varepsilon\} < \iota(\varepsilon, n)$$

where as $n \to \infty$, $\iota(\varepsilon, n)$ decreases to 0 fast (e.g. exponentially or sub-exponentially).

We call the threshold n_{00} and the rate of decay $\iota(\varepsilon, n)$ LDT parameters.

Introducing our 'appropriate' version of large deviations

Let (X, μ, T) be an ergodic base dynamical system.

Let $A \colon X \to \mathrm{Mat}(m, \mathbb{R})$ be a linear cocycle over T. Recall the notations

$$A^{(n)}(x) := A(T^{n-1}x) \, \ldots \, A(Tx) \, A(x)$$

$$L_1^{(n)}(A) := \int_X \frac{1}{n} \log\|A^{(n)}(x)\| \mu(dx)$$

We say that the cocycle A satisfies a **fiber LDT** with parameters n_{00} and $\iota(\varepsilon, n)$ if for all $n \geqslant n_{00}$ we have:

$$\mu\{x \in X \colon \left| \frac{1}{n} \log\|A^{(n)}(x)\| - L_1^{(n)}(A) \right| > \varepsilon\} < \iota(\varepsilon, n)$$

The fiber LDT is called uniform if the estimate above holds with the same parameters for all nearby cocycles.

The abstract continuity pseudo-theorem

Given are an ergodic dynamical system (X, μ, T) and a metric space $\mathcal{C}_m$ of measurable cocycles $A \colon X \to \mathrm{Mat}(m, \mathbb{R})$, satisfying some natural integrability assumptions.

We assume the following:

- ◇ A base LDT holds for a 'large enough' set of observables ξ.
- ◇ A uniform fiber LDT holds for all cocycles A with $L_1(A) > L_2(A)$.

Then each Lyapunov exponent is a continuous function of the cocycle.

Moreover, if $L_1(A) > L_2(A)$, then near A, the maximal Lyapunov exponent has a modulus of continuity whose strength depends on the sharpness of the LDT estimate, i.e. on how fast the sequence $\{\iota(\varepsilon, n)\}$ decays.

Applicability of the abstract continuity theorem

We have derived the appropriate LDTs for

- ⋄ Quasi-periodic models under progressively more general assumptions (but only for Diophantine translations).
- ⋄ Random models (both Bernoulli and Markov shifts of a very general type) under an irreducibility condition.
- ⋄ LDTs are available for spaces of cocycles with other types of base dynamics, but with greater limitations.

The abstract continuity result then applies to all of these models.

In particular this leads to continuity properties of the Lyapunov exponents as functions of the energy, for different types of discrete Schrödinger-like operators.

The abstract continuity theorem, deriving appropriate large deviations, a general avalanche principle and more

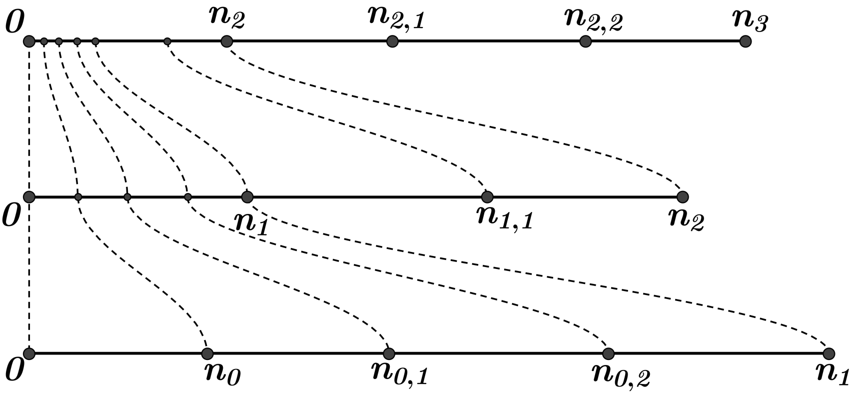
Reference:

P. Duarte and S.K., Lyapunov exponents of linear cocycles: continuity via large deviations, Atlantis Studies in Dynamical Systems, 2016.

P Duarte and S.K., Large deviation type estimates for iterates of linear cocycles, a survey to appear in Stochastics and Dynamics, 2016.

P Duarte and S.K., various recent preprints on the arXiv.

The proof of continuity



Blocks of different time scales in an inductive procedure.

The Avalanche Principle: ideal situation

Consider a long chain of matrices $g_0, g_1, \ldots, g_{n-1}$ in $\mathrm{Mat}(m, \mathbb{R})$.

In general it is not true that

$$\|g_{n-1} \, \ldots \, g_1 \, g_0\| \asymp \|g_{n-1}\| \, \ldots \, \|g_1\| \, \|g_0\|$$

unless some atypically sharp alignment of the singular directions of the matrices g_j occurs.

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. . . but let us proceed as if it were true.
Hence the above product linearizes:

$$\log \|g_{n-1} \,\ldots\, g_1\, g_0\| \asymp \log \|g_{n-1}\| + \ldots + \log \|g_1\| + \log \|g_0\|$$

The Avalanche Principle: definitions

We call gap ratio of a matrix $g \in \mathrm{Mat}(m, \mathbb{R})$ the quotient

$$\rho(g) = \frac{s_1(g)}{s_2(g)}$$

between its largest and second largest singular values.

Given $g_0, g_1 \in \mathrm{Mat}(m, \mathbb{R})$, we call the number

$$\theta(g_0, g_1) := \frac{\|g_1\, g_0\|}{\|g_1\|\, \|g_0\|} \in [0, 1]$$

the expansion rift of g_0, g_1. It measures the break of expansion in the matrix product $g_1\, g_0$.
More generally, we define the expansion rift of a chain of matrices $g_0, g_1, \ldots, g_{n-1}$ in $\mathrm{Mat}(m, \mathbb{R})$ to be the number

$$\theta(g_0, g_1, \ldots, g_{n-1}) := \frac{\|g_{n-1}\, \ldots\, g_1\, g_0\|}{\|g_{n-1}\|\, \ldots\, \|g_1\|\, \|g_0\|}$$

The Avalanche Principle: formulation

With this terminology, the AP says that given any long chain of matrices $g_0, g_1, \dots, g_{n-1} \in \mathrm{Mat}(m, \mathbb{R})$, where

- the gap ratio of each matrix is large and
- the expansion rift of any pair of consecutive matrices is never too small,

then the expansion rift of the product is almost multiplicative:

$$\theta(g_0, g_1, \dots, g_{n-1}) \asymp \theta(g_0, g_1)\, \theta(g_1, g_2)\, \dots\, \theta(g_{n-2}, g_{n-1})$$

Equivalently,

$$\frac{\|g_{n-1}\, \dots\, g_1\, g_0\|\, \|g_1\|\, \dots\, \|g_{n-2}\|}{\|g_1\, g_0\|\, \dots\, \|g_{n-1}\, g_{n-2}\|} \asymp 1$$

or

$$\|g_{n-1}\, \dots\, g_1\, g_0\| \asymp \frac{\|g_1\, g_0\|\, \dots\, \|g_{n-1}\, g_{n-2}\|}{\|g_1\|\, \dots\, \|g_{n-2}\|}$$