Finitely many Measures Maximizing the Entropy for C^∞ Smooth Diffeos of Compact Surfaces with positive entropy

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Why and How

Entropy fundamental invariant (Kolmogorov 1958, Shannon 1948)

- counts orbits of length $n \to \infty$ given resolution ϵ
- measure-preserving $h(f,\mu)$ and topological $h_{\mathrm{top}}(f)$
- classifies (Ornstein 1971, Adler-Weiss 1979)
- selecting principle

How to count orbits segments? $T : X \to X$ C^0, compact metric

Dynamical ball

$B_T(x,\epsilon,n) := \{y \in X : \forall k = 0,\dots,n-1 \; d(f^k y, f^k x) < \epsilon\}$

Covering number:

$$r_T(\epsilon,n,Y) := \min\{\#C : \bigcup_{x\in C} B_T(x,\epsilon,n) \supset Y\}$$

$$r_T(\epsilon,n,\mu) := \inf_{\mu(Y)>1/2} r_T(\epsilon,n,Y)$$

Kolmogorov-Sinai and topological Entropy

For a **subset** $Y \subset X$:

$$h_{\mathrm{top}}(T, Y) := \lim_{\epsilon \to 0} h(T, Y, \epsilon)$$

$$h_{\mathrm{top}}(T, Y, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log r_T(\epsilon, n, Y)$$

For a **compact t.d.s.** $T : X \to X$: $h_{\mathrm{top}}(T) := h_{\mathrm{top}}(T, X)$

For **measure** $\mu \in \mathbb{P}_{\mathrm{erg}}(T)$ (*ergodic*, invariant, Borel proba):

$$h(T, \mu) := \lim_{\epsilon \to 0} h(T, \mu, \epsilon)$$

$$h(T, \mu, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log r_T(\epsilon, n, \mu)$$

Variational Principle and Measures Maximizing Entropy

Theorem (Goodman 1968)

For any continuous T of compact metric X

$$h_{\mathrm{top}}(T) = \sup\{h(T,\mu) : \mu \in \mathbb{P}_{\mathrm{erg}}(T)\}$$

Proof (Misiurewicz)

Equidistributed μ at fixed small scale $\to h(T,\mu) \approx h_{\mathrm{top}}(T)$

Definition

$\mu \in \mathbb{P}_{\mathrm{erg}}(T)$ is a **measure maximizing the entropy (mme)**:

$$h(T,\mu) = \sup\{h(T,\mu) : \mu \in \mathbb{P}_{\mathrm{erg}}(T)\}$$

Existence of MME if...

0. NO: continuity
1. Expansivity: $\forall x \; B_T(x,\epsilon,\mathbb{Z}) = \{x\}$, e.g.: subshifts, mpm on I
2. C^∞ maps: Newhouse 1987
4. NO: C^r, $r < \infty$: Misiurewicz 1973 $\mathbb{T}^4$, B 2014 M^2

Multiplicity of Measures Maximizing the Entropy

Uncountable MME for...$f \times \mathrm{Id}_{\mathbb{S}^1}$, f C^∞

Finiteness of MME if...

1. Subshifts of **finite type** (Parry)
2. uniformly hyperbolic diffeos (Sinai, Ruelle, Bowen)
3. Non-uniformly hyperbolic???
   - (irreducible) countable state Markov shifts (Gurevic 1970)
   - for C^r interval maps with positive entropy:
     – NO if $r < \infty$ (B 1997)
     – YES if $r = \infty$ (or mpm) (B 1997, Hofbauer 1980)

Theorem (Ruelle for interval maps, surface diffeos)

For $\mu \in \mathbb{P}_{\mathrm{erg}}(f)$, $h(f,\mu) \leq \lambda_1(f,\mu)^+$

Remark If $h_{\mathrm{top}}(f) = 0$, $\mathrm{MME}(f) = \mathbb{P}_{\mathrm{erg}}(f) \rightsquigarrow$ unique ergodicity...

Finite Multiplicity Theorem

Theorem (B 1995)

If $f \in C^\infty([0,1])$ *and* $h_{\text{top}}(f) > 0$, $\text{MME}(f)$ *finite*

Theorem (Sarig 2013)

If $f \in \text{Diff}^{1+\alpha}(M^2)$, $\alpha > 0$, *and* $h_{\text{top}}(f) > 0$, $\text{MME}(f)$ *countable*

Main Theorem (B-Crovisier-Sarig 2015)

If $f \in \text{Diff}^\infty(M^2)$ *and* $h_{\text{top}}(f) > 0$*:*
- $\text{MME}(f)$ *is finite*
- *topological transitivity* $\implies$ *uniqueness*

Two Generalizations

Theorem (B-Crovisier-Sarig 2015)

Let $f \in \mathrm{Diff}^\infty(M^2)$ *and* $\phi : M \to \mathbb{R}$ *Hölder-continuous*
If $\sup \phi - \inf \phi < h_{\mathrm{top}}(f)$ *then:*

- *finite # ergodic* **eq. measures** *maximizing* $h(\mu) + \mu(\phi)$
- *topological transitivity* $\implies$ *uniqueness*

Theorem (B-Crovisier-Sarig 2015)

If $f \in \mathrm{Diff}^r(M^2)$ *and*

$$h_{\mathrm{top}}(f) > \frac{\log \min(\mathrm{Lip}(f), \mathrm{Lip}(f^{-1}))}{r}$$

Then

- $\mathrm{MME}(f)$ *is finite*
- *topological transitivity* $\implies$ *uniqueness*

Remark

$\log \min(\mathrm{Lip}(f), \mathrm{Lip}(f^{-1})) \rightsquigarrow \lim_{n\to\infty} \frac{1}{n} \log \min(\mathrm{Lip}(f^n), \mathrm{Lip}(f^{-n}))$

Strategy of Proof

1. **Spectral decomposition** into homoclinic measure classes
2. **Local version** of Sarig's theorem: uniqueness inside each homoclinic class
3. **Entropy at infinity** in this spectral decomposition

Remark

related ideas in RH-RH-T-U (CMP 2011, Uniqueness of SRB...)

Ingredients

1. Sarig's coding and its control through hyperbolicity
2. Rectangles bounded by segments of stable and unstable curves (planar topology)
3. Pesin theory, Katok's horseshoe theorem and positive transverse dim ($\geq h_{\text{top}}(f,\Lambda)/\log \operatorname{Lip}(f)$)
4. Sard's theorem for dynamical foliations:
$$\text{transverse dim of } \{\text{leaves tangent to curve}\} \leq 1/r$$
5. Yomdin-Newhouse-Burguet's bound on tail entropy:
$h_*(f) := \lim_\epsilon \sup_{x\in M} h_{\text{top}}(f, B_f(x,\epsilon,\infty)) \leq \frac{\log \min(\operatorname{Lip}(f),\operatorname{Lip}(f^{-1}))}{r}$

Homoclinic relations adapted from Newhouse, Smale,...

Two **points** are homoclinically related ($p \sim q$) if:
- for x=p,q $W^s(f,x) := \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^n y, f^n x) < 0\}$ & $W^u(f,x) := W^s(f^{-1},x)$ are immersed submanifolds
- $W^u(f,p) \pitchfork W^s(f,q) \neq \emptyset$ ($p \rightsquigarrow q$)
- $W^u(f,q) \pitchfork W^s(f,p) \neq \emptyset$ ($q \rightsquigarrow p$)

Two **subsets** $X, Y \subset M$ are homoclinically related ($X \sim Y$) if:
- $\exists p \geq 1$ $\exists X = X_1 \cup \cdots \cup X_p$ and $\exists Y = Y_1 \cup \cdots \cup Y_p$
- $\forall 1 \leq i \leq p$ $\forall (x,y) \in X_i \times Y_i \quad x \sim y$

Applies to **horseshoes** Λ_i (transitive, loc. maximal, 0-dim, uniformly hyperbolic, invariant compact sets)
- Equivalence relation (λ-lemma)
- $(\exists x_i \in \Lambda_i,\ x_1 \sim x_2) \implies \Lambda_1 \sim \Lambda_2$

Classical case (Newhouse 1973)
Homoclinic relation on set $\mathrm{Per}_h(f)$ of hyperbolic periodic *orbits*
Homoclinic class: $H(f,\mathcal{O}) :=$ **closure**$\{\mathcal{O}' \in \mathrm{Per}_h(f) : \mathcal{O}' \sim \mathcal{O}\}$

Homoclinic relations

$f \in C^{1+}$

For **hyperbolic measures** $\mu, \nu \in \mathbb{P}_h(f)$ (ergodic, no 0 exponent):

$\mu \sim \nu$ iff $\exists X, Y \subset M$ $\mu(X) = \nu(Y) = 1$ and $X \sim Y$

Homoclinic measure class: $H(f, \mu) := \{\nu \in \mathbb{P}_h(f) : \nu \sim \mu\}$

Lemma

$\sim$ is an equivalence relation on $\mathbb{P}_h(f)$
For $\mathcal{O}_1, \mathcal{O}_2 \in \mathrm{Per}_h(f)$: $\delta_{\mathcal{O}_1} \sim \delta_{\mathcal{O}_2} \iff \mathcal{O}_1 \sim \mathcal{O}_2$
if $\Lambda_1 \sim \Lambda_2$, $\forall \mu_i \in \mathbb{P}_{\mathrm{erg}}(f, \Lambda_i)$ $\mu_1 \sim \mu_2$

Proposition For any $h(f, \mu) > 0$:

$\exists \mathcal{O} \in \mathrm{Per}_h(f)$ $\delta_{\mathcal{O}} \sim \mu$ also $\mu(H(f, \mathcal{O})) = 1$

If C^∞, for $h(f, \nu) > 0$:

$$\nu \in H(f, \mu) \iff \nu \in \mathbb{P}_{\mathrm{erg}}(f, H(f, \mathcal{O}))$$

Corollary

If $f \in C^\infty$, $h(f, \mu) > 0$, then: $\sup\{h(f, \nu) : \nu \in H(f, \mu)\}$ achieved

Sarig's Markov extension

Let $f \in \mathrm{Diff}^{1+}(M^2)$
Set $\chi_*(\mu) := \min(\lambda_1(\mu), -\lambda_2(\mu)) > 0$ for $\mu \in \mathbb{P}_{\mathrm{erg}}(f)$
Markov shift $\Sigma(G) :=$ { biinfinite paths on countable oriented G }

Theorem (Sarig 2013)

$\forall \chi > 0$ $\exists$ *Markov shift extension* $\pi_\chi : (\Sigma_\chi, \sigma) \to (M, f)$ C^β, $\beta > 0$
and $\forall \mu \in \mathbb{P}_{\mathrm{erg}}(f)$ *with* $\chi_*(\mu) > \chi$
π_χ *is a.e. onto, finite-to-1 restricted on "regular"* $\Sigma_\chi^\#$
Moreover
$\exists Q : M \to [1, \infty)$ *a.e. measuring "Pesin hyperbolicity" and*
$\exists$ *finite* $\mathcal{A}(N) \uparrow$ *alphabet of* Σ_χ
such that: α *"regular" lift of* $x \implies \alpha_0 \in \mathcal{A}\left(\sum_{n\in\mathbb{Z}} e^{-\epsilon|n|} Q(f^n x)\right)$

Local Uniqueness Theorem

Markov shift $\Sigma(G) :=$ { biinfinite paths on countable oriented G }
Its **irreducible components** $=$

$\Sigma(H)$ st $H \subset G$ inclusion-maximal strongly connected

Proposition

Let $f \in \mathrm{Diff}^{1+}(M^2)$, $\mu \in \mathbb{P}_{\mathrm{erg}}(f)$ with $h(f,\mu) > 0$.
For any $\chi > 0$, $\exists$ irreducible component $\Sigma_{\chi,0}$ of Σ_χ such that:

$$\pi_\chi(\mathbb{P}_{\mathrm{erg}}(\Sigma_{\chi,0})) \supset \{\nu \in H(f,\mu) : \chi_*(\nu) > \chi\}.$$

Proof (1) horseshoes; (2) homoclinically related periodic orbits; (3) related hyperbolic measures □

Theorem (B-Crovisier-Sarig (2015))

If C^{1+}, $\mu \in \mathbb{P}_{\mathrm{erg}}(f)$ with $h(f,\mu) > 0$,
$\exists$ at most one $\mu_{\max} \in H(f,\mu)$ st

$$h(f,\mu_{\max}) = \sup\{h(f,\nu) : \nu \in H(f,\mu)\}$$

(assuming C^∞) $\qquad = h_{\mathrm{top}}(f, H(f,\mathcal{O})) \qquad (\delta_{\mathcal{O}} \sim \mu)$

Small entropy at infinity

Theorem (B-Crovisier-Sarig 2015)

Let $f \in \mathrm{Diff}^\infty(M^2)$
For any $\delta > 0$*, the collection of homoclinic measure classes*

$$\{H(f,\mu) : \mu \in \mathbb{P}_h(f) \text{ and } h(f,\mu) \geq \delta\}$$

is finite

Remark

Together with the local uniqueness theorem $\implies$ *Main Theorem*

Proof of the Multiplicity Theorem

A **rectangle** R for a horseshoe Λ = open disk bounded by four segments $\partial^{\sigma,i}R$ of leaves of $\mathcal{F}^s(\Lambda)$, $\mathcal{F}^u(\Lambda)$

By contradiction:

$\mu_1, \mu_2, \dots \in \mathbb{P}_{\mathrm{erg}}(f)$ with $h(f,\mu_n) > \delta$ and $n \neq m \Longrightarrow \mu_n \not\sim \nu_m$
compactness $\rightsquigarrow \quad \mu_n \to \nu \in \mathbb{P}(f)$

for convenience: ν ergodic and hyperbolic

1. $C^\infty \implies \exists \epsilon_1 > 0 \; h_*(f,\epsilon_1) < \delta/10$ (Yomdin-...-Burguet)
2. $\exists\ R_1, \dots, R_N$ s.t. $\mathrm{diam}(R_i) < \epsilon_1 \& \, \nu(R_1 \cup \dots \cup R_N) > 1 - \mu_1$ (Pesin theory and Katok's horseshoe theorem)
3. For μ close to ν, let $\delta_{\mathcal{O}(p)} \sim \mu$: let $\sigma = u, s$,
$$\left[\forall i,j \; f^j(W^\sigma(p)) \text{ does not cross } \partial R_i \right]$$
$\implies h(f,\mu) \le h_*(f,\epsilon_1) + o(1)$
4. Otherwise, $\mu_n \sim \Lambda_i$ for some $1 \le i \le N$ (dynamical Sard's theorem) $\square$

Conclusion

Questions

1. Find $f \in \mathrm{Diff}^r(M^2)$, $r < \infty$, with infinitely many MME
2. Combinatorial or geometric properties of the measure spectral decomposition. What combination of entropies and periods can be achieved by C^∞ surface diffeomorphisms?
3. Finite multiplicity for C^∞ nonsingular 3-dimensional flows using Lima-Sarig?
4. In higher dimensions, does hyperbolicity of the high entropy ergodic measures + zero tail entropy imply finite multiplicity?

Thank you!