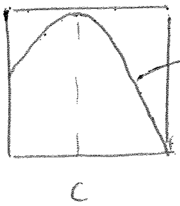
Reference for R of circle diffeos:

J. Stark: smooth conjugacy and renormalization for diffeos of the circle, Nonlinearity **1** (1988), 541–575.

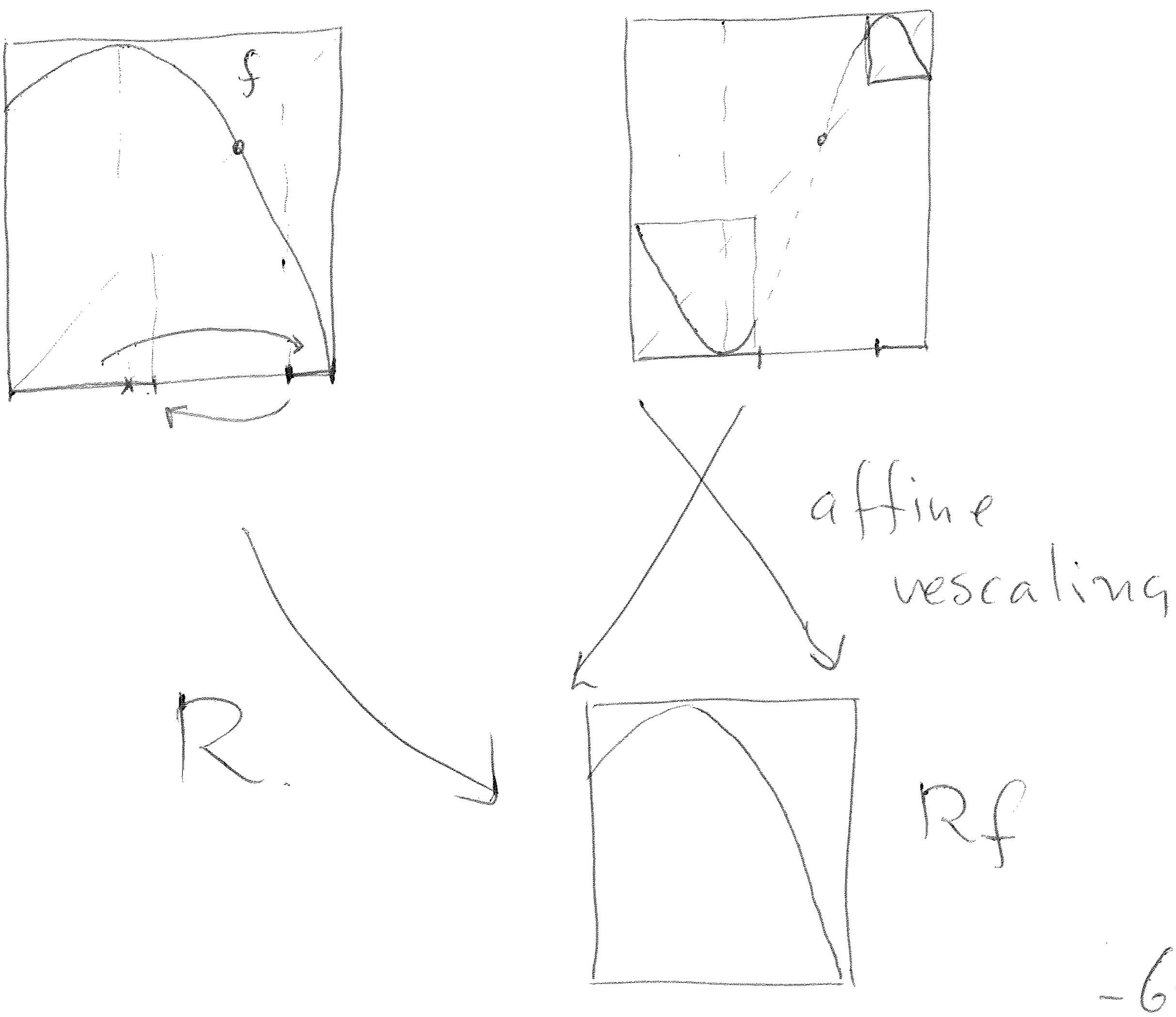
Renormalization for Unimodal maps (of Period Doubling type)

$$\mathcal{U} = \{ f : [0,1] \circlearrowleft \mid$$

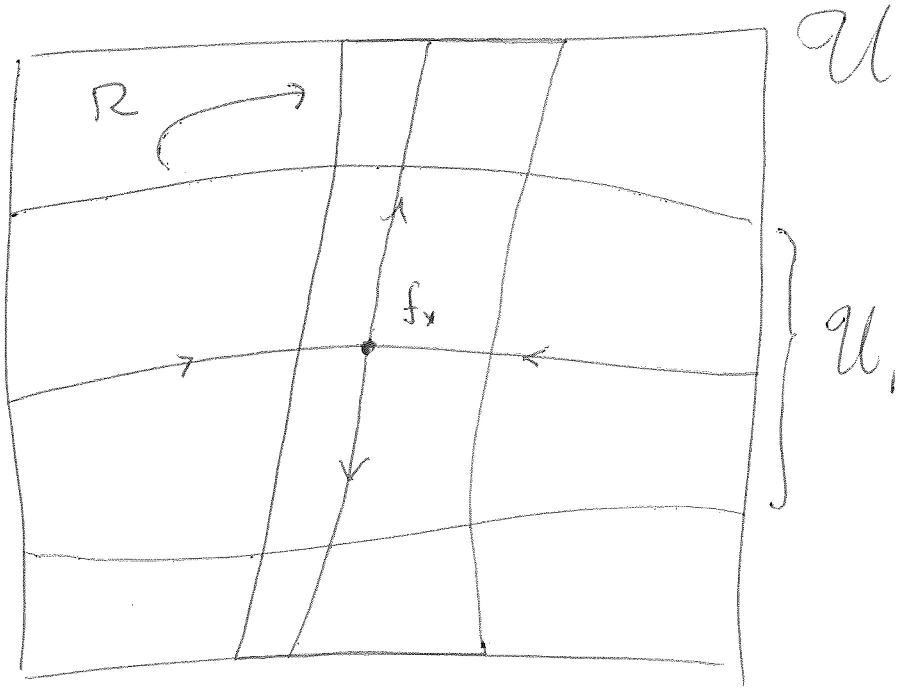


$$C^3,\ D^2 f(c) < 0 \}.$$

$\mathcal{U}_1 = \{ f \in \mathcal{U} \mid \text{renormalizable} \}$:

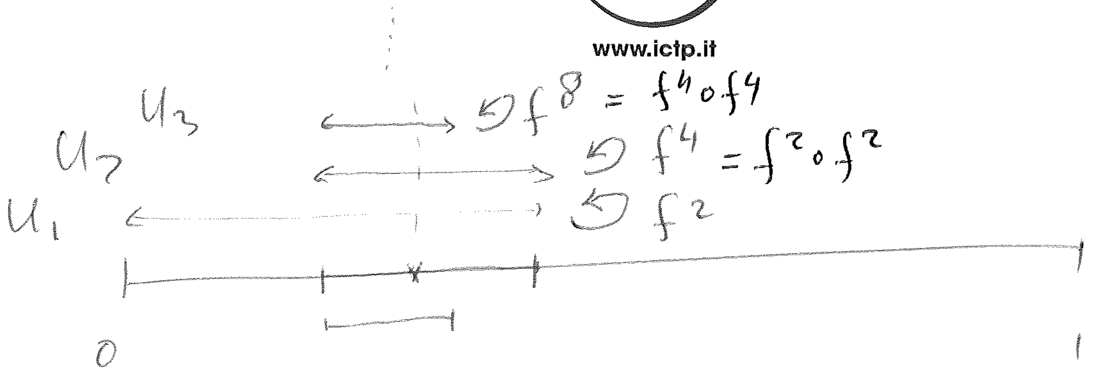


$R : \mathcal{U}_1 \longrightarrow \mathcal{U}$



<u>Thm</u>: R has a unique fixed pt f_*.

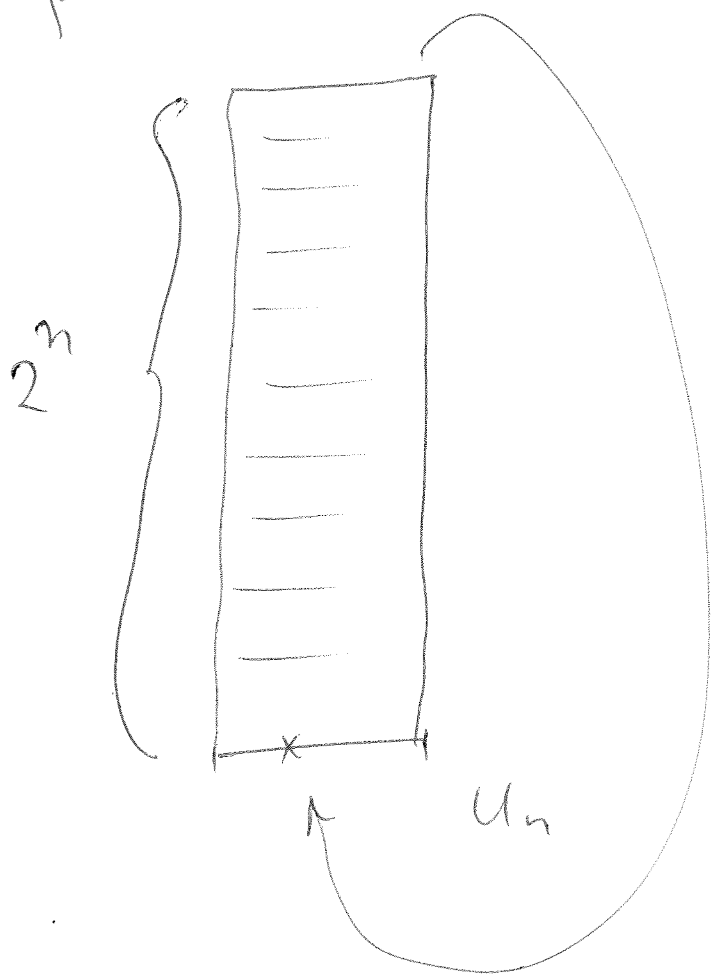
- f_* is hyperbolic.
- $\dim W^u_{f_*} = 1$ $\qquad \delta = 4.669\ldots$
- $\operatorname{codim} W^s_{f_*} = 1$
- $W^s_{f_*} = \{ \infty\text{-reno} \} = \partial \text{Chaos}.$



So $R^n f$ is a rescaled version of

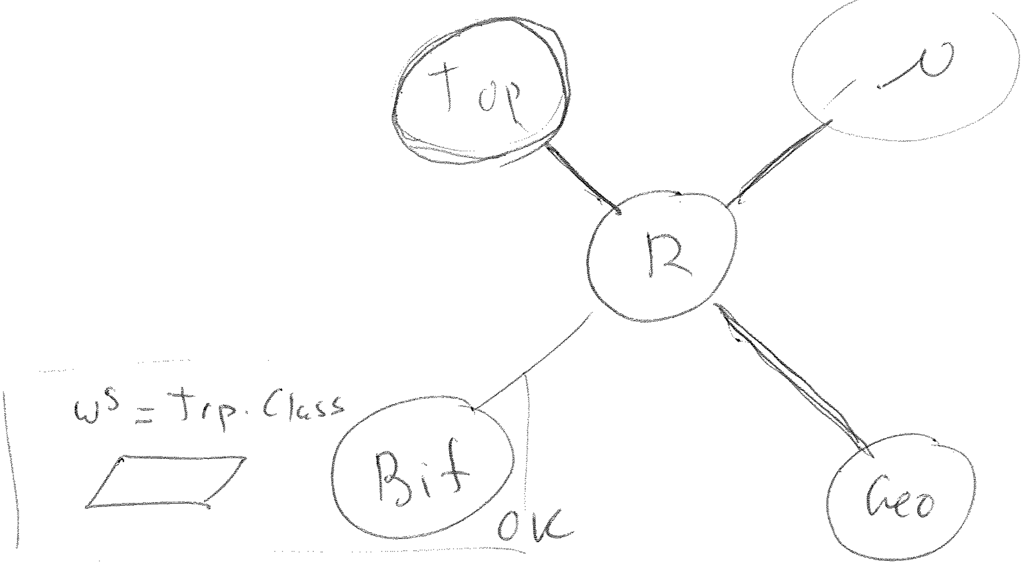
$$f^{2^n} : U_n \circlearrowleft$$

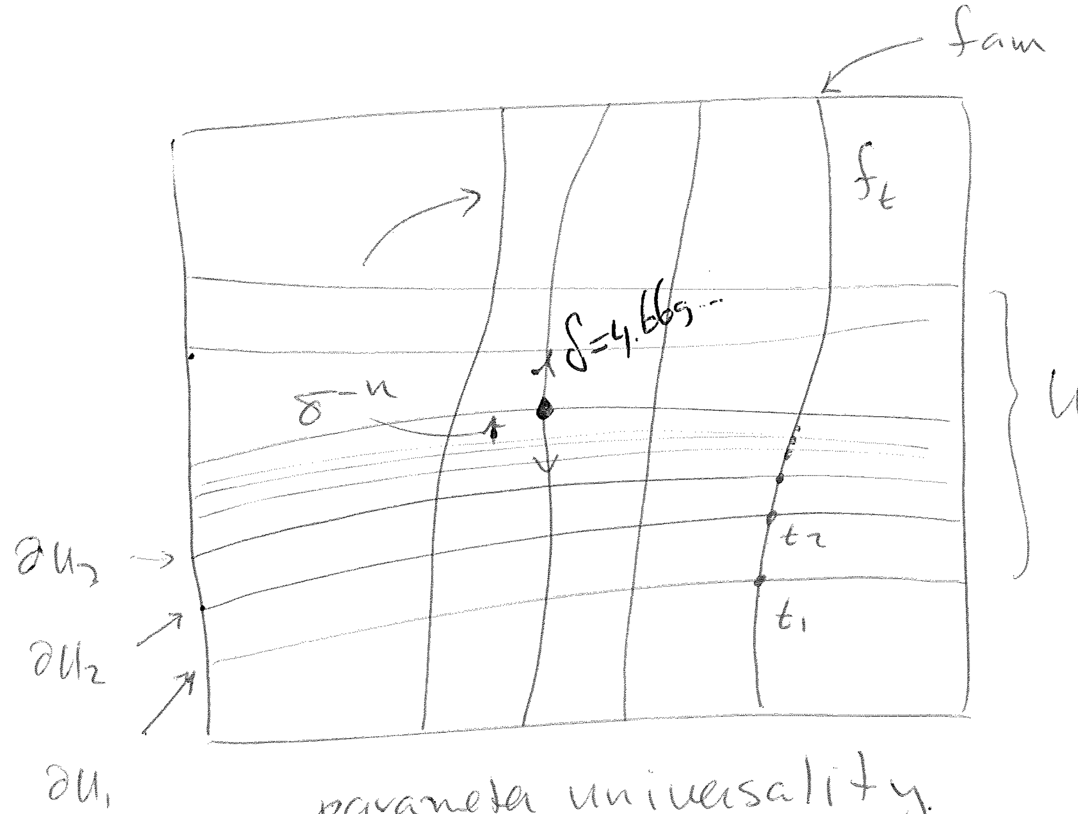
Simple Tower



$\mathcal{P}_n$:
Dynamical partition.

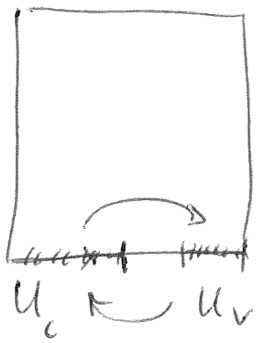
This Theorem has a long history:
Coullet Tresser & Feigenbaum, Epstein & Lanford,
Sullivan, McMullen, M, Avila & Lyubich

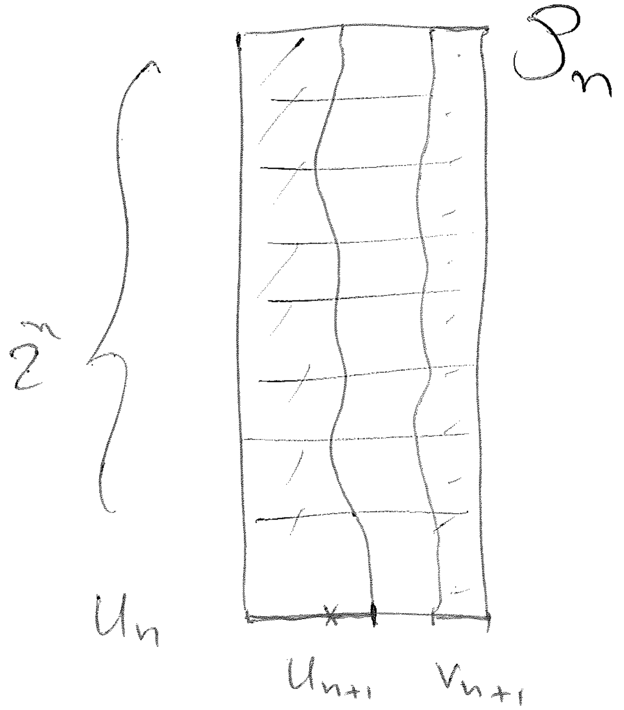


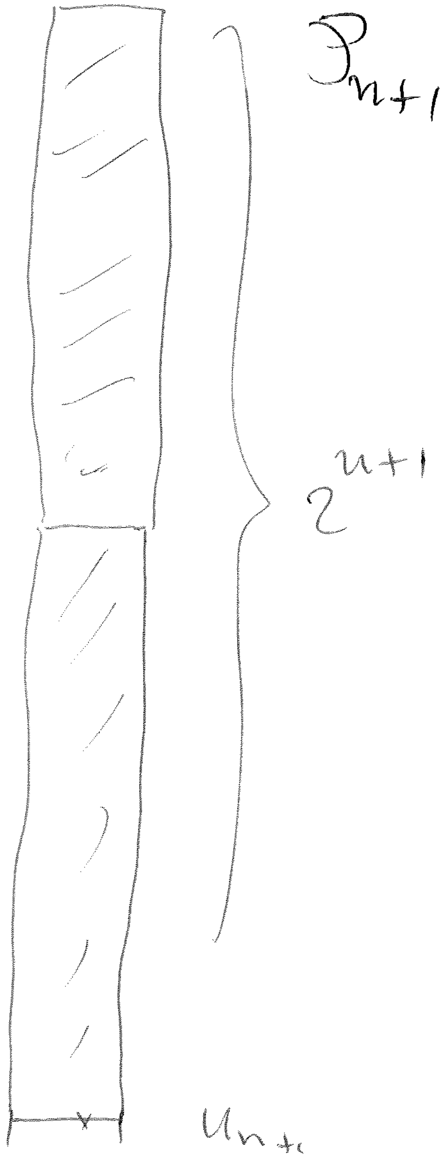


parameter universality

How does Renormalization ~~acts~~ acts on Towers.

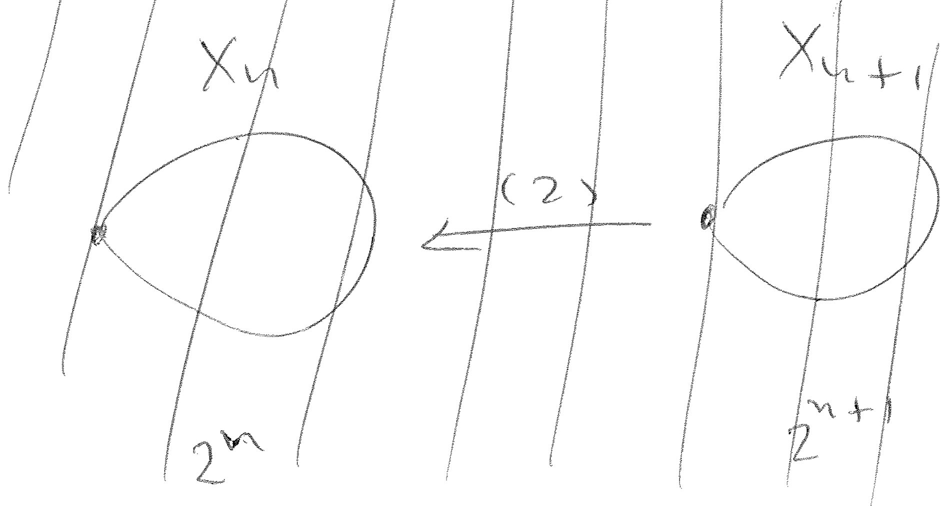




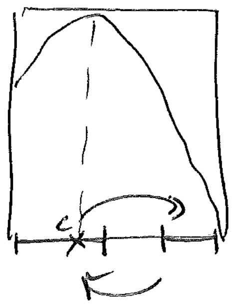


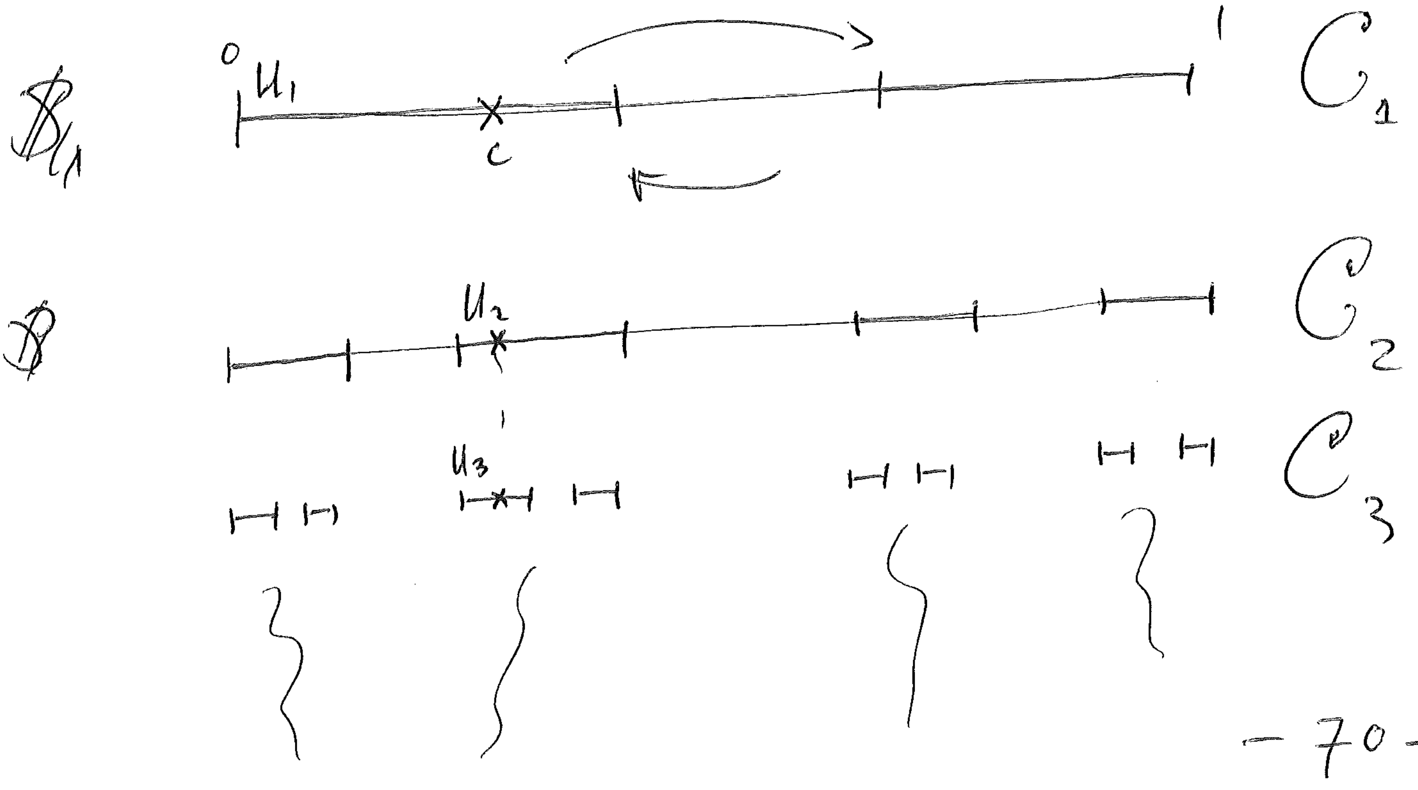
Remark 1: in the case of circle diffeos: the limits of renormalization are rotation. and the limiting towers are affine - towers. What makes unimodal renormalization difficult is that the limiting maps are not affine or simple.

Measure Theory



The dynamical "Partition" $\mathcal{P}_n$ is not a partition of $[0,1]$. It consists of disjoint intervals but they don't fill up $[0,1]$. They are not called partitions but cycles $\mathcal{C}_n (= \mathcal{P}_n)$.





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<u>Thm</u> f ∞-renormalizable.

let $\mathcal{C} = \bigcap \mathcal{C}_n$

Then

- $\mathcal{C}$ is a Cantor set.
- $\forall n\ \exists!$ periodic orbit $\underset{P_n}{\wedge}$ of period 2^n in $\mathcal{C}_n$.
- if $x \notin \bigcup P_n$ then $\omega(x) = \mathcal{C}$
  $\uparrow$ limit set.

Topology

Thm: if f, g are ∞-reno.

then $f \sim_{top} g$.

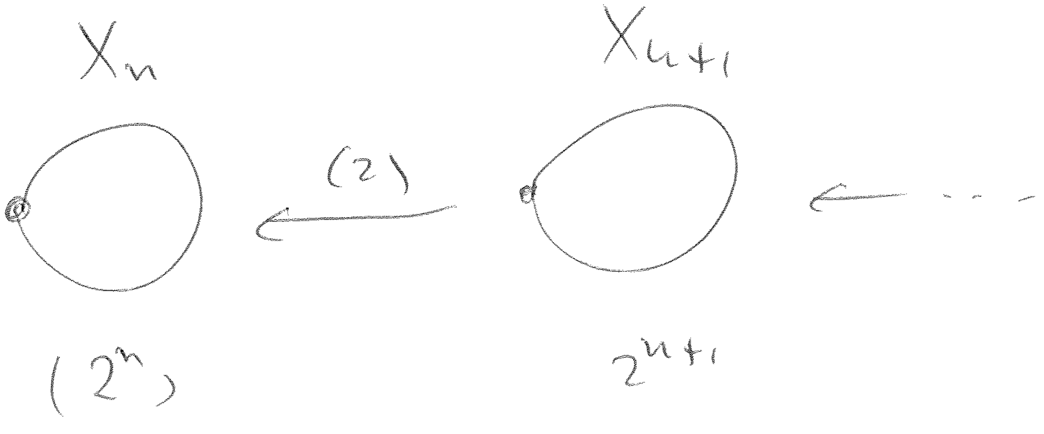
$$\begin{array}{ccc} [0,1] & \xrightarrow{h} & [0,1] \\ \circlearrowleft_f & & \circlearrowleft_g \end{array}$$

$$h \circ f = g \circ h.$$

Measure Theory

Thm: $\mathcal{C}$ is uniquely ergodic.

Pf

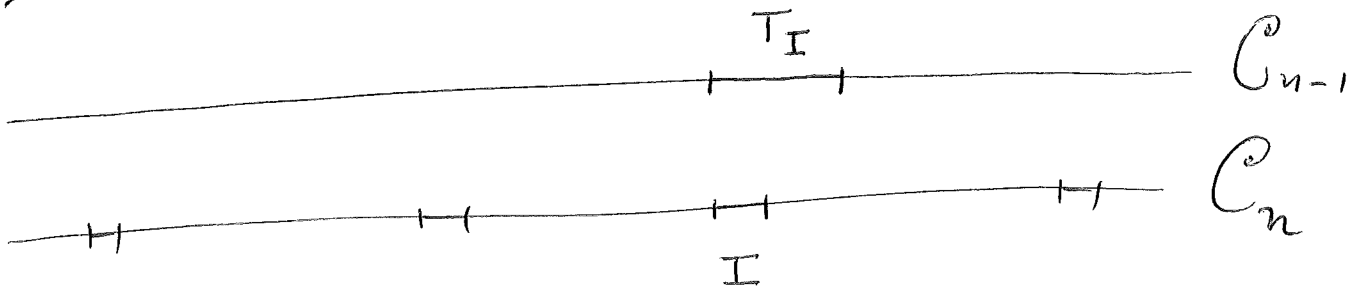


$\square$.

Geometry

let $f \in \mathcal{U}$ ∞-reno

Scaling Ratios



$$\sigma_I = \frac{|I|}{|T_I|}$$

$$\Sigma_f(n) = \{ \sigma_I \mid I \in \mathcal{C}_n \}.$$

$$\Sigma_f = \bigcap_{N>0} \overline{\bigcup_{n \geq N} \Sigma_f(n)}$$

Example: If $\mathcal{C}$ is the middle $1/3$ Cantor set then $\Sigma(n) = \{\frac{1}{3}\}$ and $\Sigma = \{\frac{1}{3}\}$

Thm (Birkhof, M Tresser).

- $\forall f \quad \Sigma_f = \Sigma_{f_*} \quad (f_* = Rf_*)$
- $\Sigma_{f_*} \subset (0,1)$ Cantor Set. | very rich scaling structure

Rmk 1) $HD(C_f) \in (0,1)$.

2) C_f is very far from being a "fractal"/self similar: there are not two places where you see the same asymptotic scaling structure.

Thm: f ∞-reno (C^3)

$$h : C_f \longrightarrow C_{f_*} \quad \text{conjugation}$$

then h is $C^{1+\alpha}$.

Rigidity