Regularity and bifurcation phenomena in simple families of maps.

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June 2015

Work with

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Sefano Marmi (SNS)

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Sefano Marmi (SNS) Alessandro Profeti (SNS)

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Sefano Marmi (SNS) Alessandro Profeti (SNS)
Henk Bruin (Wien)

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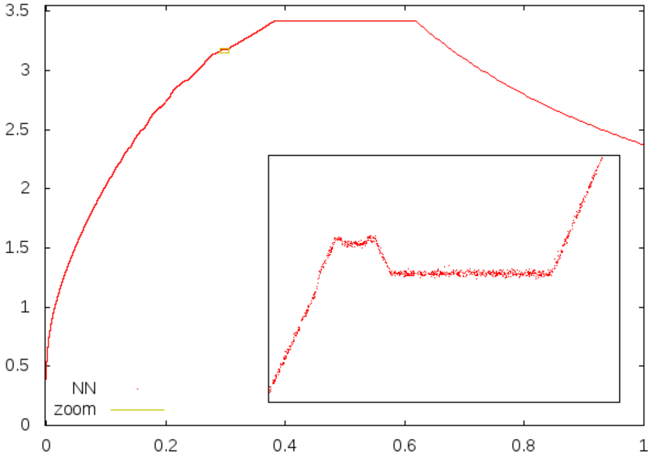

Nakada, Kraaikamp, Marmi-Luzzi, Nakada-Natsui, C-Tiozzo, etc.

A (much simpler) simple affine model

[BSORG] V. Botella-Soler, J. A. Oteo, J. Ros, P. Glendinning, *Families of piecewise linear maps with constant Lyapunov exponents*, J. Phys. A: Math. Theor. **46** (2013)

Piecewise affine maps (c.f. [BSORG])

Let $s > 1$ be a fixed slope, and $T_\beta(x) = x + 2$ if $x < 0$,
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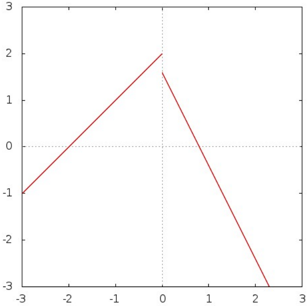

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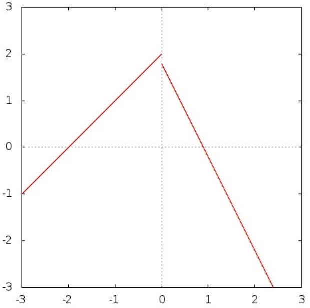

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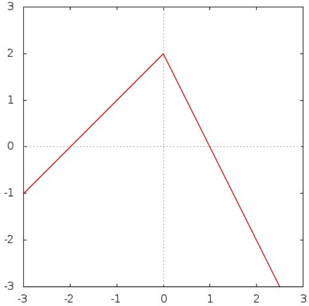

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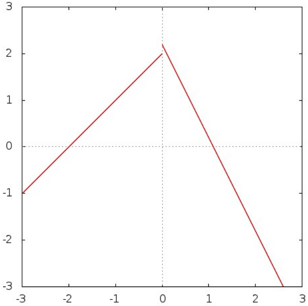

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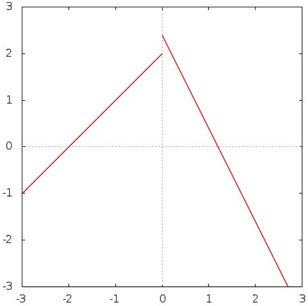

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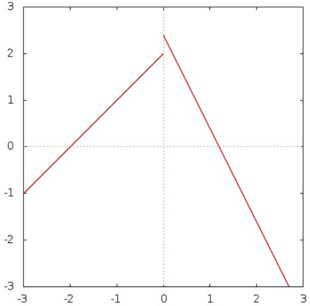

In this picture the slope of the expanding branch is $s = 2$.

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- algebraic values (main example $s = \frac{\sqrt{5}+1}{2}$).

Entropy for piecewise affine maps with two branches

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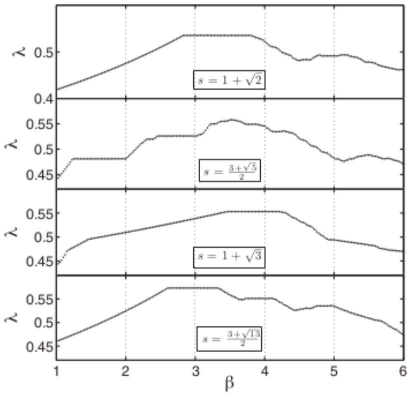

Figure 4. Lyapunov exponents as a function of β for the case of s taking the value of different quadratic Pisot numbers.

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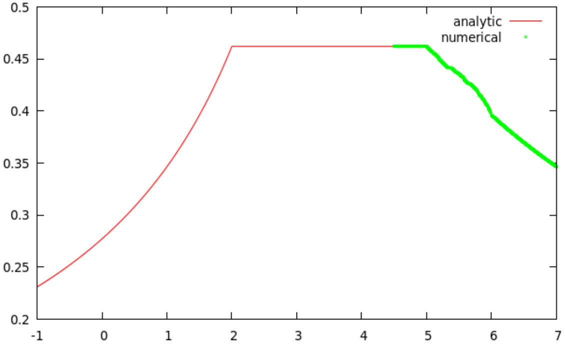

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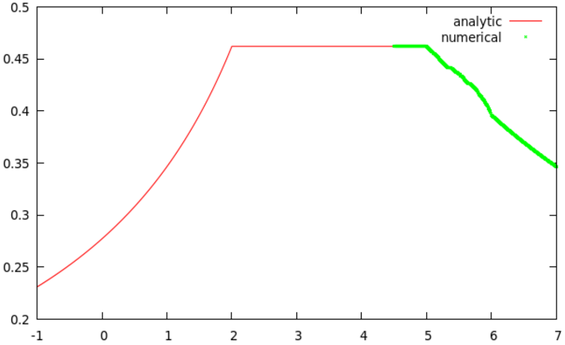

Tool: explicit computation (the invariant density is a simple function).

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Remark: the matching property occurs also in other families of transformations (for instance generalized β-transformations).

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Matching: general definition.

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Let $T : S \to S$ be a piecewise smooth map. We say that the map T satisfies the *matching condition* if for every discontinuity point γ of T (or T') there exist integers $k^-, k^+ \in \mathbb{N}$ (called *matching exponents*) such that

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where the union is taken on γ ranging on the discontinuities of T and T'.

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Theorem

Let T be a piecewise affine, eventually expanding map satisfying the matching property. Then T admits an invariant density which is locally constant outside the prematching set.

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- If $s \in \mathbb{Q} \setminus \mathbb{Z}$ then matching doesn't occur.
- For all integer slopes $s \geq 2$ matching actually occurs.

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- we will call *matching interval* a connected component of the set where stable matching holds; here the matching exponents (and matching index) are constant.

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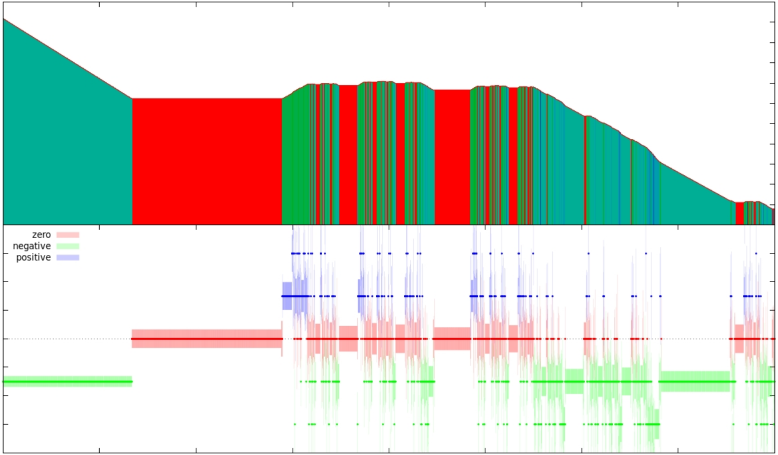
The complement of the set where stable matching holds is called **bifurcation set**.

Stable matching vs. monotonicity

Zoom of entropy function for $\beta \in [5.32, 5.40]$.



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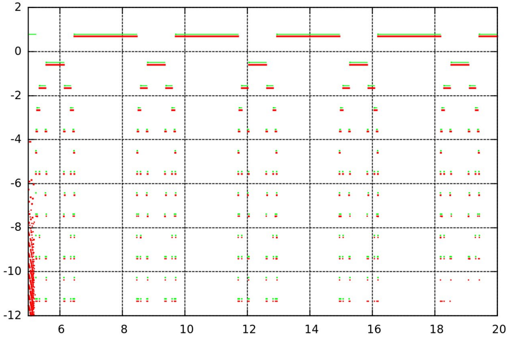

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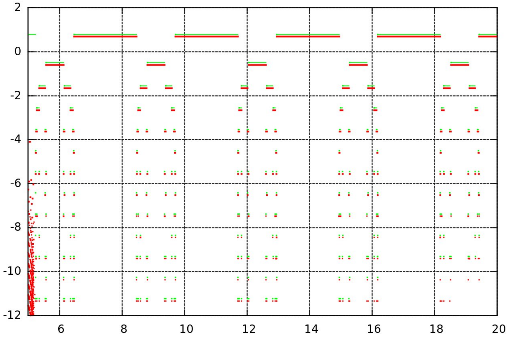

However in this case the bifurcation set is **unbounded**.

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The same question can be asked for the family of generalized β-transformations $(T_\alpha)_{\alpha \in [0,1]}$

$$T_\alpha(x) := sx + \alpha \pmod 1$$

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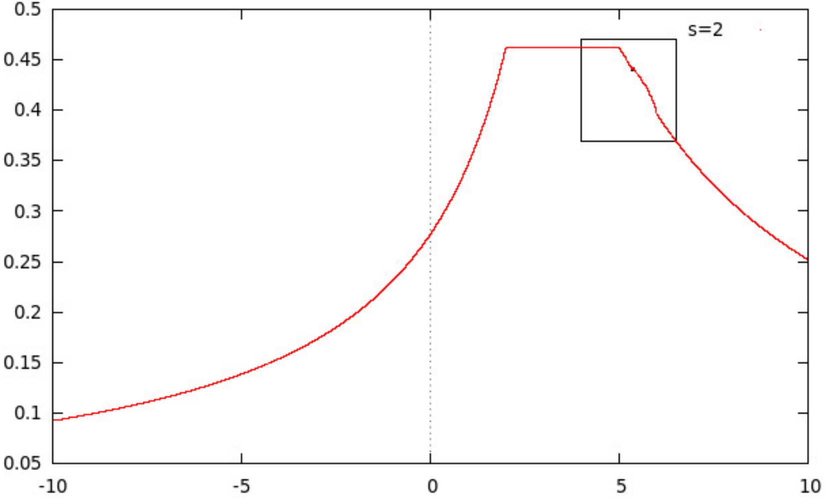
Being bolder, one could also ask the following question:
Is the graph of the entropy self-similar? Is this a consequence of **renormalization**?
c.f. C-Tiozzo: "Tuning and plateaux for the entropy of α-continued fractions", Nonlinearity **26**, 2013.

Zooming in

Let us look at the graph for $s = 2$ more closely:



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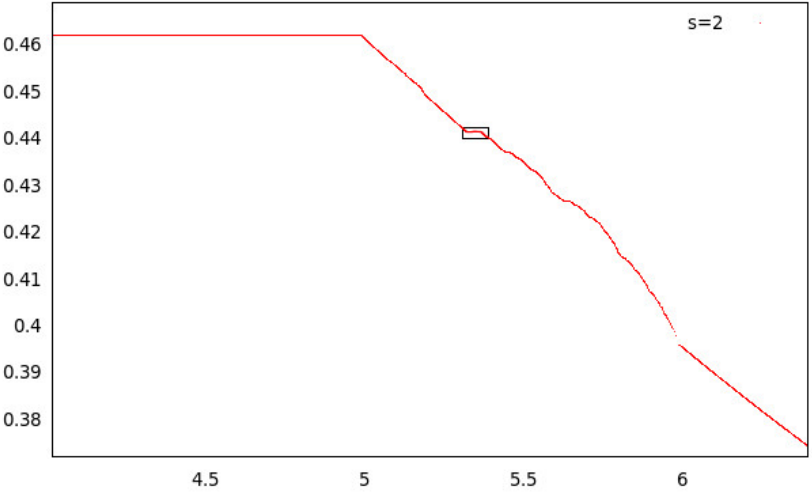

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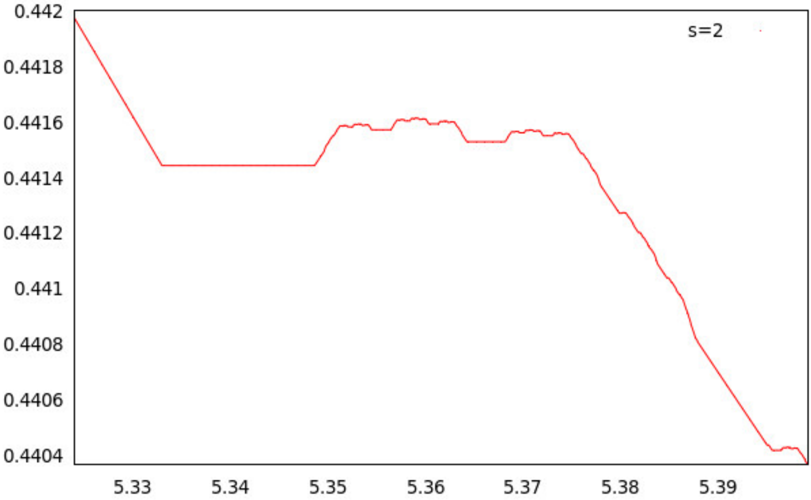

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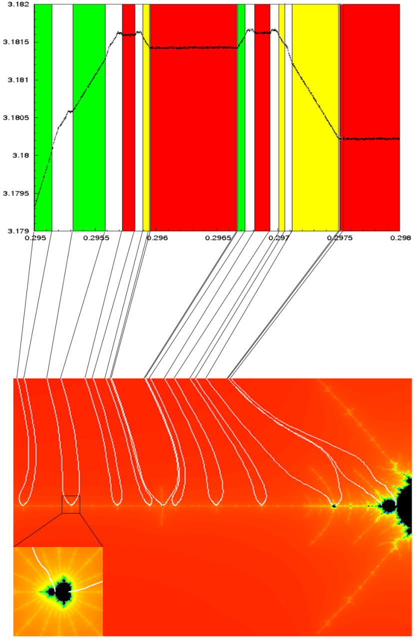
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