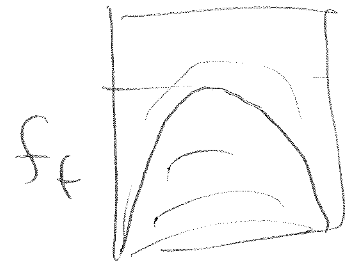
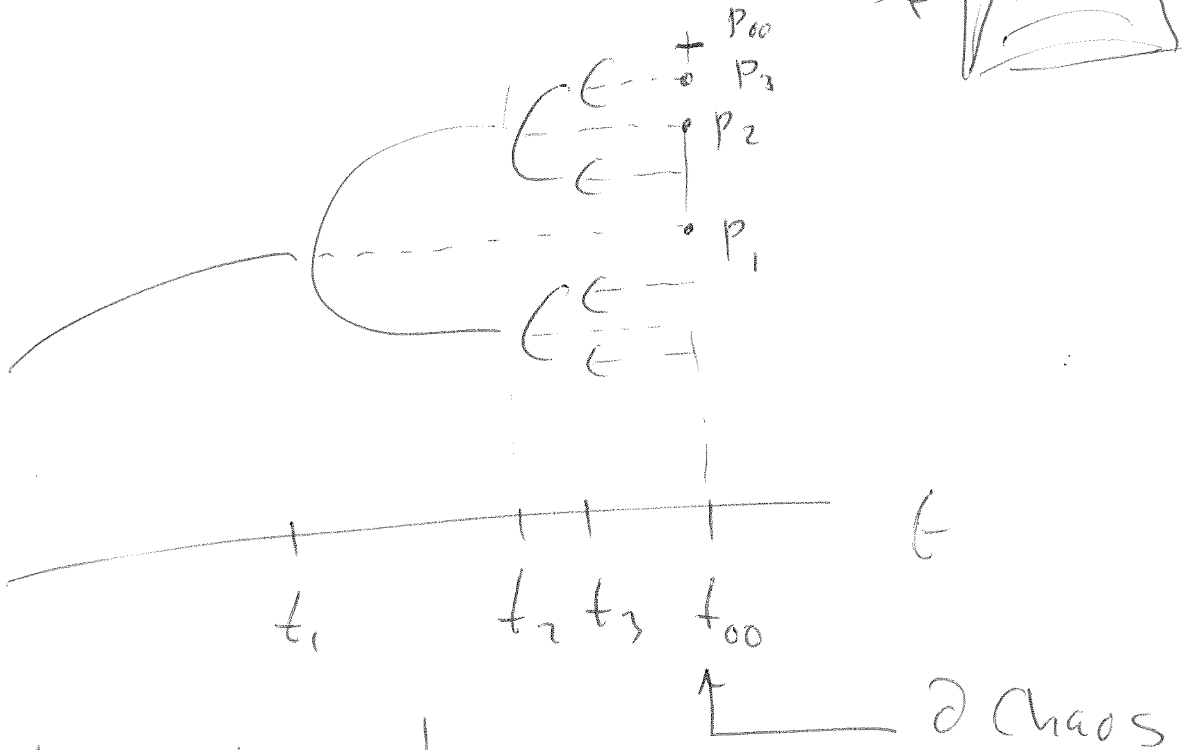
III Renormalization of Hénon Maps

Recall Period Doubling Cascade for a Unimodal family





$t_n \xrightarrow{exp} t_\infty$ rate $= \dfrac{1}{\delta = 4.669\ldots}$

Parameter Universality

$$p_n \xrightarrow{\exp} p_\infty \quad \text{rate} = \frac{1}{\delta^2} \quad = (2.6..)^2$$

Phase Space Universality

(Dynamical Geometry Universality).

$$\left(\Sigma_f = \Sigma_{f_*} \right).$$

Rigidity

$$h : \mathcal{O}_f \xrightarrow{C^{1+\alpha}} \mathcal{O}_{f_*}$$

So far Renormalization was a unifying tool to study top, geo, μ, Bif.

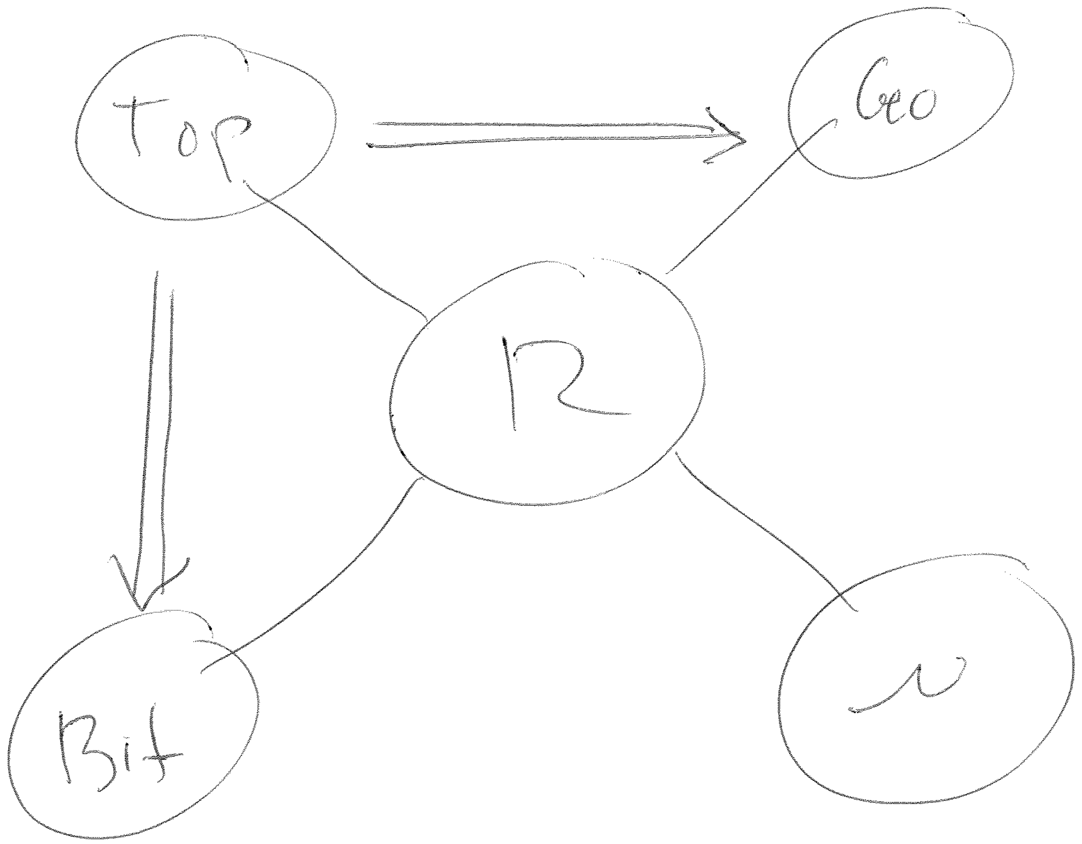
But it adds : something :

Main Conclusion

$$\boxed{\text{Topology} \Longrightarrow \text{Geometry}}$$

<u>Example</u>: • Period Doubling Cascade $\longrightarrow \begin{cases} \delta = 4.6. \\ \delta^2 = (2.6)^2 \\ h \in C^{1+\alpha} \end{cases}$

• $a_n \equiv 1 \longrightarrow h \in C^{1+\alpha}$

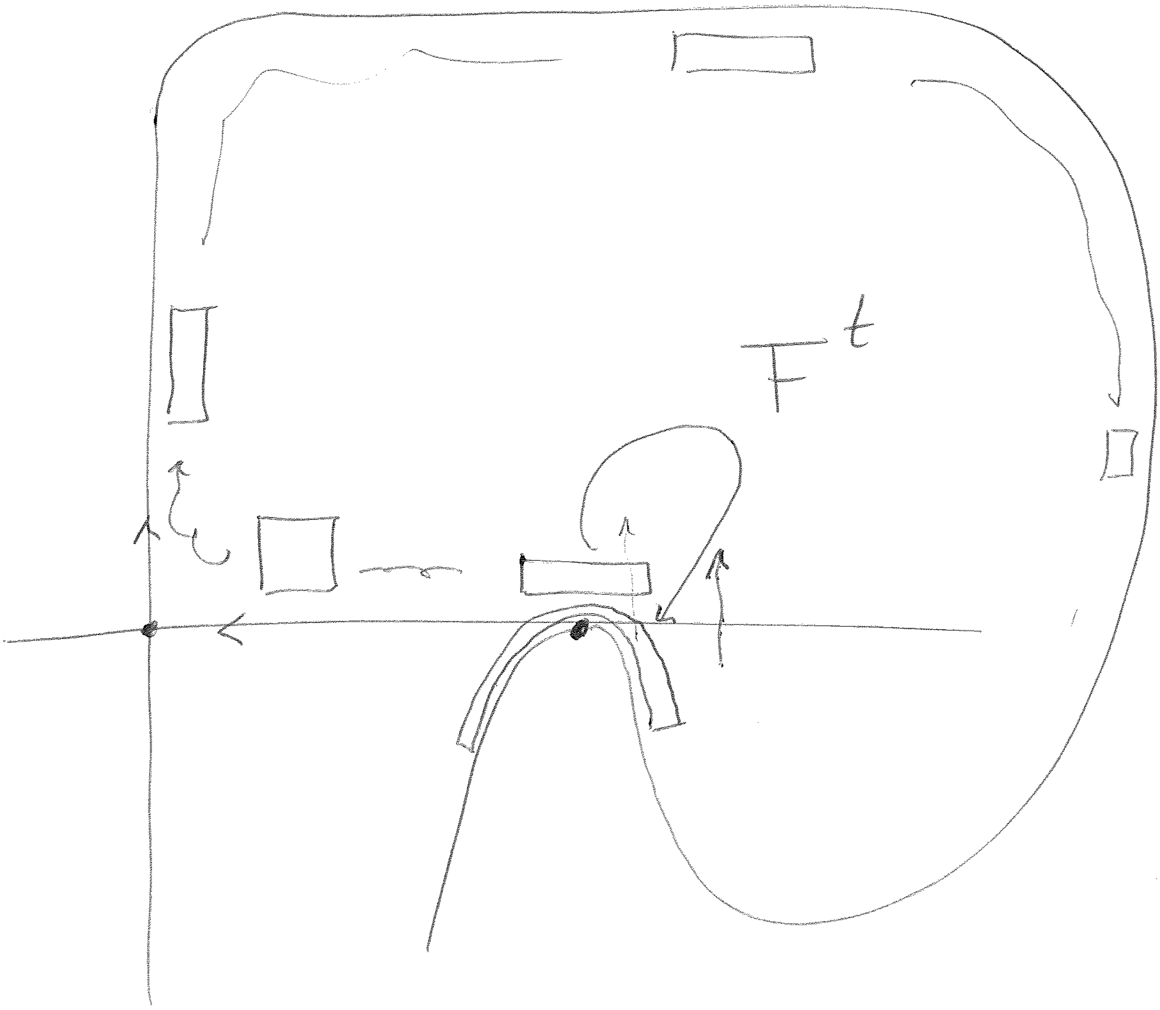


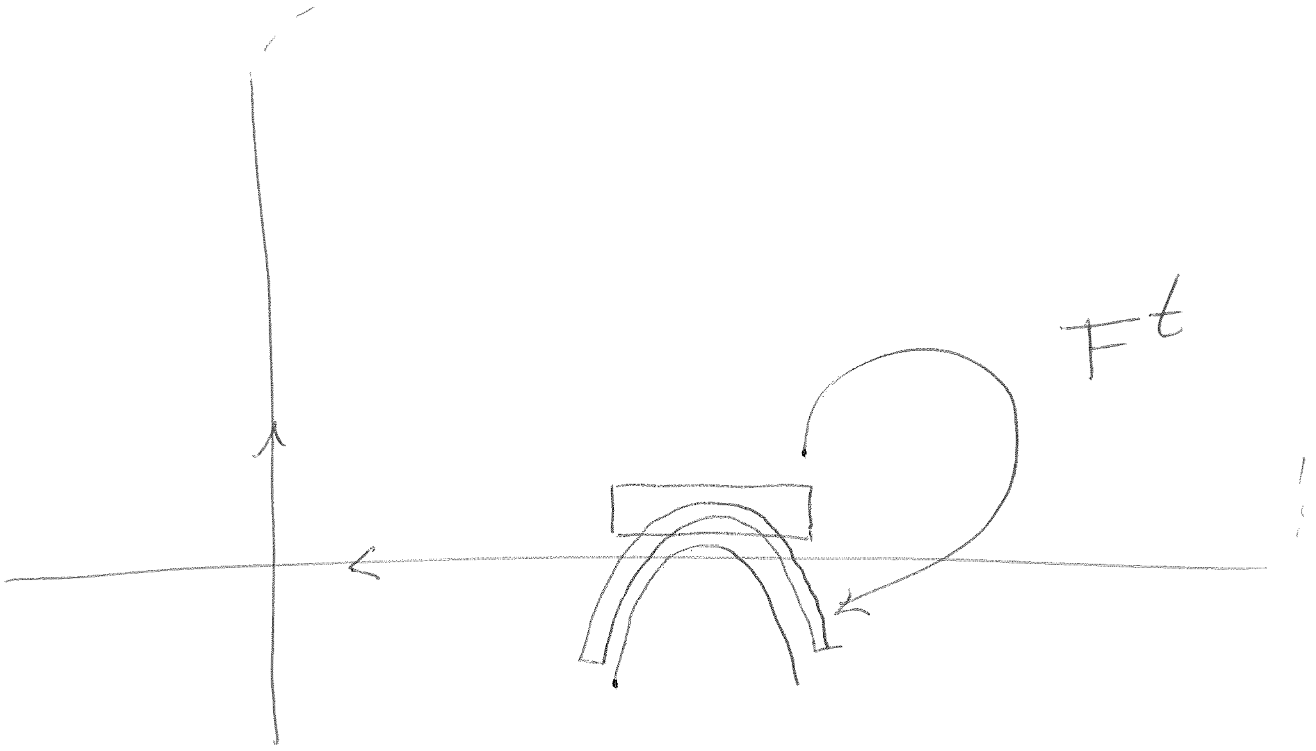
That was 1 Dim. Dyn.
Lets move on to 2 Dim Dyn.

Hénon-Maps.

Hénon-Maps describe
Homoclinic bifurcations

(Topological changes in Dynamics
are associated with homoclinic
bifurcations)


F^t


F^t

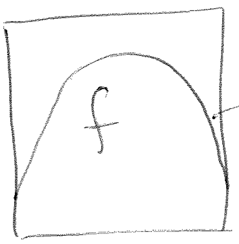
The return map F^t are of the form (Up to a smooth change of coordinates)

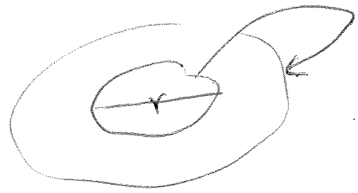
Hénon-like map

$$F : [0,1]^2 \longrightarrow [0,1]^2$$

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x) - \varepsilon(x,y) \\ x \end{pmatrix}$$

where

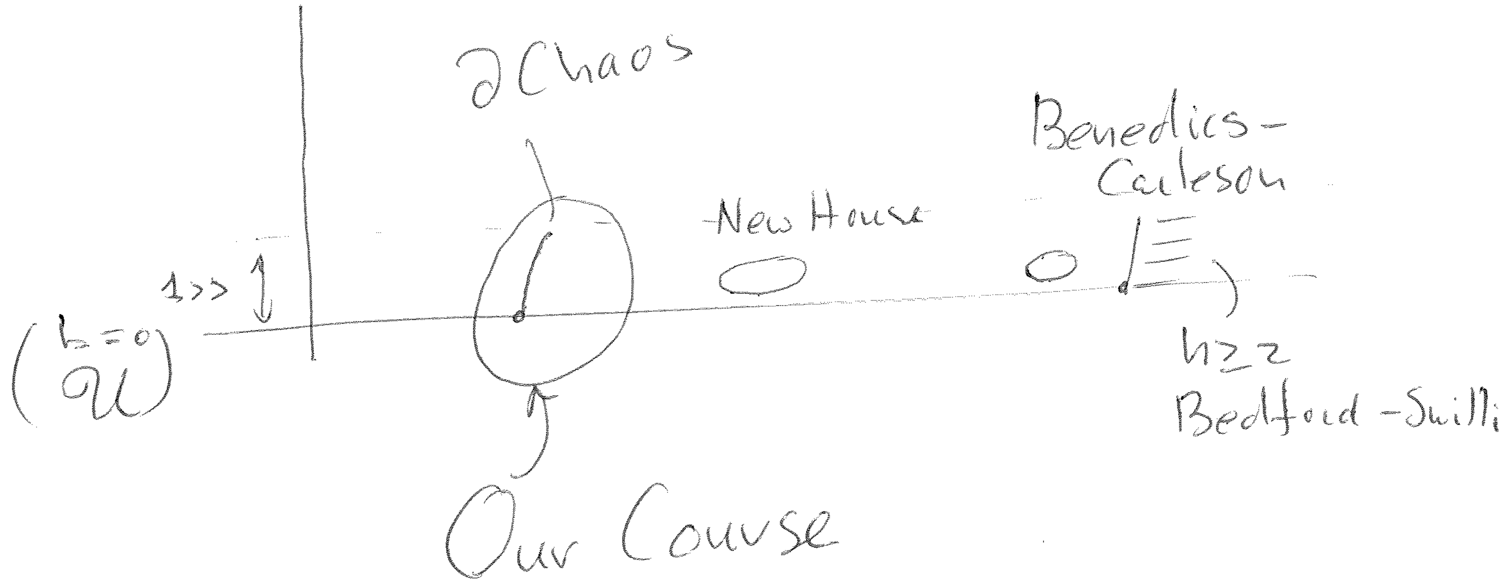
- 

(Quad. Like )

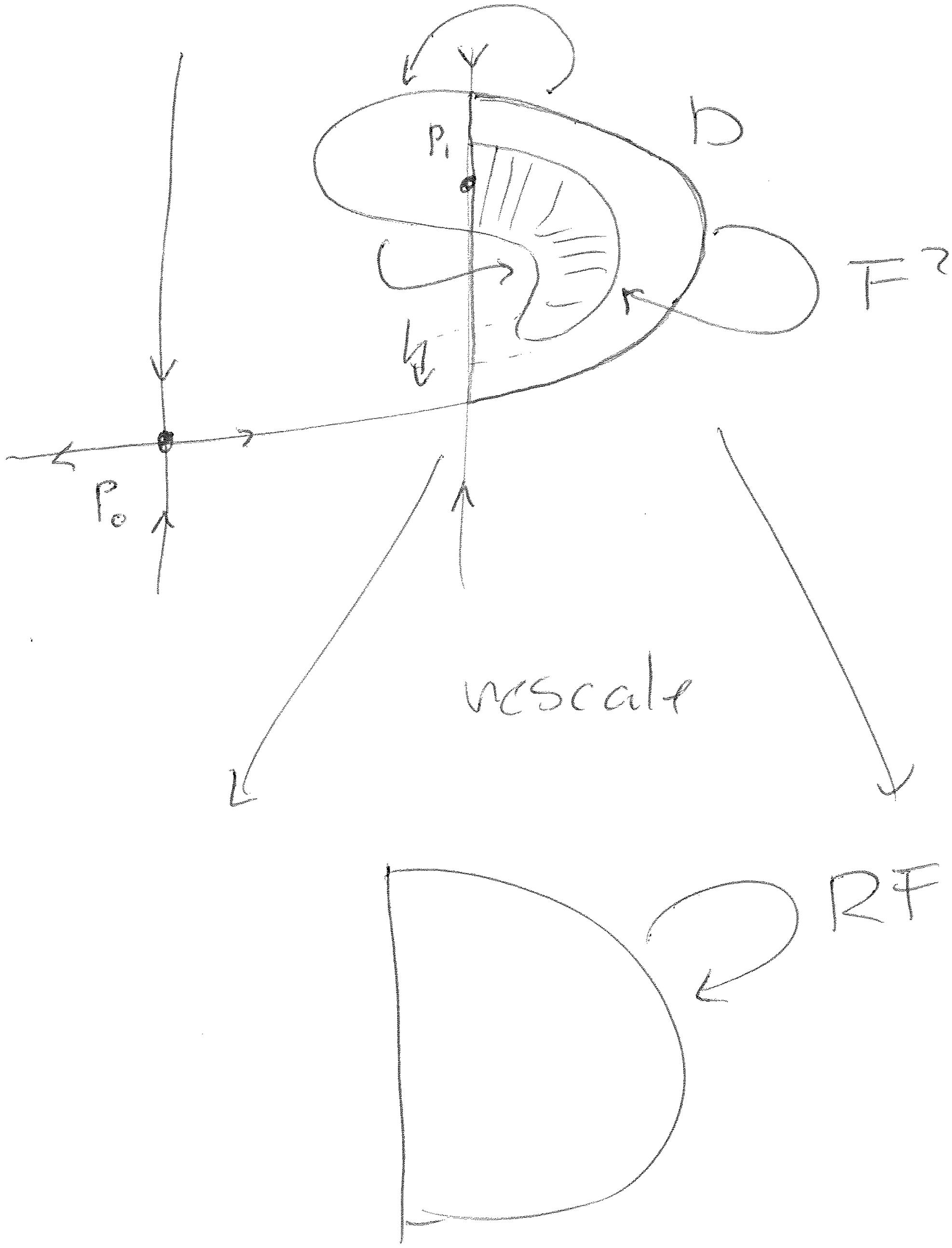
- $\varepsilon : \mathbb{D}^2 \longrightarrow \mathbb{C}$ $\quad |\varepsilon| \ll 1$

Example The Hénon Family

$$F_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - x^2 - by \\ x \end{pmatrix}$$

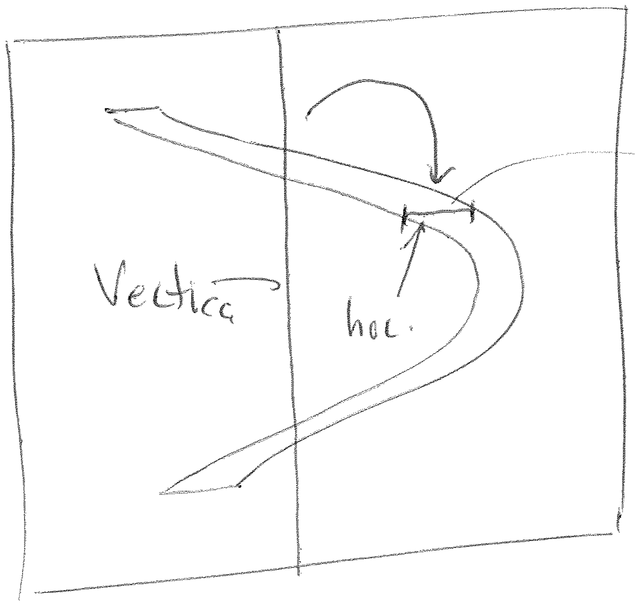


__Def__: F Hénon-like is renormalizable if.



<u>Problem</u>: Shape of Hénon-Maps

$$F\binom{x}{y} = \begin{pmatrix} f(x) - \varepsilon(x,y) \\ x \end{pmatrix}$$



<u>Remark</u>: $\mathrm{Jac}(F) = \det DF = \frac{\partial \varepsilon}{\partial y} \ll 1.$

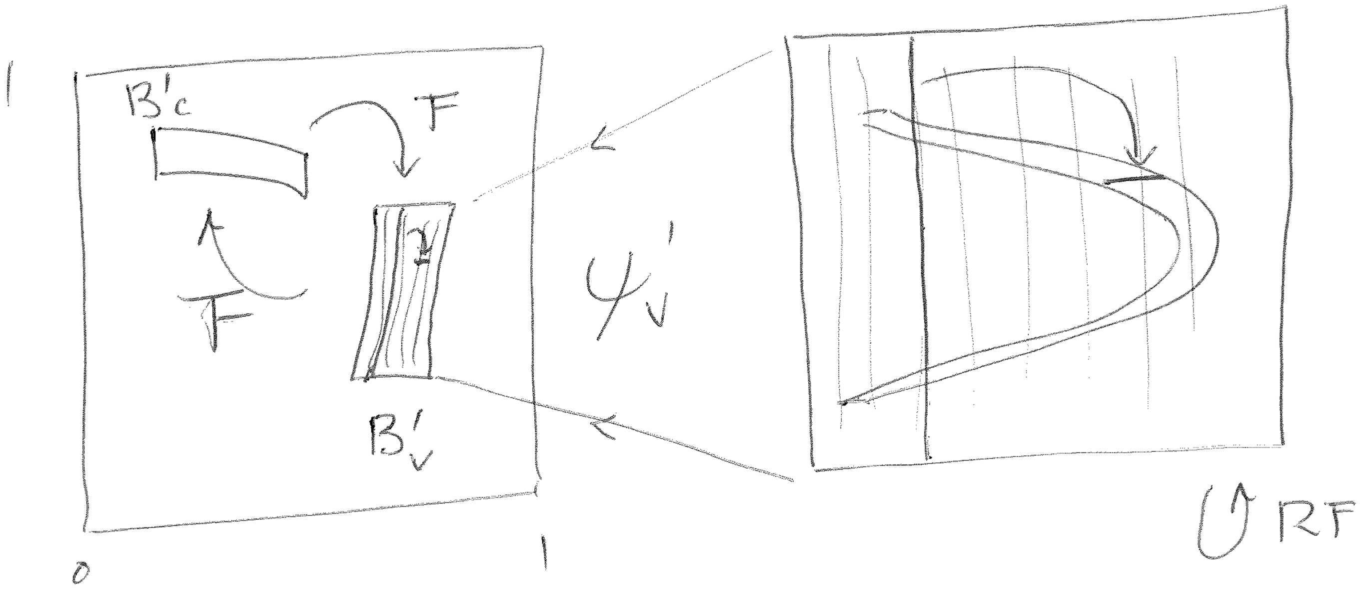
Our Hénon have very small Jacobian, they are <u>strongly dissipative</u>.

The map $F^2|D$ will not map vertical lines to horizontal straight lines.

So there is no affine rescaling that brings $F^2|D$ back to Hénon-Shape: $\begin{pmatrix} - & - & - \\ & x & \end{pmatrix}$

We need non-affine rescalings

The way Renormalization is defined: F is renormalizable if



In B_v there is a foliation which is mapped by F^2 into B_v and becomes horizontal.

ψ'_v straightens the foliation in B_v.

let $\psi'_c = F \circ \psi'_v$

$$RF = (\psi_v^1)^{-1} F^2 \psi_v^1$$

$$B_v^1 = \psi_v^1([0,1]^2)$$

$$B_c^1 = \psi_c^1([0,1]^2).$$

$$\begin{array}{ccc} R: \mathcal{A}_0 & \longrightarrow & \mathcal{A} \\ \uparrow & & \uparrow \\ \text{reno Hénon-like} & & \text{Hénon-like.} \end{array}$$

<u>Thm</u> (de Carvalho, Lyubich, M) (Similar result by Collet Eckmann Koch)

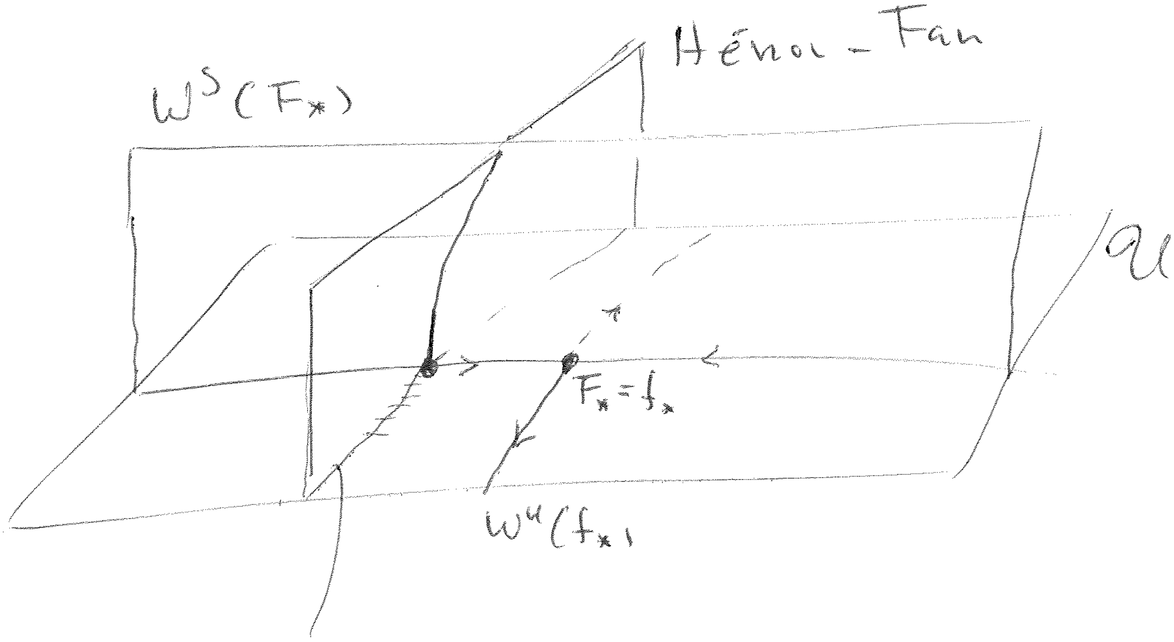
$$R: \mathcal{A}_0(\bar{\varepsilon}) = \left\{ F = \begin{pmatrix} f - \varepsilon \\ x \end{pmatrix} \;\middle|\; |\varepsilon| \leq \bar{\varepsilon}(\|f\|) \right\}.$$

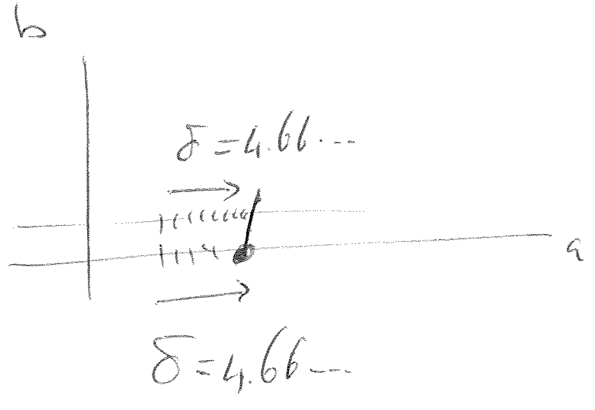
$$\longrightarrow \mathcal{A}.$$

Has a unique fixed Pt $F_* = \begin{pmatrix} f_*(x) \\ x \end{pmatrix}$.

where f_* is the unimodal fixed Pt.

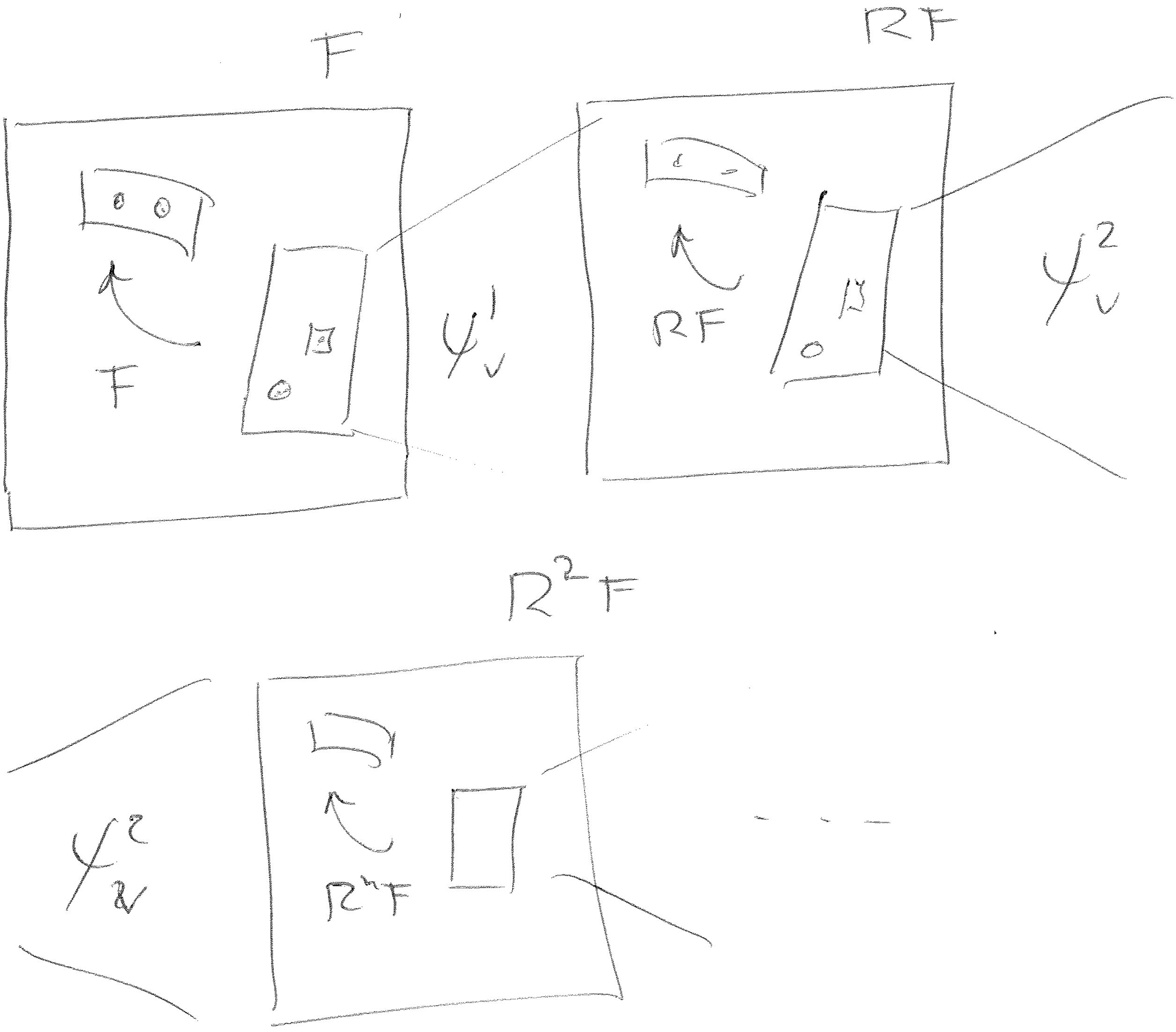
- F_* is hyp.
- $W^u_{F_*} = W^u_{f_*} \subset \mathcal{U}$.
- codim $W^s(F_*) = 1$





parameter Universality as in 1D. ($\delta = 4.6$).

Dynamics / Attractor



Cycles

$$C_1 = \{ B^1_v , B^1_c \} = \{ \psi^1_v([0,1]^2) , \psi^1_c([0,1]^2) \}$$

$$\cup$$

$$C_2 = \{ B^2_{vv} , B^2_{vc} , B^2_{cv} , B^2_{cc} \}$$

$$\| $$

$$= \{ \quad , \quad , \psi^1_c \circ \psi^2_v([0,1]^2) , \cdots \}$$

$$\cup$$

$$C_3 = \{ \cdots\cdots , B^3_{cvc} , \cdots \}$$

$$\|$$

$$= \{ \cdots , \psi^1_c \circ \psi^2_v \circ \psi^3_c([0,1]^2) , \cdots \}.$$

$$\vdots$$

$$C_1 \supset C_2 \supset C_3 \supset \cdots$$

Let $\mathcal{C} = \bigcap \bigcup \mathcal{C}_n$

Similar result (Gambaudo, v. Strien, Tresser)

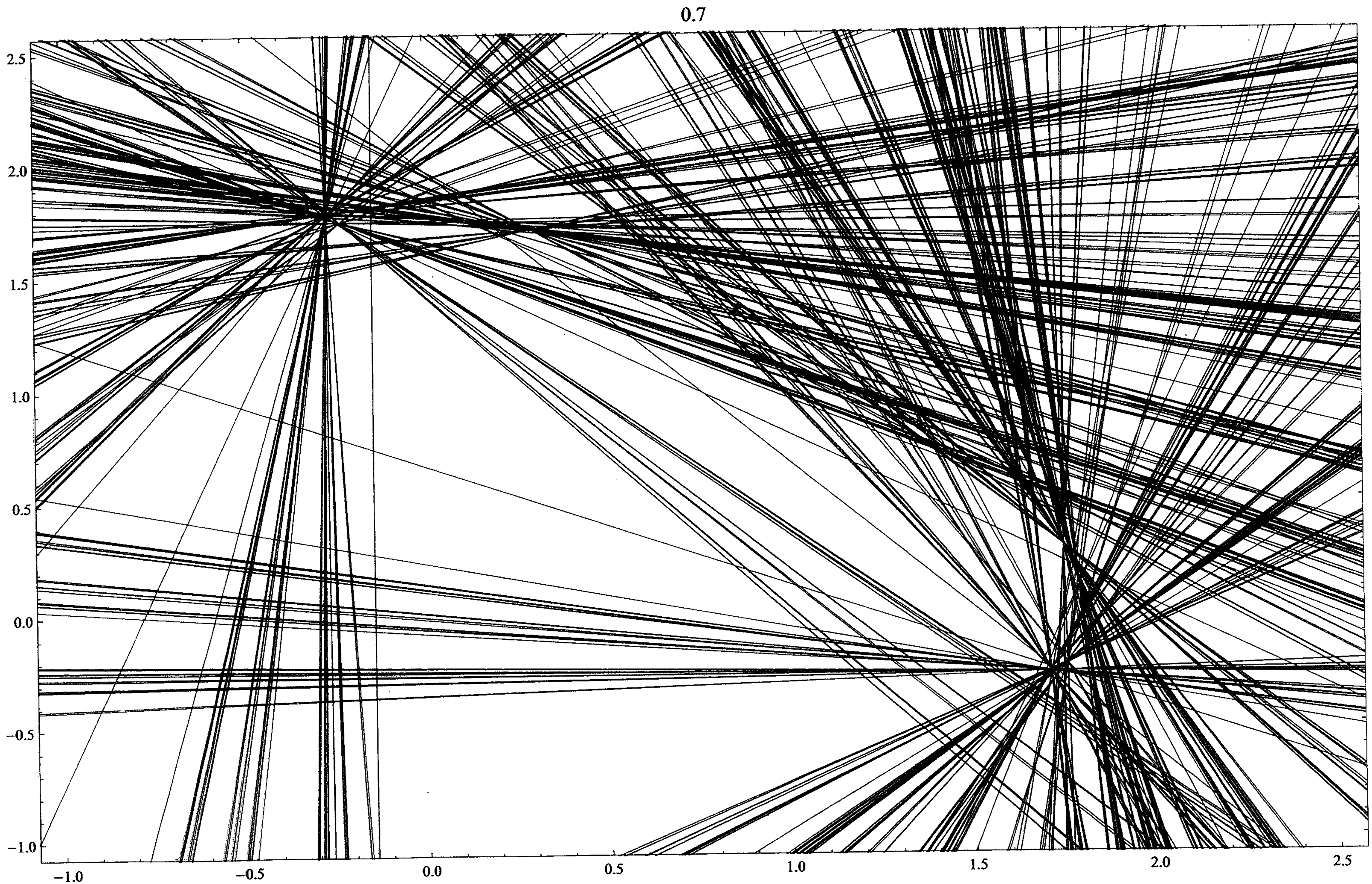
<u>Thm</u> (de Carvalho, Lyubich, M)

- $\mathcal{C}$ is a Cantor set.
- $\forall n\ \exists!$ periodic orbit P_n of period 2^n.
- $x \in [0,1]^2 : \exists_n\ \omega(x) = P_n$ or

$$\omega(x) = \mathcal{C}.$$

- $\exists!\ \mu$.

Pic.


0.7
2.5
2.0
1.5
1.0
0.5
0.0
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−1.0
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
2.5