Ergodic theory of expanding Thurston maps

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Thurston's Theorem

Thurston's theorem on characterization of rational maps among topological self-branched covering of 2-sphere.
[Douady & Hubbard 1993]

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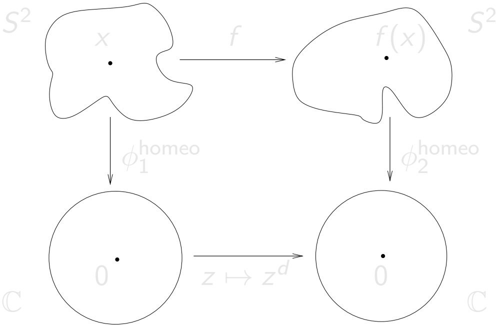
Branched covering map on S^2: A continuous map $f : S^2 \to S^2$ that satisfies



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x is a *critical point* if $\deg_f(x) = d \geq 2$

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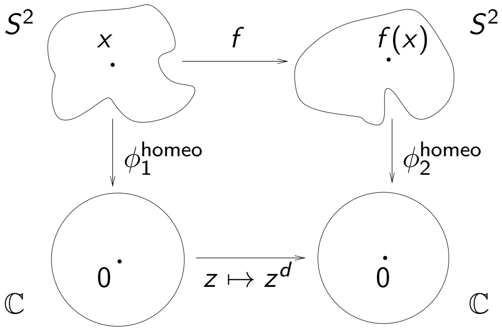

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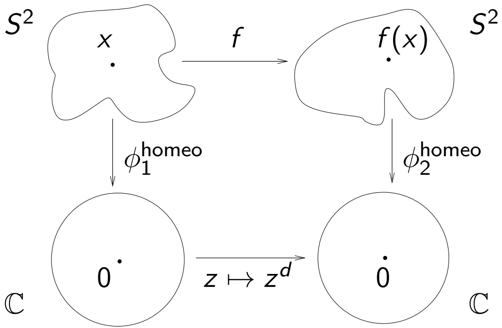

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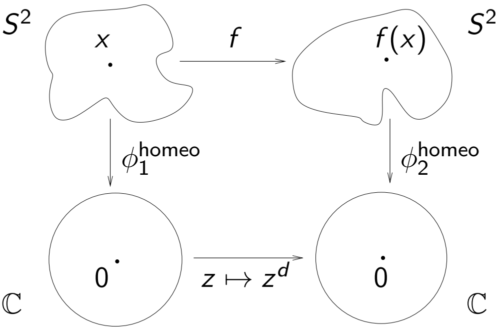

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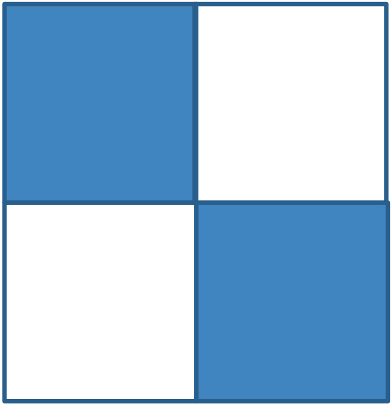
Thurston maps

Thurston map: A non-homeomorphic branched covering map $f\colon S^2 \to S^2$ with $\operatorname{card}(\operatorname{post} f) < +\infty$.

Postcritical set:
$\operatorname{post} f = \{f^n(x) \mid n \in \{1, 2, \dots\}, x \text{ is a critical point of } f\}$.

Example

Pillow: f, a Lattès map



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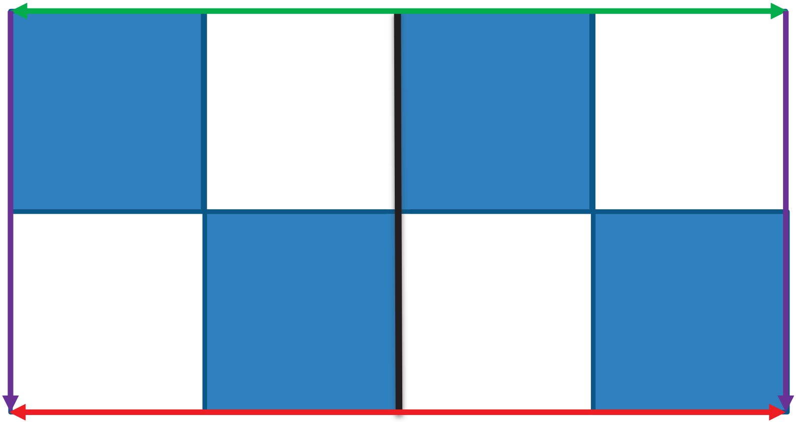

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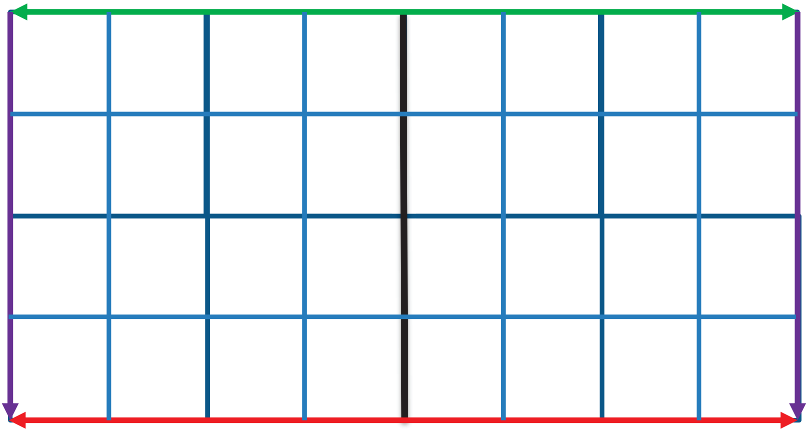

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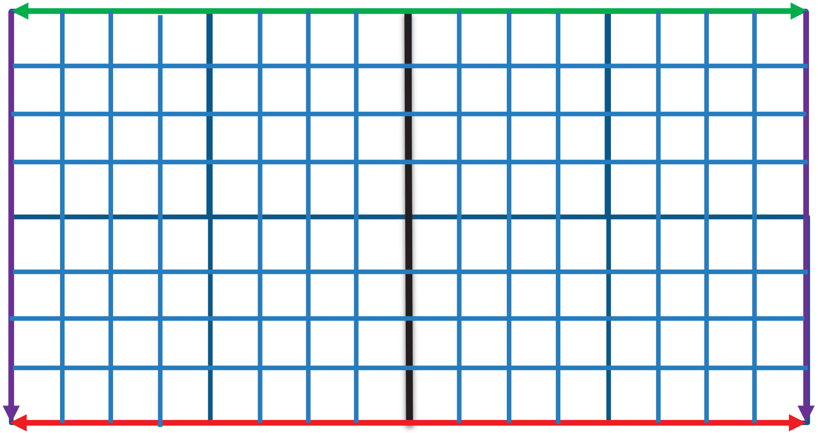
Pillow: f^3



Expanding Thurston maps

A Thurston map $f\colon S^2 \to S^2$ is *expanding* if there exist

- a metric d on S^2 that induces the standard topology on S^2,
- a Jordan curve $\mathcal{C} \subseteq S^2$ containing post f

such that

$$\lim_{n\to+\infty} \max\{\mathrm{diam}_d(X) \,|\, X \text{ is a conn. comp. of } S^2 \setminus f^{-n}(\mathcal{C})\} = 0.$$

Remark: the definition is independent of the choices of d and $\mathcal{C}$.

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Rational expanding Thurston maps

Proposition. Let $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational Thurston map. Then the following are equivalent:

1. R is expanding,
2. the Julia set of R is $\widehat{\mathbb{C}}$,
3. R has no periodic critical points.

Invariant Jordan curves

Theorem (Cannon, Floyd, & Parry 01, Bonk & Meyer 10)

Let f be an expanding Thurston map. For each $n \in \mathbb{N}$ sufficiently large, there exists an f^n-invariant Jordan curve $\mathcal{C} \subseteq S^2$ containing $\operatorname{post} f$.

Visual metrics

f - expanding Thurston map
$\mathcal{C}$ - Jordan curve on S^2 containing post f

d - visual metric (with expansion factor $\Lambda > 1$) characterized by

- $\operatorname{diam}_d(X^n) \asymp \Lambda^{-n}$,
- $d(X^n, Y^n) \gtrsim \Lambda^{-n}$ if $X^n \cap Y^n = \emptyset$.

If f is rational, then $d \overset{\text{q.s.}}{\simeq}$ spherical metric.

$\overset{\text{q.s.}}{\simeq}$ - quasisymmetrically equivalent

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Sullivan's dictionary

Geometric group theory	**Complex dynamics**
cocompact Kleinian group	rational expanding Thurston map
Gromov hyperbolic group with boundary S^2	expanding Thurston map
Cannon's Conjecture	Characterization Theorems of rational maps

Cannon's Conjecture The boundary at infinity of a Gromov hyperbolic group is homeomorphic to S^2 iff
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Theorem (Bonk & Meyer 10, Haïssinsky & Pilgrim 09)

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- Existence & uniqueness of the measure of maximal entropy ([Haïssinsky & Pilgrim 09], [Bonk & Meyer 10])
- Ergodic properties of the measure of maximal entropy
- Existence & uniqueness of the equilibrium states
- Distribution of preimages and periodic points

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Theorem (L. 13)

Let f be an **expanding Thurston map**, *with μ_0 its measure of maximal entropy, $p \in S^2$.*

Choose $w_n(x)$ to be 1 or $\deg_{f^n}(x)$. Then as $n \longrightarrow +\infty$,

$$\frac{1}{(\deg f)^n} \sum_{x \in f^{-n}(p)} w_n(x) \delta_x \xrightarrow{w^*} \mu_0, \qquad \textit{(preimage pts)}$$

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Theorem (L. 13)

Each expanding Thurston map f has exactly $1 + \deg f$ fixed points (counted with local degree $\deg_f(x)$).

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Thermodynamical formalism

"Thermodynamic formalism is the part of ergodic theory which studies measures under assumptions on the regularity of their Jacobian functions."

— Omri Sarig

$f : X \to X$, μ Borel measure
J is the *Jacobian function* of f w.r.t. μ if
$\mu(f(A)) = \int_A J \, \mathrm{d}\mu$ for any $A \subseteq X$ on which f is injective.

Write $J = c \exp(-\phi)$, where ϕ is called *potential*.

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Let $f: X \to X$ be a continuous map.

$\phi: X \to \mathbb{R}$ is a Hölder-continuous function (called *potential*).

For an f-invariant Borel probability measure μ,

$h_\mu(f) + \int \phi \, \mathrm{d}\mu$ - measure-theoretic pressure

$P(f, \phi)$ - (topological) pressure

Theorem (Variational Principle)

$$P(f, \phi) = \sup \left\{ h_\mu(f) + \int \phi \, \mathrm{d}\mu \,\middle|\, f\text{-invariant Borel prob. measure } \mu \right\}$$

μ is an *equilibrium state* for f and ϕ if $h_\mu(f) + \int \phi \, \mathrm{d}\mu = P(f, \phi)$.

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Rational maps on $\hat{\mathbb{C}}$ with Hölder continuous potentials:

[Denker & Urbański 1991] - potential $\phi < P(f, \phi)$
[Przytycki & Urbański 10], [Comman & Rivera-Letelier 11] - special classes of rational maps
[Inoquio-Renteria & Rivera-Letelier 12] - "hyperbolic" potentials

Theorem (L. 14)

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Then there exists a unique equilibrium state μ_ϕ for f and ϕ.

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(f, μ_ϕ) is exact,

i.e., for each Borel set $E \subseteq S^2$ with $\mu_\phi(E) > 0$,

$$\lim_{n \to +\infty} \mu_\phi(f^n(E)) = 1.$$

In particular, (f, μ_ϕ) is mixing,

i.e., for any Borel sets $A, B \subseteq S^2$,

$$\lim_{n \to +\infty} \mu_\phi \left(f^{-n}(A) \cap B \right) = \mu_\phi(A) \cdot \mu_\phi(B).$$

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Ruelle operator

The Ruelle operator $\mathcal{L}_\phi \colon C(S^2) \to C(S^2)$:

$$\mathcal{L}_\phi(u)(x) = \sum_{y \in f^{-1}(x)} \deg_f(y) u(y) \exp(\phi(y)).$$

Co-homologous potentials

f - an expanding Thurston map $\quad \phi, \phi'$ - Hölder cont. potentials
$\mu_\phi, \mu_{\phi'}$ - the corresponding equilibrium states

Theorem (L. 14)

$\mu_\phi = \mu_{\phi'}$ if and only if there exists $K \in \mathbb{R}$ s.t. $\phi - \phi'$ and K are **co-homologous**, *i.e.,*

$$\phi - \phi' - K = u \circ f - u \qquad \text{for some } u \in C(S^2).$$

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Equidistribution w.r.t. equilibrium states

f - an expanding Thurston map ϕ - a Hölder cont. potential
μ_ϕ - the unique equilibrium state for f and ϕ
$w_n(x) = \deg_{f^n}(x) \exp\left(\sum_{i=0}^{n-1} \widetilde{\phi}(f^i(x))\right)$ - the weight

Theorem (L. 14)

For each $p \in S^2$, as $n \longrightarrow +\infty$,

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What about periodic points?

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Large deviation principles and equidistribution

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equidistribution (w.r.t. the equilibrium state μ_ϕ)

$\Uparrow$

large deviation principles (w.r.t. μ_ϕ)

$\Uparrow$ [Kifer 90, Comman & Rivera-Letelier 11]

(i) existence and uniqueness of the equilibrium state

(ii) certain characterization of the topological pressure $P(f, \phi)$

(iii) upper semi-continuity of $\mu \mapsto h_\mu(f)$

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Weak expansion properties

expansive

$\Downarrow$

h-expansive (Bowen '72)

$\Downarrow$

asymptotic h-expansive (Misiurewicz '73)

$\Downarrow$

upper semi-continuity of $\mu \mapsto h_\mu(f)$

E.T.M. $\nearrow$ w/o periodic critical pts $\overset{\text{no}}{\rightarrow}$ (h-expansive); $\searrow$ yes (asymptotic h-expansive)

E.T.M. $\searrow$ w/ periodic critical pts $\nearrow$ no (asymptotic h-expansive); $\underset{?}{\rightarrow}$ (upper semi-continuity)

Theorem (L. 14)

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E.T.M. $\searrow$ w/ periodic critical pts $\overset{\text{no}}{\nearrow}$ asymptotic *h*-expansive; w/ periodic critical pts $\underset{?}{\rightarrow}$ upper semi-continuity of $\mu \mapsto h_\mu(f)$

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E.T.M. ↗ w/o periodic critical pts $\overset{no}{\rightarrow}$; ↘ yes

E.T.M. ↘ w/ periodic critical pts ↗ no ; $\underset{?}{\rightarrow}$

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w/o periodic critical pts — no → h-expansive; yes ↘ asymptotic h-expansive

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			$\searrow$ yes	$\Downarrow$
E.T.M.	$\nearrow$ $\searrow$			asymptotic h-expansive (Misiurewicz '73)
			no $\nearrow$	$\Downarrow$
		w/ periodic critical pts	$\xrightarrow[?]{}$	upper semi-continuity of $\mu \mapsto h_\mu(f)$

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Existence of equilibrium states - continuous potentials

Corollary

f - an expanding Thurston map **without periodic critical points**
$\psi\colon S^2 \to \mathbb{R}$ **continuous**
Then there **exists at least one equilibrium state** *for f and ψ.*

Proof.

The space of f-invariant Borel probability measures is **compact** in the weak* topology.
$\mu \mapsto h_\mu(f) + \int \psi \, d\mu$ is **upper semi-continuous**. □

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$w_n(x,\phi) = \exp\left(\sum_{i=0}^{n-1}\phi(f^i(x))\right)$ or $\deg_{f^n}(x)\exp\left(\sum_{i=0}^{n-1}\phi(f^i(x))\right)$

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Thank you!