On quasi-conformal (in-) compatibility of satellite copies of the Mandelbrot set

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Quadratic polynomials on $\widehat{\mathbb{C}}$

- $P_c(z) = z^2 + c$, ∞ (super)attracting fixed point, with basin $\mathcal{A}(\infty)$.
- Filled Julia set $K_c = K_{P_c} = \widehat{\mathbb{C}} \setminus \mathcal{A}(\infty)$, $J_P = \partial K_P = \partial \mathcal{A}(\infty)$
- Mandelbrot set: set of parameters for which K_{P_c} is connected (**connectedness locus** for the family P_c).

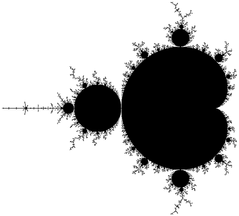


Figure : The Mandelbrot set.

Figure : K_0, 0 center of the main component

Figure : $K_{1/4}$, 1/4 root of the main component.

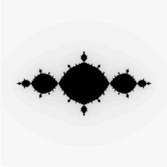


Figure : K_c, c center period 2 component.

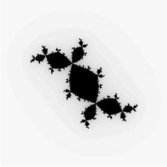


Figure : K_c, c center period 3 component.

Polynomial-like mappings

- A (dg d) polynomial-like map is a triple (f, U', U), where $U' \subset\subset U$ and $f : U' \to U$ is a (dg d) proper and holomorphic map.
- **Straightening theorem (Douady-Hubbard, '85)** Every (dg d) polynomial-like map $f : U' \to U$ is hybrid equivalent to a (dg d) polynomial.

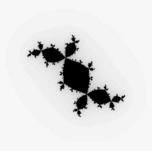
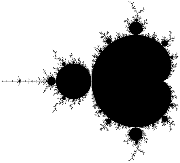


Figure : K_c, c center of the period 3 component

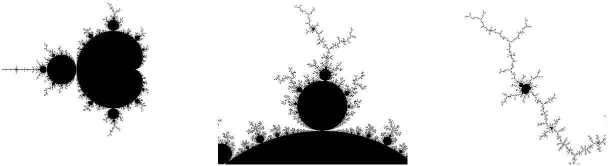
Figure : K_0, 0 center of the main component



- **Theorem (D-H,'85)** (Under some conditions) there exists a homeomorphism χ between the connectedness locus of a family of polynomial-like maps and the Mandelbrot set M.

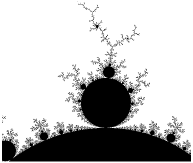
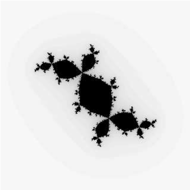
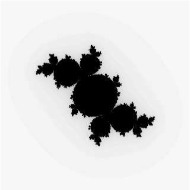
Consequence: little copies of M inside M

- Satellite copies of M (attached to some hyperbolic component of M): χ homeomorphism except at the root.
- H primitive (non satellite): χ homeomorphism,
- Haissinsky ('00): χ homeomorphism at the root in the satellite case.



M and its little copies

- Conjecture (D-H,'85) χ is the restriction of a quasi-conformal map in the primitive case, and away from neighborhoods of the root in the satellite case.
- Lyubich ('99): χ is qc in the primitive case, and outside a neighborhood of the root in the satellite case.
- *Are the satellite copies mutually qc homeomorphic?*
- L. ('14): the root of any two satellite copies have restrictions q-c conjugate.

Satellite copies, result

- $M_{p/q}$ satellite copy attached to M_0 at c, where P_c has a fixed point with multiplier $\lambda = e^{2\pi i p/q}$
- **Theorem (L-Petersen, 2015)**: For p/q and P/Q irreducible rationals with $q \neq Q$,

$$\xi := \chi_{P/Q}^{-1} \circ \chi_{p/q} : M_{p/q} \to M_{P/Q}$$

is not quasi-conformal, *i.e.* it does not admit a quasi-conformal extension to any neighborhood of the root.

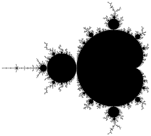


Figure : M.

Main idea

- **Proposition:** $c \in (M_{p/q} \setminus \{0\})$, $f_\lambda : U' \to U$ polynomial-like restriction of P_c, $\xi(c) \in M_{P/Q}$ and $g_\nu : V' \to V$ polynomial-like restriction of $P_{\xi(c)}$. Any quasi-conformal conjugacy ϕ between f_λ and g_ν has:

$$\limsup_{z \to \beta_f} \mathrm{Log} K_\phi(z) \geq d_{\mathbb{H}_+}(\Lambda, N),$$

where $\Lambda = \mathrm{Log}(\mathit{multiplier}(\beta_f))$, $N = \mathrm{Log}(\mathit{multiplier}(\beta_g))$

- Proof of the Proposition:
  1. $(U \setminus \{\beta_f\})/f$ and $(V \setminus \{\beta_g\})/g$ (marked) quotient tori.
  2. ϕ induces a qc homeomorphism between the corresponding (marked) quotient tori.
  3. Teichmüller extremal theorem for complex tori:

$$d_{\mathbb{H}_+}(\Lambda, M) = d_T(T_\Lambda, T_M) =: \mathit{inf}_\varphi \mathrm{Log} K_\phi,$$

     where $\varphi : T_\Lambda \to T_M$ qc homeo (respecting the marking).
  4. So $\limsup_{z \to 0} \mathrm{Log} K_\phi(z) \geq \mathit{inf}_\phi \mathrm{Log} K_\phi = d_T(T_\Lambda, T_M) = d_{\mathbb{H}_+}(\Lambda, M)$.

Lower bound for qc conjugacy, parameter plane

1. Generalization of the Teich. extr. thm for a non-compact setting and a holomorphic motion argument give: **Theorem:** $\Lambda^* \in \Lambda(M_{p/q})$ Misiurewicz parameter s.t. the critical value is prefixed to 0, $M^* = \hat{\xi}(\Lambda^*)$. Then

$$\limsup_{\Lambda \to \Lambda^*} \operatorname{Log} K_{\hat{\xi}}(\Lambda) \geq d_{\mathbb{H}_+}(\Lambda^*, M^*).$$

2. Yoccoz inequality gives that he hyperbolic size of the limbs of the considered limbs shrink to 0 going to the root,
3. ρ multiplier of the α f.p., computations (using Res iter) give: For $q \neq Q$, and $\rho = e^{it} \in \mathbb{S}^1$,

$$d_{\mathbb{H}_+}(\Lambda(\rho), M(\rho)) \xrightarrow{\rho \to 1} \infty$$

Combining $1, 2, 3$ we have the result.

Thank you for your attention!